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## Some new directions in $p$ -adic Hodge theory

par KIRAN S. KEDLAYA

RÉSUMÉ. Nous rappelons quelques constructions fondamentales de la théorie de Hodge  $p$ -adique, et décrivons ensuite quelques résultats nouveaux dans ce domaine. Nous traitons principalement la notion de  $B$ -paire, introduite récemment par Berger, qui fournit une extension naturelle de la catégorie des représentations Galoisienne  $p$ -adiques. (Sous une autre forme, cette extension figure dans les travaux récents de Colmez, Bellaïche et Chenevier sur les représentations triangulables.) Nous discutons aussi quelques résultats de Liu qui indiquent que le formalisme de la cohomologie Galoisienne, y compris la dualité locale de Tate, se prolonge aux  $B$ -paires.

ABSTRACT. We recall some basic constructions from  $p$ -adic Hodge theory, then describe some recent results in the subject. We chiefly discuss the notion of  $B$ -pairs, introduced recently by Berger, which provides a natural enlargement of the category of  $p$ -adic Galois representations. (This enlargement, in a different form, figures in recent work of Colmez, Bellaïche, and Chenevier on trianguline representations.) We also discuss results of Liu that indicate that the formalism of Galois cohomology, including Tate local duality, extends to  $B$ -pairs.

### 1. Setup and overview

Throughout,  $K$  will denote a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers, and  $G_K = \text{Gal}(\overline{\mathbb{Q}_p}/K)$  will denote the absolute Galois group of  $K$ . We will write  $\mathbb{C}_p$  for the completion of  $\overline{\mathbb{Q}_p}$ ; it is algebraically closed, and complete for a nondiscrete valuation. For any field  $F$  carrying a valuation (like  $K$  or  $\mathbb{C}_p$ ), we write  $\mathfrak{o}_F$  for the valuation subring.

One may think of  $p$ -adic Hodge theory as the  $p$ -adic analytic study of  $p$ -adic representations of  $G_K$ , by which we mean finite dimensional  $\mathbb{Q}_p$ -vector spaces  $V$  equipped with continuous homomorphisms  $\rho : G_K \rightarrow \text{GL}(V)$ . (One might want to allow  $V$  to be a vector space over a finite extension of  $\mathbb{Q}_p$ ; for ease of exposition, I will only retain the  $\mathbb{Q}_p$ -structure in this discussion.) A typical example of a  $p$ -adic representation is the (geometric)  $p$ -adic étale cohomology  $H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  of an algebraic variety  $X$  defined over  $K$ . Another typical example is the restriction to  $G_K$  of a global Galois

representation  $G_F \rightarrow \mathrm{GL}(V)$ , where  $F$  is a number field,  $K$  is identified with the completion of  $F$  at a place above  $p$ , and  $G_K$  is identified with a subgroup of  $G_F$ ; this agrees with the previous construction if the global representation itself arises as  $H_{\mathrm{et}}^i(X_{\overline{F}}, \mathbb{Q}_p)$  for a variety  $X$  over  $F$ .

Examples of this sort may be thought of as having a “geometric origin”; it turns out that there are many  $p$ -adic representations without this property. For instance, there are several constructions that start with the  $p$ -adic representations associated to classical modular forms (which do have a geometric origin), and produce new  $p$ -adic representations by  $p$ -adic interpolation. These constructions include the  $p$ -adic families of Hida [11], and the eigencurve of Coleman and Mazur [5]. (Note that these are *global* representations, so one has to first restrict to a decomposition group to view them within our framework.)

Our purpose here is to first recall the basic framework of  $p$ -adic Hodge theory, then describe some new results. One important area of application is Colmez’s work on the  $p$ -adic local Langlands correspondence for 2-dimensional representations of  $G_{\mathbb{Q}_p}$  [6, 7, 8, 9].

### 2. The basic strategy

The basic methodology of  $p$ -adic Hodge theory, as introduced by Fontaine, is to linearize the data of a  $p$ -adic representation  $V$  by tensoring with a suitable topological  $\mathbb{Q}_p$ -algebra  $\mathbf{B}$  equipped with a continuous  $G_K$ -action, then forming the space

$$D_{\mathbf{B}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B})^{G_K}$$

of Galois invariants. One usually asks for  $\mathbf{B}$  to be *regular*, which means that  $\mathbf{B}$  is a domain,  $(\mathrm{Frac} \mathbf{B})^{G_K} = \mathbf{B}^{G_K}$  (so the latter is a field), and any  $b \in \mathbf{B}$  for which  $\mathbb{Q}_p \cdot b$  is stable under  $G_K$  satisfies  $b \in \mathbf{B}^\times$ . It also forces the map

$$(2.1) \quad D_{\mathbf{B}}(V) \otimes_{\mathbf{B}^{G_K}} \mathbf{B} \rightarrow V \otimes_{\mathbb{Q}_p} \mathbf{B}$$

to be an injection; one says that  $V$  is  *$\mathbf{B}$ -admissible* if (2.1) is a bijection, or equivalently, if the inequality  $\dim_{\mathbf{B}^{G_K}} D_{\mathbf{B}}(V) \leq \dim_{\mathbb{Q}_p} V$  is an equality.

In particular, Fontaine defines period rings  $\mathbf{B}_{\mathrm{crys}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ ; we say  $V$  is *crystalline*, *semistable*, or *de Rham* if it is admissible for the corresponding period ring. We will define these rings shortly; for now, consider the following result, conjectured by Fontaine and Jannsen, and proved by Fontaine-Messing, Faltings, Tsuji, et al.

