Analytic Assessment of Eigenvalues of the Neutron Transport Equation

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Articles for Current Study

- Primarily reviewed article

- References
This presentation will focus on the following key topics

- Introduction: Physical Meaning of the Decay Constants (Temporal and Spatial Eigenvalues)
- Three Methods of Solution: Diffusion Approximation, Spherical Harmonics, and Boltzmann Approximation
- Time-dependent Decay Constants in a Pulsed Neutron Problem (Gas Model)
- Time-dependent Decay Constants in a Pulsed Neutron Problem (Polycrystalline Model)
- Spatially dependent Decay Constants in a Diffusion Problem
INTRODUCTION

- Temporal eigenvalues (time decay constants, $\lambda_n$)
  - $\Phi(x, E, \mu, t) = \psi(E, \mu) e^{-\lambda t + iBx}$
  - Parameter of natural phenomena of asymptotic behavior
  - Dimension of reciprocal of time: $[\lambda] = [1/\text{sec}]$
  - The higher eigenvalues $\Rightarrow$ the faster decay of neutron density
  - Only the lowest value of $\lambda$, viz. $\lambda_o$ has the physical meaning

- Spatial eigenvalues (space decay constants, $K_n$)
  - $\Phi(x, E, \mu, t) = \psi(E, \mu) e^{-Kx}$
  - Parameter of natural phenomena of asymptotic behavior
  - Dimension of reciprocal of distance: $[K] = [1/\text{cm}]$
  - The higher eigenvalues $\Rightarrow$ the smaller diffusion lengths
  - Only the lowest value of $K$, viz. $K_o$ has the physical meaning
Three Methods of Solution: Diffusion Approximation

\[ \frac{1}{v} \frac{\partial \Phi(E, \Omega)}{\partial t} = -\Omega \cdot \nabla \Phi(E, \Omega) - \Sigma(E) \Phi(E, \Omega) + S(E, \Omega) + \int dE' d\Omega' \Phi(E', \Omega') \Sigma_s(E' \rightarrow E; \Omega' \rightarrow \Omega) \]

- Balance between the decrease and increase of neutron density in the system
- Neutron gain: Scattering and source
- Neutron loss: Collision and streaming
- Isotropic scattering: Cross section does not depend on position
- Inclusion of anisotropic scattering:
  \[ \Sigma_s(E' \rightarrow E; \Omega' \rightarrow \Omega) \Rightarrow \Sigma_s(x, E' \rightarrow E; \Omega' \rightarrow \Omega) \]
Three Methods of Solution: Spherical Harmonics ($P_3$)

\[ \Sigma_s (E' \rightarrow E; \Omega' \rightarrow \Omega) = \sum_l s_l (E' \rightarrow E) P_l (\cos \theta_o) \]

\[ \Phi(x, E, \Omega, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm} (x, E, t) P_{lm} (\Omega) \]

- Angular dependence of scattering cross section by spherical harmonics
- Modification of scattering term by Legendre expansion $P_l (\cos \theta_o)$
- Expansion by spherical harmonics, $P_{lm} (\Omega)$
- Variable dependence: $(x, y, z, E, \theta, \phi, t) \rightarrow (x, y, z, E, t)$
Three Methods of Solution:
Boltzmann Approximation \((B_o)\)

\[
\Psi_l(B, E) = \int e^{-iBx} \Phi_{l0}(x, E) \, dz
\]

- Multiplied with \(\exp (-iBx)\) and integrating over \(x\)
- Setting all \(\Psi_l = 0\) (for \(l > L\)) \(\rightarrow B_L\) approximation
- Comparable to \(P_L\) approximation
- Faster convergence than \(P_L\) approximation
Temporal Eigenvalues (Gas Model)

- Comparison of $\lambda_o/(\nu \Sigma)_{\text{min}}$ for three methods of solution
- Data points taken from Wood, 1965
- $B_o$ approximation is exact in principle
- There exists a theoretical limit for $\lambda_o < (\nu \Sigma)_{\text{min}}$
- According maximum buckling is bounded, $B^2_{\text{max}}=5.9 \Sigma_f^2$
Temporal Eigenvalues (Polycrystalline Model)


