Analytic Assessment of Eigenvalues of the Neutron Transport Equation*

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Articles for Current Study

- Primarily reviewed article
  - J. Wood, “Eigenvalues of the neutron transport equation”, Proceedings of Physics Society, Vol. 85, 1965.
- References
  - J. Wood, “The existence of a discrete decay constant exceeding $(v\Sigma(E))_{min}$ in the pulsed neutron experiment”, British Journal of Applied Physics, Ser. 2, Vol. 2, 1969.
  - N. Corngold, “Some transient phenomena in thermalization, I. Theory”, Nuclear Science and Engineering, Vol. 19, pp. 80-90, 1964.
  - N. Corngold and P. Michael, “Some transient phenomena in thermalization, I. Implications for experiment”, Nuclear Science and Engineering, Vol. 19, pp. 91-94, 1964.
  - M. Nelkin, “Asymptotic solutions of the transport equation for thermal neutrons”, Physica, Vol. 29, pp. 261-273, 1963.
  - A. M. Weinberg and E. P. Wigner, “The physical theory of neutron chain reactors”, The university of Chicago press, 1958.

This presentation will focus on the following key topics

- ❑ Introduction: Physical Meaning of the Decay Constants (Temporal and Spatial Eigenvalues)
- ❑ Three Methods of Solution: Diffusion Approximation, Spherical Harmonics, and Boltzmann Approximation
- ❑ Time-dependent Decay Constants in a Pulsed Neutron Problem (Gas Model)
- ❑ Time-dependent Decay Constants in a Pulsed Neutron Problem (Polycrystalline Model)
- ❑ Spatially dependent Decay Constants in a Diffusion Problem

INTRODUCTION

- Temporal eigenvalues (time decay constants, λ_n)
  - $\Phi(x,E,\mu,t)=\psi(E,\mu)e^{-\lambda t+iBx}$
  - Parameter of natural phenomena of asymptotic behavior
  - Dimension of reciprocal of time: [λ]=[1/sec]
  - The higher eigenvalues → the faster decay of neutron density
  - Only the lowest value of λ, viz. λ_o ***has the physical meaning***
- Spatial eigenvalues (space decay constants, K_n)
  - $\Phi(x,E,\mu,t)=\psi(E,\mu)e^{-Kx}$
  - Parameter of natural phenomena of asymptotic behavior
  - Dimension of reciprocal of distance: [K]=[1/cm]
  - The higher eigenvalues → the smaller diffusion lengths
  - Only the lowest value of K, viz. K_o ***has the physical meaning***

Three Methods of Solution: Diffusion Approximation

$$\frac{1}{v}\frac{\partial\Phi(E,\Omega)}{\partial t} = -\Omega\cdot\nabla\Phi(E,\Omega) - \Sigma(E)\Phi(E,\Omega) + S(E,\Omega) + \int dE'\,d\Omega'\,\Phi(E',\Omega')\Sigma_s(E'\to E;\Omega'\to\Omega)$$

- Balance between the decrease and increase of neutron density in the system
- Neutron gain: Scattering and source
- Neutron loss: Collision and streaming
- Isotropic scattering: Cross section does not depend on position
- Inclusion of anisotropic scattering:

$$\Sigma_s(E'\to E;\Omega'\to\Omega) \Rightarrow \Sigma_s(x,E'\to E;\Omega'\to\Omega)$$

Three Methods of Solution: Spherical Harmonics (P_3)

$$\Sigma_s\left(E' \to E; \Omega' \to \Omega\right) = \sum_l s_l\left(E' \to E\right) P_l\left(\cos\theta_o\right)$$

$$\Phi\left(x, E, \Omega, t\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm}\left(x, E, t\right) P_{lm}\left(\Omega\right)$$