**Theorem 2.2.** *Let  $V = H_{\mathrm{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  for  $X$  a smooth proper scheme over  $K$ .*

- (a) *The representation  $V$  is de Rham, and there is a canonical isomorphism of filtered  $K$ -vector spaces*

$$D_{\mathrm{dR}}(V) \cong H_{\mathrm{dR}}^i(X, K).$$

- (b) If  $X$  extends to a smooth proper  $\mathfrak{o}_K$ -scheme, then  $V$  is crystalline.
- (c) If  $X$  extends to a semistable  $\mathfrak{o}_K$ -scheme, then  $V$  is semistable.

In line with the previous statement, the following result was conjectured by Fontaine; its proof combines a result of Berger with a theorem concerning  $p$ -adic differential equations due to André, Mebkhout, and the author.

**Theorem 2.3.** *Let  $V$  be a de Rham representation of  $G_K$ . Then  $V$  is potentially semistable; that is, there exists a finite extension  $L$  of  $K$  such that the restriction of  $V$  to  $G_L$  is semistable.*

### 3. Period rings

We now describe some of the key rings in Fontaine’s theory. (We recommend [3] for a more detailed introduction.) Keep in mind that everything we write down will carry an action of  $G_{\mathbb{Q}_p}$ , so we won’t say this explicitly each time.

Write  $\mathfrak{o}$  for  $\mathfrak{o}_{\mathbb{C}_p}$ . The ring  $\mathfrak{o}/p\mathfrak{o}$  admits a  $p$ -power Frobenius map; let  $\tilde{\mathbf{E}}^+$  be the inverse limit of  $\mathfrak{o}/p\mathfrak{o}$  under Frobenius. That is, an element of  $\tilde{\mathbf{E}}^+$  is a sequence  $\bar{x} = (\bar{x}_n)_{n=0}^\infty$  of elements of  $\mathfrak{o}/p\mathfrak{o}$  with  $\bar{x}_n = \bar{x}_{n+1}^p$ . By construction, Frobenius is a bijection on  $\tilde{\mathbf{E}}^+$ .

If  $\bar{x} \in \tilde{\mathbf{E}}^+$  is nonzero, then  $p^n v_{\mathbb{C}_p}(\bar{x}_n)$  is constant for those  $n$  for which  $\bar{x}_n \neq 0$ . The resulting function

$$v_E(\bar{x}) = \lim_{n \rightarrow \infty} p^n v_{\mathbb{C}_p}(\bar{x}_n)$$

is a valuation, and  $\tilde{\mathbf{E}}^+$  is a valuation ring for this valuation; in particular,  $\tilde{\mathbf{E}}^+$  is an integral domain (even though  $\mathfrak{o}/p\mathfrak{o}$  is nonreduced). It will be convenient to fix the choice of an element  $\epsilon \in \tilde{\mathbf{E}}^+$  such that  $\epsilon_n$  is a primitive  $p^n$ -th root of unity; then  $\tilde{\mathbf{E}}^+$  is the valuation ring in a completed algebraic closure of  $\mathbb{F}_p((\epsilon - 1))$ .

Given  $\bar{x} \in \tilde{\mathbf{E}}^+$ , let  $x_n \in \mathfrak{o}$  be any lift of  $\bar{x}_n$ . The elementary fact that

$$a \equiv b \pmod{p^m} \implies a^p \equiv b^p \pmod{p^{m+1}}$$

implies first that the sequence  $(x_n^{p^n})_{n=0}^\infty$  is  $p$ -adically convergent, and second that the limit is independent of the choice of the  $x_n$ . We call this limit  $\bar{\theta}(\bar{x})$ ; the resulting function  $\bar{\theta} : \tilde{\mathbf{E}}^+ \rightarrow \mathfrak{o}$  is not a homomorphism, but it is multiplicative. In particular,  $\bar{\theta}(\epsilon) = 1$ .

Let  $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$  be the ring of  $p$ -typical Witt vectors with coefficients in  $\tilde{\mathbf{E}}^+$ . Although this ring is non-noetherian (because the valuation on  $\tilde{\mathbf{E}}^+$  is not discrete), one should still think of it as a “two-dimensional” local ring, since  $\tilde{\mathbf{A}}^+/p\tilde{\mathbf{A}}^+ \cong \tilde{\mathbf{E}}^+$ . By properties of Witt vectors, the multiplicative map  $\bar{\theta}$  lifts to a ring homomorphism  $\theta : \tilde{\mathbf{A}}^+ \rightarrow \mathfrak{o}$  taking a Teichmüller lift  $[\bar{x}]$  to  $\bar{\theta}(\bar{x})$ . (The Teichmüller lift  $[\bar{x}]$  is the unique lift of  $\bar{x}$  which has  $p^n$ -th roots for all  $n \geq 0$ .) Also, there is a Frobenius map  $\phi : \tilde{\mathbf{A}}^+ \rightarrow \tilde{\mathbf{A}}^+$  lifting

the usual Frobenius on  $\tilde{\mathbf{E}}^+$ . Put  $\tilde{\mathbf{B}}^+ = \tilde{\mathbf{A}}^+[\frac{1}{p}]$ ; then  $\theta$  extends to a map  $\theta : \tilde{\mathbf{B}}^+ \rightarrow \mathbb{C}_p$ , and  $\phi$  also extends.