- Experimental results cited from Corngold and Michael, 1964
- Exceeding behavior of $\lambda_0 > (\nu \Sigma)_{\text{min}}$
- $(\nu \Sigma)_{\text{min}} \rightarrow$ An index of the amount of inelastic scattering experienced by a neutron of low energy
  $\rightarrow$ Corresponding to Bragg cut-off energy
- Presumably it was attributed by a measurements uncertainty
- High buckling $\rightarrow$ small system $\rightarrow$ high leakage rate $\rightarrow$ lack of sufficient neutron intensity
Spatial Eigenvalues (Gas Model)

- Comparison of $K_o$ for three methods of solution
- Data points taken from Wood, 1965
- $K_o$ disappears at sufficiently strong concentration of absorber
- Extrapolated to $K_o=1$ for $B_0$ approximation
  $\Rightarrow$ Maximum $\sum_a(v_o)=0.31 \Sigma_f$
Spatial Eigenvalues

- Corngold and Michael, 1964
  - Diffusion length $L$ must be larger than $1/\Sigma_{\text{min}}$
  - In non-crystalline moderators (H$_2$O)
    -> the value of $(\Sigma_s)_{\text{min}}$ will be the “free atom” value, at $E=0.1$ eV
  - Crystalline moderators (Be and C)
    -> the minimum will lie on the low side of Bragg cut-off side

- Miller (1961) and Starr and Koppel (1962)
  - Diffusion length in light water with heavy absorber, boron
    - $\{0.73 \text{ cm} < L < 2.82 \text{ cm}\} > \{1/\Sigma_{\text{min}} = 0.65 \text{ cm}\}$ -> acceptable range
  - Miller (1961)
    - Diffusion length in light water with cadmium
      - $\{0.22 \text{ cm} < L < 0.55 \text{ cm}\} < \{1/\Sigma_{\text{min}} = 0.65 \text{ cm}\}$ -> contradicted to theoretical limit
Summary and Conclusions

- Physical meaning of the decay constants has been implemented for the pulsed neutron and diffusion length problems.
- Three methods of solutions for the decay constants have been introduced briefly.
- Only the fundamental eigenvalue has the physical meaning.
- The temporal eigenvalues for the gas model has been limited by the minimum collision rate of \((v\Sigma)_{\text{min}}\).
- The temporal eigenvalues exceeding the limit can be attributed by the lack of sufficient neutron density originated from the small system.
- The spatial eigenvalues must be greater than \(1/\Sigma_{\text{min}}\).
- The existence of spatial eigenvalues can be restricted by strong concentration of absorber, “die-away”.
* APPENDIX

Theoretical Limit for Temporal Eigenvalues

\[
\psi(E, \Omega) = \frac{1}{4\pi} \int_0^\infty \Sigma(E' \rightarrow E) \psi(E') dE'
\]

\[
\psi(E) = \frac{1}{4\pi} \int_\Omega \frac{d\Omega}{-\lambda/\nu + \Sigma(E) + i\Omega \cdot B} \int_0^\infty \Sigma(E' \rightarrow E) \psi(E') dE'
\]

\[
\psi(E) = \frac{1}{B} \tan^{-1}\left( \frac{\nu B}{-\lambda + \nu \Sigma(E)} \right) \int_0^\infty \Sigma(E' \rightarrow E) \psi(E') dE'
\]

\[
\psi = A(\lambda, B) \psi
\]

\[
a_{ij} = \frac{1}{B} \tan^{-1}\left( \frac{\nu_i B}{-\lambda + \nu_j \Sigma(E)} \right) w_j \Sigma(E_j' \rightarrow E_i)
\]

“The existence of a discrete decay constant exceeding \((\nu \Sigma)_{min}\) in the pulsed neutron experiment”, J. Wood, 1968