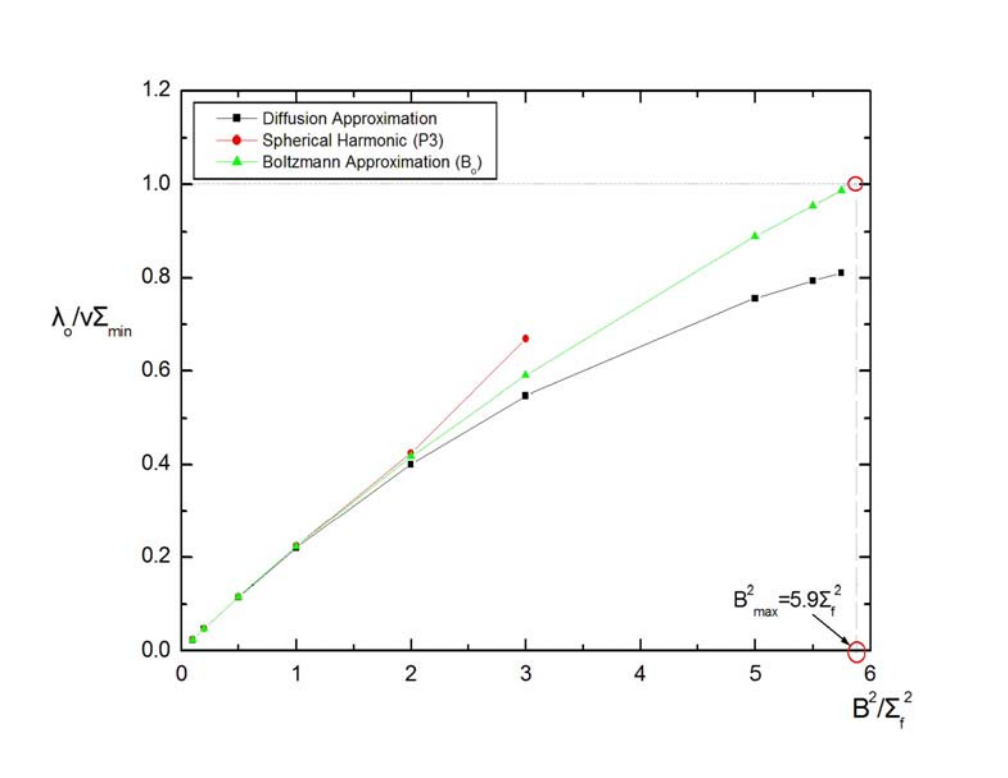
- Angular dependence of scattering cross section by spherical harmonics
- Modification of scattering term by Legendre expansion $P_l(\cos\theta_o)$
- Expansion by spherical harmonics, $P_{lm}(\Omega)$
- Variable dependence: $(x, y, z, E, \theta, \phi, t) \rightarrow (x, y, z, E, t)$

Three Methods of Solution: Boltzmann Approximation (B_o)