With this, we are ready to make Fontaine’s rings. It can be shown that in  $\tilde{\mathbf{B}}^+$ ,  $\ker(\theta)$  is principal and generated by  $([\epsilon] - 1)/([\epsilon]^{1/p} - 1)$ . Also,  $\tilde{\mathbf{A}}^+$  is complete for the  $\ker(\theta)$ -adic topology. However,  $\tilde{\mathbf{B}}^+$  is not (despite being  $p$ -adically complete); denote its completion by  $\mathbf{B}_{\text{dR}}^+$ . This ring does not admit an extension of  $\phi$ , because  $\phi$  is not continuous for the  $\ker(\theta)$ -adic topology. Instead, choose  $\tilde{p} \in \tilde{\mathbf{E}}^+$  with  $\bar{\theta}(\tilde{p}) = p$ , form the  $p$ -adically completed polynomial ring  $\tilde{\mathbf{B}}^+\langle x \rangle$ , and let  $\mathbf{B}_{\text{max}}^+ \subset \mathbf{B}_{\text{dR}}^+$  be the image of the map  $\tilde{\mathbf{B}}^+\langle x \rangle \rightarrow \mathbf{B}_{\text{dR}}^+$  under  $x \mapsto [\tilde{p}]/p$  (that makes sense because  $[\tilde{p}]/p - 1 \in \ker(\theta)$ ). Then  $\phi$  does extend to  $\mathbf{B}_{\text{max}}^+$ , and we can define

$$\tilde{\mathbf{B}}_{\text{rig}}^+ = \bigcap_{n \geq 0} \phi^n(\mathbf{B}_{\text{max}}^+).$$

(Note for experts: we will substitute  $\mathbf{B}_{\text{max}}^+$  for  $\mathbf{B}_{\text{crys}}^+$ , and this is okay because they give the same notion of admissibility.) Put  $\mathbf{B}_{\text{st}}^+ = \mathbf{B}_{\text{max}}^+[\log[\tilde{p}]]$ ; this logarithm is only a formal symbol, but there is a logical way to extend  $\phi$  to it, namely  $\phi(\log[\tilde{p}]) = p \log[\tilde{p}]$ . To embed  $\mathbf{B}_{\text{st}}^+$  into  $\mathbf{B}_{\text{dR}}^+$ , one must choose a branch of the  $p$ -adic logarithm (i.e., a value of  $\log(p)$ ), and then map

$$\log[\tilde{p}] \mapsto \log(p) - \sum_{i=1}^{\infty} \frac{(1 - [\tilde{p}]/p)^i}{i}.$$

One defines a *monodromy operator*  $N = d/d(\log[\tilde{p}])$  on  $\mathbf{B}_{\text{st}}^+$ , satisfying  $N\phi = p\phi N$ .

Finally, the non-plus variants  $\mathbf{B}_{\text{max}}, \mathbf{B}_{\text{st}}, \mathbf{B}_{\text{dR}}$  are obtained from their plus counterparts by adjoining  $1/t$  for

$$t = \log([\epsilon]) = \sum_{i=1}^{\infty} -\frac{(1 - [\epsilon])^i}{i} \in \ker(\theta).$$

Note that  $\mathbf{B}_{\text{dR}}$  is a complete discrete valuation field with uniformizer  $t$ ; it is natural to equip it with the decreasing filtration

$$\text{Fil}^i \mathbf{B}_{\text{dR}} = t^i \mathbf{B}_{\text{dR}}^+.$$

Also, the ring  $\mathbf{B}_e = \mathbf{B}_{\text{max}}^{\phi=1}$  sits in the so-called *fundamental exact sequence*:

$$0 \rightarrow \mathbb{Q}_p \rightarrow \mathbf{B}_{\text{max}}^{\phi=1} \rightarrow \mathbf{B}_{\text{dR}}/\mathbf{B}_{\text{dR}}^+ \rightarrow 0.$$

This is loosely analogous to the exact sequence

$$0 \rightarrow k \rightarrow k[u] \rightarrow k((t))/k[[t]] \rightarrow 0$$

for  $u \in t^{-1}k[[t]] \setminus k[[t]]$ .

#### 4. Permuting the steps

We note in passing that it is possible to permute the steps of the above constructions, with the aim of postponing the choice of the prime  $p$ . This might help one present the theory so that the infinite place becomes a valid choice of prime, under which one should recover ordinary Hodge theory. So far, this is more speculation than reality; we report here only the first steps, leaving future discussion to another occasion.

For starters,  $\tilde{\mathbf{A}}^+$  can be constructed as the inverse limit of  $W(\mathfrak{o})$  under the Witt vector Frobenius map, and  $\theta$  can be recovered as the composition

$$\tilde{\mathbf{A}}^+ \rightarrow W(\mathfrak{o}) \rightarrow \mathfrak{o}$$

of the first projection of the inverse limit with the residue map on Witt vectors. (This observation was made by Lars Hesselholt and verified by Chris Davis.) This still involves the use of  $p$  in constructing  $\mathfrak{o} = \mathfrak{o}_{\mathbb{C}_p}$ ; that can be postponed as follows. Let  $\overline{\mathbb{Z}}$  be the ring of algebraic integers. Let  $R$  be the inverse limit of  $W(\overline{\mathbb{Z}})$  under the Witt vector Frobenius; then we get a map  $\theta : R \rightarrow W(\overline{\mathbb{Z}}) \rightarrow \overline{\mathbb{Z}}$  by composing as above. Choose a prime  $\mathfrak{p}$  of  $\overline{\mathbb{Z}}$  above  $p$  (thus determining a map  $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_{\mathfrak{p}}$ , up to an automorphism of  $\overline{\mathbb{Q}}_{\mathfrak{p}}$ ), and put  $(\tilde{p}) = \overline{\mathbb{Z}} \cap p\overline{\mathbb{Z}}_{\mathfrak{p}}$ ; in other words,  $(\tilde{p})$  are the algebraic integers whose  $\mathfrak{p}$ -adic valuation is at least that of  $p$  itself. Then  $\tilde{\mathbf{A}}^+$  is the completion of  $R$  for the  $\theta^{-1}(\tilde{p})$ -adic topology.

We are still using  $p$  via the definition of  $W$  using  $p$ -typical Witt vectors. One can postpone the choice of  $p$  further by using the big Witt vectors, taking the inverse limit under all of the Frobenius maps, instead of the  $p$ -typical Witt vectors. In this case, the completion for the  $\theta^{-1}(\tilde{p})$ -adic topology splits into copies of  $\tilde{\mathbf{A}}^+$  indexed by positive integers coprime to  $p$ , which are shifted around by the prime-to- $p$  Frobenius maps.

#### 5. $B$ -pairs as $p$ -adic Hodge structures

One of the principal points of departure for ordinary Hodge theory is the comparison isomorphism

$$H_{\text{Betti}}^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\text{dR}}^i(X, \mathbb{C}).$$

One then defines a *Hodge structure* as a  $\mathbb{C}$ -vector space carrying the extra structures brought to the comparison isomorphism by the extra structures on both sides: the integral structure on the left side, and the Hodge decomposition on the right side.

The notion of a  $B$ -pair, recently introduced by Berger [4], performs an analogous function for the comparison isomorphism

$$H_{\text{et}}^i(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}} \cong H_{\text{dR}}^i(X, K) \otimes_K \mathbf{B}_{\text{dR}},$$

where the extra structures are a Hodge filtration on the right side, and the compatible Galois actions on both sides.

A *B-pair* over  $K$  is a pair  $(W_e, W_{\text{dR}}^+)$ , where  $W_e$  is a finite free  $\mathbf{B}_e$ -module equipped with a continuous semilinear  $G_K$ -action, and  $W_{\text{dR}}^+$  is a finite free  $\mathbf{B}_{\text{dR}}^+$ -submodule of  $W_{\text{dR}} = W_e \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{dR}}$  stable under  $G_K$  such that  $W_{\text{dR}}^+ \otimes_{\mathbf{B}_{\text{dR}}^+} \mathbf{B}_{\text{dR}} \rightarrow W_{\text{dR}}$  is an isomorphism. (Note that  $\mathbf{B}_e$  is a Bézout domain, i.e., an integral domain in which finitely generated ideals are principal [4, Proposition 1.1.9].) The *rank* of a  $B$ -pair  $W$  is the common quantity  $\text{rank}_{\mathbf{B}_e} W_e = \text{rank}_{\mathbf{B}_{\text{dR}}^+} W_{\text{dR}}^+$ .

**Lemma 5.1.** *Every  $B$ -pair  $W$  of rank 1 over  $K$  can be uniquely written as  $(\mathbf{B}_e(\delta), t^i \mathbf{B}_{\text{dR}}^+(\delta))$  for some character  $\delta : G_K \rightarrow \mathbb{Q}_p^\times$  and some  $i \in \mathbb{Z}$ .*

*Proof.* See [4, Lemme 2.1.4]. □

We refer to the integer  $i$  as the *degree* of  $W$ . If  $W$  has rank  $d > 1$ , we define the degree of  $W$  as the degree of the determinant  $\det W = \wedge^d W$ . We refer to the quotient  $\mu(W) = (\text{deg } W)/(\text{rank } W)$  as the *average slope* of  $W$ . (This is meant to bring to mind the theory of vector bundles on curves; see the next section for an extension of this analogy.)

There is a functor

$$V \mapsto W(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_e, V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+),$$

from  $p$ -adic representations to  $B$ -pairs; it has the one-sided inverse

$$W \mapsto V(W) = W_e \cap W_{\text{dR}}^+,$$

and so is fully faithful. However, not every  $B$ -pair corresponds to a representation; for instance, in Lemma 5.1, only those  $B$ -pairs with  $i = 0$  correspond to representations. We will return to the structure of a general  $B$ -pair in the next two sections.

One can generalize many definitions and results from  $p$ -adic representations to  $B$ -pairs. For instance, we set

$$\begin{aligned} D_{\text{max}}(W) &= (W_e \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{max}})^{G_K} \\ D_{\text{st}}(W) &= (W_e \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{st}})^{G_K} \\ D_{\text{dR}}(W) &= (W_e \otimes_{\mathbf{B}_e} \mathbf{B}_{\text{dR}})^{G_K}; \end{aligned}$$

for  $* \in \{\text{max}, \text{st}, \text{dR}\}$ , the map

$$(5.2) \quad D_*(W) \otimes_{\mathbf{B}_*^{G_K}} \mathbf{B}_* \rightarrow V \otimes_{\mathbb{Q}_p} \mathbf{B}_*$$

is injective, and we say that  $W$  is *\*-admissible* if (5.2) is a bijection. In these terms, we have the following extension of Theorem 2.3, again due to Berger [2]. (The closest analogue of this result in ordinary Hodge theory is a theorem of Borel [18, Lemma 4.5], that any polarized variation of rational Hodge structures is quasi-unipotent.)