$$\Psi_l(B,E) = \int e^{-iBx}\Phi_{l0}(x,E)dz$$

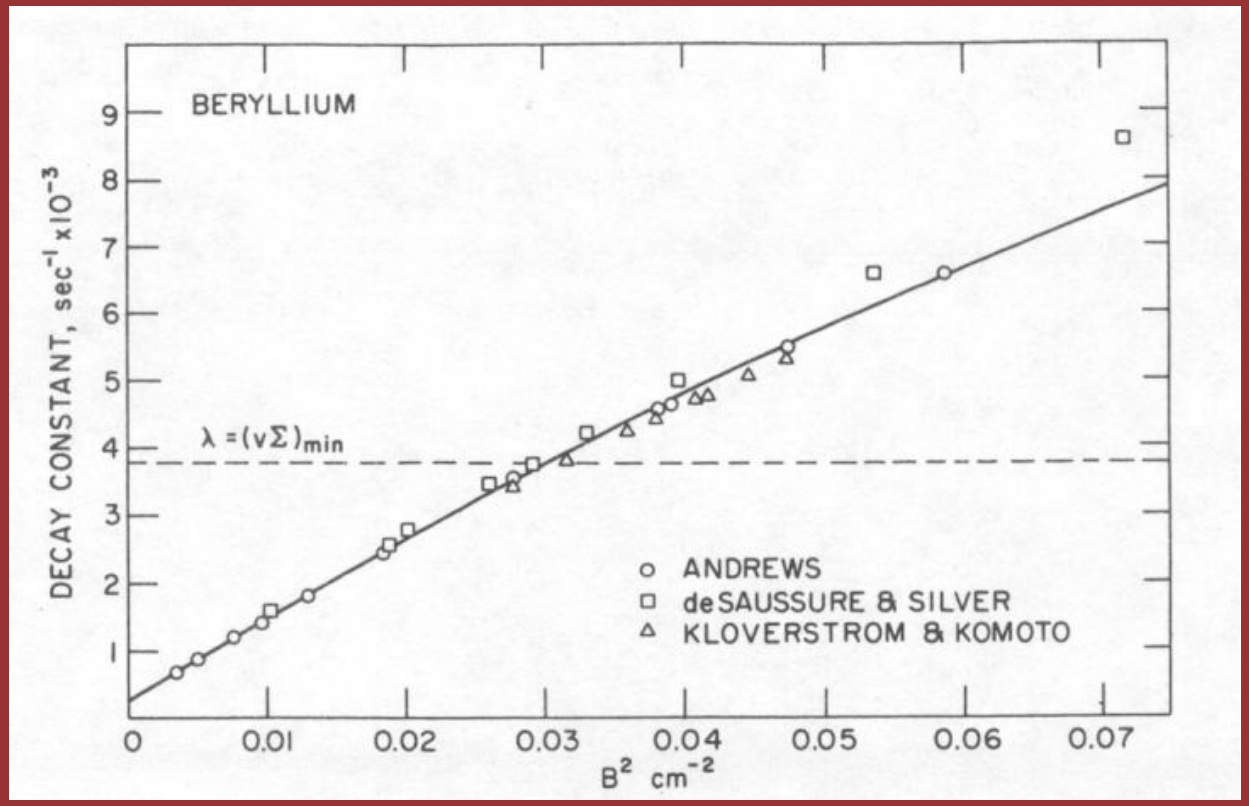
- Multiplied with *exp (-iBx)* and integrating over x
- Setting all $\Psi_l = 0$ (for $l > L$) → B_L approximation
- Comparable to P_L approximation
- Faster convergence than P_L approximation

Temporal Eigenvalues (Gas Model)



- Comparison of $\lambda_o/(v\Sigma)_{min}$ for three methods of solution
- Data points taken from Wood, 1965
- B_o approximation is exact in principle
- There exists a theoretical limit for $\lambda_o < (v\Sigma)_{min}$
- According maximum buckling is bounded, $B^2_{max}=5.9\Sigma_f^2$

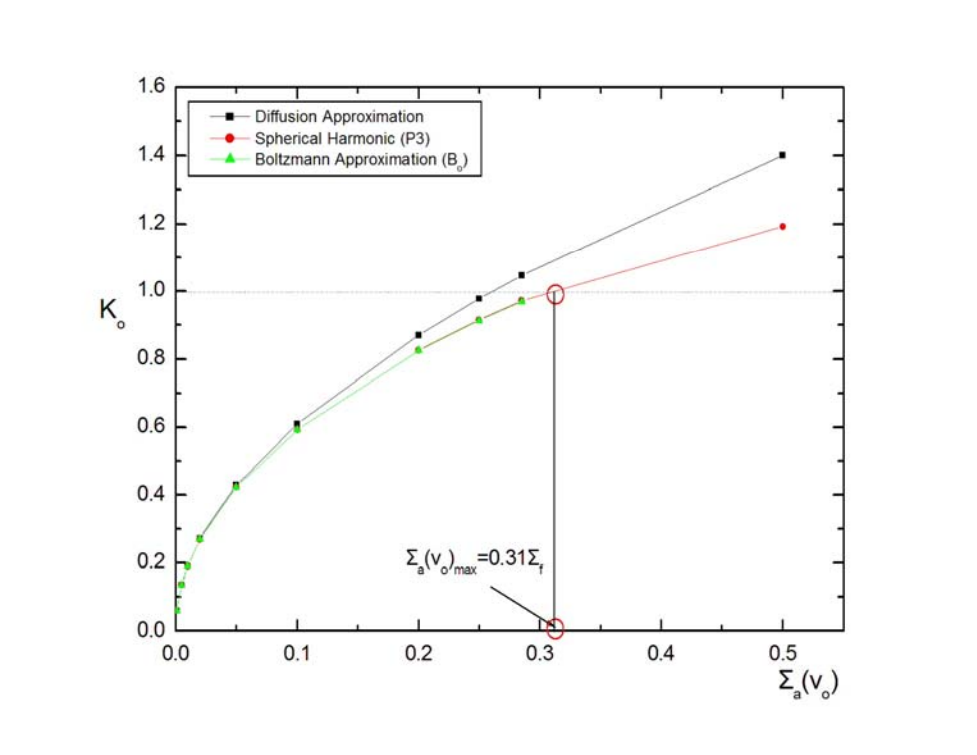
Temporal Eigenvalues (Polycrystalline Model)