**Theorem 5.3.** *Let  $W$  be a de Rham  $B$ -pair over  $K$ . Then there exists a finite extension  $L$  of  $K$  such that the restriction of  $W$  to  $L$  is semistable.*

Moreover, the  $B$ -pairs which are already crystalline over  $K$  can be described more explicitly; they form a category equivalent to the category of filtered  $\phi$ -modules over  $K$ . (Such an object is a finite dimensional  $K_0$ -vector space  $V$ , for  $K_0$  the maximal unramified extension of  $K$ , equipped with a semilinear  $\phi$ -action and an exhaustive decreasing filtration  $\text{Fil}^i V_K$  of  $V_K = V \otimes_{K_0} K$ .)

### 6. Sloped representations

For  $h$  a positive integer and  $a \in \mathbb{Z}$  coprime to  $h$ , we define an  $(a/h)$ -representation as a finite-dimensional  $\mathbb{Q}_{p^h}$ -vector space  $V_{a,h}$  equipped with a semilinear  $G_K$ -action and a semilinear Frobenius action  $\phi$  commuting with the  $G_K$ -action, satisfying  $\phi^h = p^a$ . For instance, we may view a  $p$ -adic representation as a 0-representation by taking  $\phi = \text{id}$ .

We say that the  $B$ -pair  $W$  is *isoclinic of slope  $a/h$*  if it occurs in the essential image of  $W_{a,h}$ ; we say that  $W$  is *étale* if it is isoclinic of slope 0.

**Theorem 6.1.** *The functor*

$$V_{a,h} \mapsto W_{a,h}(V_{a,h}) = ((\mathbf{B}_{\max} \otimes_{\mathbb{Q}_{p^h}} V_{a,h})^{\phi=1}, \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbb{Q}_{p^h}} V_{a,h})$$

*from  $(a/h)$ -representations to  $B$ -pairs of slope  $a/h$  is fully faithful. (Here  $\mathbb{Q}_{p^h}$  denotes the unramified extension of  $\mathbb{Q}_p$  with residue field  $\mathbb{F}_{p^h}$ .)*

*Proof.* See [4, Théorème 3.2.3] for the construction of a one-sided inverse. □

Using the equivalence of categories between  $B$ -pairs and  $(\phi, \Gamma)$ -modules (Theorem 9.1), one deduces the following properties of isoclinic  $B$ -pairs.

- Lemma 6.2.**
- (a) *If  $0 \rightarrow W_1 \rightarrow W \rightarrow W_2 \rightarrow 0$  is exact and any two of  $W_1, W_2, W$  are isoclinic of slope  $s$ , then so is the third.*
  - (b) *Let  $W_1, W_2$  be isoclinic  $B$ -pairs of slopes  $s_1 < s_2$ . Then  $\text{Hom}(W_1, W_2) = 0$ .*
  - (c) *Let  $W_1, W_2$  be isoclinic  $B$ -pairs of slopes  $s_1, s_2$ . Then  $W_1 \otimes W_2$  is isoclinic of slope  $s_1 + s_2$ .*

*Proof.* See Theorem 1.6.6, Corollary 1.6.9, and Corollary 1.6.4, respectively, of [12]. □

The following is a form of the author’s slope filtration theorem for Frobenius modules over the Robba ring.

**Theorem 6.3.** *Let  $W$  be a  $B$ -pair. Then there is a unique filtration  $0 = W_0 \subset \dots \subset W_l = W$  of  $W$  by  $B$ -pairs with the following properties.*

- (a) For  $i = 1, \dots, l$ , the quotient  $W_i/W_{i-1}$  is a  $B$ -pair which is isoclinic of slope  $s_i$ .
- (b) We have  $s_1 < \dots < s_l$ .

*Proof.* See [12, Theorem 1.7.1]. □

A corollary is that a  $B$ -pair  $W$  is “semistable” (in the sense of vector bundles, up to a reversal of the sign convention), meaning that

$$0 \neq W_1 \subseteq W \implies \mu(W_1) \geq \mu(W),$$

if and only if  $W$  is isoclinic.

**Corollary 6.4.** *The property of a  $B$ -pair having all slopes  $\geq s$  (resp.  $\leq s$ ) is stable under extensions.*

## 7. Trianguline $B$ -pairs

One might reasonably ask why the study of  $B$ -pairs should be relevant to problems only involving  $p$ -adic representations. One answer is that there are many  $p$ -adic representations which become more decomposable when viewed as  $B$ -pairs.

Specifically, following Colmez [6], we say that a  $B$ -pair  $W$  is *trianguline* (or *triangulable*) if  $W$  admits a filtration  $0 = W_0 \subset \dots \subset W_l = W$  in which each quotient  $W_i/W_{i-1}$  is a  $B$ -pair of rank 1. For instance, by a theorem of Kisin [13], if  $V$  is the two-dimensional representation corresponding to a classical or overconvergent modular form of finite slope, then  $W(V)$  is trianguline.

As one might expect, the extra structure of a triangulation makes trianguline  $B$ -pairs easier to classify. For example, Colmez has classified all two-dimensional trianguline  $B$ -pairs over  $\mathbb{Q}_p$ , by calculating  $\text{Ext}(W_1, W_2)$  whenever  $W_1, W_2$  are  $B$ -pairs of rank 1. He has also shown that their  $\mathcal{L}$ -invariants (in the sense of Fontaine-Mazur) can be read off from the triangulation.

A study of the general theory of trianguline  $B$ -pairs has been initiated by Bellaïche and Chenevier [1], with the aim of applying this theory to the study of Selmer groups associated to Galois representations of dimension greater than 2 (e.g., those arising from unitary groups). It is also hoped that this study will give insight into questions like properness of the Coleman-Mazur eigencurve. One feature apparent in the work of Bellaïche and Chenevier, connected to the results of the next section, is that the trianguline property of a representation is reflected by the structure of the corresponding deformation ring. (A related notion is Pottharst’s definition of a *triangulordinary* representation [17], which generalizes the notion of an ordinary representation in a manner useful when considering duality of Selmer groups.)

### 8. Cohomology of *B*-quotients

In this section, we describe a theorem of Liu [15] generalizing Tate’s fundamental results on the Galois cohomology of *p*-adic representations, and also the Ext group calculations of Colmez mentioned above. However, to do this properly, we must work with a slightly larger category than the *B*-pairs, because this category is not abelian: it contains kernels but not cokernels.

One can construct a minimal abelian category containing the *B*-pairs as follows. Define a *B*-quotient as an inclusion  $(W_1 \hookrightarrow W_2)$  of *B*-pairs. We put these in a category in which the morphisms from  $(W_1 \hookrightarrow W_2)$  to  $(W'_1 \hookrightarrow W'_2)$  consist of subobjects  $X$  of  $W_2 \oplus W'_2$  containing  $W_1 \oplus W'_1$ , such that the composition  $X \rightarrow W_2 \oplus W'_2 \rightarrow W_2$  is surjective and the inverse image of  $W_1$  is  $W_1 \oplus W'_1$ . (One must also define addition and composition of morphisms, but the reader should have no trouble reconstructing them.) It can be shown that this yields an abelian category, into which the *B*-pairs embed by mapping  $W$  to  $0 \hookrightarrow W$ .

For  $W = (W_1 \hookrightarrow W_2)$  a *B*-quotient, we define the *rank* of  $W$  as  $\text{rank}(W_2) - \text{rank}(W_1)$ , and we say  $W$  is *torsion* if  $\text{rank}(W) = 0$ . We also write  $\omega = V(\mathbb{Q}_p(1))$ . One then has the following result, which in particular includes the Euler characteristic formula and local duality theorems of Tate [16]. (However, the proof uses these results as input, and so does not rederive them independently.)

**Theorem 8.1** (Liu). *Define the functor  $H^0$  from *B*-quotients to  $\mathbb{Q}_p$ -vector spaces by  $H^0(W) = \text{Hom}(W_0, W)$ , where  $W_0$  is the trivial *B*-pair  $(\mathbf{B}_e, \mathbf{B}_{\text{dR}}^+)$ . Then  $H^0$  extends to a universal  $\delta$ -functor  $(H^i)_{i=0}^\infty$  with the following properties.*

- (a) *For  $W$  a *B*-quotient,  $H^i(W)$  is finite dimensional over  $\mathbb{Q}_p$ .*
- (b) *For  $W$  a *B*-quotient,  $H^i(W) = 0$  for  $i > 2$ .*
- (c) *For  $W$  a torsion *B*-quotient,  $H^2(W) = 0$ .*
- (d) *For  $W$  a *B*-quotient,  $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{Q}_p} H^i(W) = -[K : \mathbb{Q}_p] \text{rank } W$ .*
- (e) *For  $W$  a *B*-pair, the pairing*

$$H^i(W) \times H^{2-i}(W^\vee \otimes \omega) \rightarrow H^2(W \otimes W^\vee \otimes \omega) \rightarrow H^2(\omega) \cong \mathbb{Q}_p$$

*is perfect for  $i = 0, 1, 2$ .*

*Moreover, on the subcategory of *p*-adic representations, the  $H^i$  are canonically naturally isomorphic to Galois cohomology (in a fashion compatible with connecting homomorphisms).*

## 9. $(\phi, \Gamma)$ -modules

It would be somewhat misleading to leave the story at that, because it would fail to give a real sense as to how one *proves* any of the results explained above. In fact, one tends to make proofs by working on the opposite side of an equivalence of categories, in which the Galois action is replaced by some more “commutative” data. (As remarked in [14] in a slightly different context, these objects bear some resemblance to the local versions of certain geometric objects, called shtukas, appearing in Drinfel’d’s approach to the Langlands correspondence for function fields. How to profit from this observation remains a mystery.)

The *Robba ring* over a coefficient field  $L$  is the ring of formal Laurent series  $\sum_{n \in \mathbb{Z}} c_n x^n$  with  $c_n \in L$ , such that the series converges on some annulus  $\epsilon < |x| < 1$  (depending on the series); this is a Bézout domain. Let  $\mathcal{R} = \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^\dagger$  be the Robba ring over  $\mathbb{Q}_p$  with the series variable  $x$  identified with  $\pi = [\epsilon] - 1$ . This ring admits a Frobenius map  $\phi$  given by  $\phi(\pi) = (1 + \pi)^p - 1$ , and an action of the group  $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$  satisfying  $\gamma(\pi) = (1 + \pi)^{\chi(\gamma)} - 1$ , for  $\chi : \Gamma \rightarrow \mathbb{Z}_p$  the cyclotomic character.

A  $(\phi, \Gamma)$ -module over  $\mathcal{R}$  is a finite free  $\mathcal{R}$ -module  $D$  equipped with an isomorphism  $\phi^* D \rightarrow D$ , viewed as a semilinear action of  $\phi$  on  $D$ , and a continuous (for a topology which we won’t describe here) semilinear  $\Gamma$ -action commuting with  $\phi$ . If we only require that  $D$  be finitely presented, we call the result a *generalized  $(\phi, \Gamma)$ -module*. One then has the following equivalence [4, Théorème 2.2.7].