Corngold N., and P. Michael. "Some Transient Phenomena in Thermalization, I. Implications for Experiment." *Nuclear Science and Engineering* 19 (1964): 91-94.

- Experimental results cited from Corngold and Michael, 1964
- Exceeding behavior of $\lambda_o > (v\Sigma)_{min}$
- $(v\Sigma)_{min}$ → An index of the amount of inelastic scattering experienced by a neutron of low energy → Corresponding to Bragg cut-off energy
- Presumably it was attributed by a measurements uncertainty
- High buckling → small system → high leakage rate → lack of sufficient neutron intensity

Spatial Eigenvalues (Gas Model)



- Comparison of K_o for three methods of solution
- Data points taken from Wood, 1965
- K_o disappears at sufficiently strong concentration of absorber
- Extrapolated to K_o=1 for B_o approximation

  → Maximum $\Sigma_a(v_o)=0.31\Sigma_f$

Spatial Eigenvalues

- Corngold and Michael, 1964
  - Diffusion length L must be larger than $1/\Sigma_{min}$
  - In non-crystalline moderators (H_2O)

    → the value of $(\Sigma_s)_{min}$ *will be the "free atom" value, at E=0.1 eV*
  - Crystalline moderators (Be and C)

    → the minimum will lie on the low side of Bragg cut-off side
- Miller (1961) and Starr and Koppel (1962)
  - Diffusion length in light water with heavy absorber, boron
  - {0.73 cm < L < 2.82 cm} > {$1/\Sigma_{min}$=0.65 cm} → acceptable range
- Miller (1961)
  - Diffusion length in light water with cadmium
  - {0.22 cm < L < 0.55 cm} < {$1/\Sigma_{min}$=0.65 cm} → contradicted to theoretical limit

Summary and Conclusions

- Physical meaning of the decay constants has been implemented for the pulsed neutron and diffusion length problems
- Three methods of solutions for the decay constants have been introduced briefly
- Only the fundamental eigenvalue has the physical meaning
- The temporal eigenvalues for the gas model has been limited by the minimum collision rate of $(v\Sigma)_{min}$
- The temporal eigenvalues exceeding the limit can be attributed by the lack of sufficient neutron density originated from the small system
- The spatial eigenvalues must be greater than $1/\Sigma_{min}$
- The existence of spatial eigenvalues can be restricted by strong concentration of absorber, “die-away”

* APPENDIX
Theoretical Limit for Temporal Eigenvalues

$$\left(-\frac{\lambda}{v}+\Sigma(E)+i\Omega\cdot B\right)\psi(E,\Omega)=\frac{1}{4\pi}\int_0^\infty \Sigma(E'\to E)\psi(E')dE'$$

$$\psi(E)=\frac{1}{4\pi}\int_\Omega \frac{d\Omega}{-\lambda/v+\Sigma(E)+i\Omega\cdot B}\int_0^\infty \Sigma(E'\to E)\psi(E')dE'$$

$$\psi(E)=\frac{1}{B}\tan^{-1}\left(\frac{vB}{-\lambda+v\Sigma(E)}\right)\int_0^\infty \Sigma(E'\to E)\psi(E')dE'$$

$$\psi = A(\lambda,B)\psi$$

$$a_{ij}=\frac{1}{B}\tan^{-1}\left(\frac{v_i B}{-\lambda+v_i\Sigma(E)}\right)w_j\Sigma(E_j'\to E_i)$$

- "The existence of a discrete decay constant exceeding $(v\Sigma)_{min}$ in the pulsed neutron experiment", J. Wood, 1968