**Theorem 9.1** (Berger). *The category of  $B$ -pairs over  $\mathbb{Q}_p$  is equivalent to the category of  $(\phi, \Gamma)$ -modules over  $\mathcal{R}$ . Under this equivalence, isoclinicity for  $B$ -pairs equates to isoclinicity for  $\phi$ -modules as in [12].*

There is an analogous equivalence over  $K$ , using a suitable finite extension of  $\mathcal{R}$  which is again isomorphic to a Robba ring (over a certain extension of  $\mathbb{Q}_p$ ); for simplicity, we omit further details.

A  $(\phi, \Gamma)$ -quotient is a generalized  $(\phi, \Gamma)$ -module of the form  $D/D_1$  for some inclusion  $D_1 \hookrightarrow D$  of  $(\phi, \Gamma)$ -modules. It is easily checked that any generalized  $(\phi, \Gamma)$ -module occurring as a submodule or quotient of a  $(\phi, \Gamma)$ -quotient is also a  $(\phi, \Gamma)$ -quotient. (We expect that not every generalized  $(\phi, \Gamma)$ -module occurs as a  $(\phi, \Gamma)$ -quotient, but we have no example.)

**Corollary 9.2.** *The category of  $B$ -quotients is equivalent to the category of  $(\phi, \Gamma)$ -quotients.*

In particular, any  $B$ -quotient  $W$  admits a torsion subobject  $X$  such that  $W/X$  is a  $B$ -pair (since the same is true of finitely presented  $\mathcal{R}$ -modules).

There is a sort of cohomology for generalized  $(\phi, \Gamma)$ -modules, computed by a complex introduced by Colmez [6] derived from work of Herr [10]. For

$p > 2$ , it can be constructed as follows. Choose a topological generator  $\gamma$  of  $\Gamma$ . Given a generalized  $(\phi, \Gamma)$ -module  $D$ , for  $i = 0, 1, 2$ , let  $H^i$  be the cohomology at position  $i$  of the complex

$$0 \rightarrow D \rightarrow D \oplus D \rightarrow D \rightarrow 0$$

where the first map is  $x \mapsto ((\gamma - 1)x, (\phi - 1)x)$  and the second map is  $(x, y) \mapsto ((\phi - 1)x - (\gamma - 1)y)$ . One also constructs cup product pairings, the only nonobvious one of which is the map  $H^1(D) \times H^1(D') \rightarrow H^2(D \otimes D')$  given by  $(x, y), (z, t) \mapsto y \otimes \gamma(z) - x \otimes \phi(t)$ . (For  $p = 2$ , let  $\gamma$  be a generator of  $\Gamma/\{\pm 1\}$ , and replace  $D$  by its invariants under  $\{\pm 1\}$  in the construction of the complex.)

We can now state what Liu actually proves in [15]. Note that the more general result working with the Robba ring corresponding to  $K$ , in which everything is the same except that the right side of (c) must be multiplied by  $[K : \mathbb{Q}_p]$ , follows from the case  $K = \mathbb{Q}_p$  by a version of Shapiro’s lemma for  $(\phi, \Gamma)$ -modules.

**Theorem 9.3** (Liu). *Let  $D$  be a  $(\phi, \Gamma)$ -module over  $\mathcal{R}$ .*

- (a) *For  $i = 0, 1, 2$ ,  $H^i(D)$  is finite dimensional over  $\mathbb{Q}_p$ .*
- (b) *If  $D$  is torsion, then  $H^2(D) = 0$ .*
- (c) *We have  $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{Q}_p} H^i(D) = -\text{rank}(D/D_{\text{tors}})$ .*
- (d) *If  $D$  is free, then for  $i = 0, 1, 2$ , the pairing*

$$H^i(D) \times H^{2-i}(D^\vee \otimes \omega) \rightarrow H^2(D \otimes D^\vee \otimes \omega) \rightarrow H^2(\omega) \cong \mathbb{Q}_p$$

*is perfect.*

*Moreover, on the subcategory of  $p$ -adic representations, the  $H^i$  are canonically naturally isomorphic to Galois cohomology (in a fashion compatible with connecting homomorphisms).*

### Appendix A. Cohomology of $B$ -pairs

For  $W$  a  $B$ -pair or  $B$ -quotient, let  $D(W)$  be the associated  $(\phi, \Gamma)$ -module or  $(\phi, \Gamma)$ -quotient (see [4, Théorème 2.2.7] for the construction), and write  $H^i(W)$  instead of  $H^i(D(W))$ . For this to be consistent with Theorem 8.1, we must have  $H^0(D(W)) = \text{Hom}(W_0, W)$ ; fortunately, Theorem 9.1 implies  $\text{Hom}(W_0, W) = \text{Hom}(D(W_0), D(W))$  and it is trivial to check that  $\text{Hom}(D(W_0), D(W)) = H^0(D(W))$ . All of the assertions of Theorem 8.1 follow from Theorem 9.3 as stated, except for the fact that the  $\delta$ -functor formed by the  $H^i$  is universal.

The purpose of this appendix is to fill in this gap, as the proof does not appear elsewhere. It is mostly meant for experts, and does not maintain the expository style we have attempted to maintain in the main text.

Besides Theorem 9.3, and the equivalence given by Theorem 9.1, we will make repeated use of the following facts.

- If  $0 \rightarrow W_1 \rightarrow W \rightarrow W_2 \rightarrow 0$  is an exact sequence of  $B$ -quotients, and  $W_1, W_2$  are  $B$ -pairs, then so is  $W$ .
- If  $D \rightarrow D_2 \rightarrow 0$  is an exact sequence of generalized  $(\phi, \Gamma)$ -modules, and  $D$  is a  $(\phi, \Gamma)$ -quotient, then so is  $D_2$ .

**Lemma A.1.** *Let  $W$  be a  $B$ -pair. Then there exists a character  $\delta : G_K \rightarrow \mathbb{Q}_p^\times$  such that  $\text{Hom}(W(\delta), W) = 0$ .*

*Proof.* We will show that all but finitely many  $\delta$  have this property, by induction on rank  $W$ . Suppose first that rank  $W = 1$ ; by Lemma 5.1, we can write  $W = t^i W_0 \otimes W(\delta')$  for some  $i \in \mathbb{Z}$  and some character  $\delta' : G_K \rightarrow \mathbb{Q}_p^\times$ . Then to have a nonzero map  $W_e(\delta) \rightarrow W_e = W_e(\delta')$ , we must have  $\delta = \delta'$ .

Suppose now that rank  $W > 1$ . If there is no  $\delta_0$  such that  $\text{Hom}(W(\delta_0), W) \neq 0$ , we are done. Otherwise, we have a short exact sequence

$$0 \rightarrow t^i W(\delta_0) \rightarrow W \rightarrow W_1 \rightarrow 0$$

for some  $i \in \mathbb{Z}$ . By the induction hypothesis, for all but finitely many  $\delta$ ,  $\text{Hom}(W(\delta), t^i W(\delta_0)) = \text{Hom}(W(\delta), W_1) = 0$ , so  $\text{Hom}(W(\delta), W) = 0$ .  $\square$

By the *slopes* of a  $B$ -pair  $W$ , we will mean those numbers  $s_1, \dots, s_l$  occurring in Theorem 6.3.

**Lemma A.2.** *Let  $0 \rightarrow W_1 \rightarrow W \rightarrow W_2 \rightarrow 0$  be a nonsplit exact sequence of  $B$ -pairs, in which  $W_1$  has positive slopes, and  $W_2$  has rank 1 and degree  $-1$ . Then  $W$  has nonnegative slopes.*

*Proof.* Suppose the contrary; then by Theorem 6.3, there is a short exact sequence  $0 \rightarrow X \rightarrow W \rightarrow W/X \rightarrow 0$  of  $B$ -pairs with  $X$  isoclinic of negative slope. Note that  $\text{deg}(X) = \text{deg}(X \cap W_1) + \text{deg}(X/(X \cap W_1))$ ; by Lemma 6.2, the first term is positive unless  $X \cap W_1 = 0$ , and the second term is at least  $-1$ . The only way to have  $\text{deg}(X) < 0$  is to have  $X \cap W_1 = 0$  and  $X \rightarrow W_2$  an isomorphism, but this only happens if the sequence splits.  $\square$

**Lemma A.3.** *If  $W$  is a  $B$ -pair with negative slopes, then  $H^2(W) = 0$ .*

*Proof.* By Theorem 9.3,  $H^2(W)$  is dual to  $H^0(W^\vee \otimes \omega)$ , which vanishes by Lemma 6.2.  $\square$

**Proposition A.4.** *Let  $W$  be a  $B$ -pair whose slopes are all nonnegative. Then there exists a short exact sequence  $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$  such that  $X$  is étale.*

*Proof.* (This argument is a variant of the argument used in [15] to reduce Tate duality to the étale case.) We induct on  $s = \text{deg}(W) \geq 0$ . If  $s = 0$ , then  $W$  is étale and we are done. Otherwise, let  $W_1$  be the maximal étale subobject of  $W$  (which may be zero) and put  $W_2 = W/W_1$ . By applying Lemma A.1, we can choose a  $B$ -pair  $T$  which is isoclinic of rank 1 and degree  $-1$ , such that  $H^0(W_1^\vee \otimes T \otimes \omega) = 0$ .

Put  $W' = W \otimes T^\vee, W'_1 = W_1 \otimes T^\vee, W'_2 = W_2 \otimes T^\vee$ . Apply Theorem 9.3 to obtain

$$\begin{aligned} \dim H^1(W') &= \dim H^0(W') + \dim H^2(W') + [K : \mathbb{Q}_p] \text{rank}(W') \\ \dim H^1(W'_1) &= \dim H^0(W'_1) + \dim H^2(W'_1) + [K : \mathbb{Q}_p] \text{rank}(W'_1). \end{aligned}$$

We claim each of the three terms in the first row is greater than or equal to the corresponding term in the second row. The inequality is an equality on the first terms, because both terms vanish by Lemma 6.2. The inequality on the second terms holds because by Theorem 9.3,  $H^2(W'_1) = H^0((W'_1)^\vee \otimes \omega)^\vee = 0$ . The inequality on the third terms is strict because  $W \neq W_1$  by assumption.

We conclude that  $\dim H^1(W') > \dim H^1(W'_1)$ , so in particular the map  $H^1(W'_1) \rightarrow H^1(W')$  cannot be surjective. Choose a class in  $H^1(W')$  not in the image of this map; then the resulting class in  $H^1(W'_2)$  is nonzero. Form the corresponding extension

$$0 \rightarrow W \rightarrow X \rightarrow T \rightarrow 0;$$

then we also have an exact sequence

$$0 \rightarrow W_1 \rightarrow X \rightarrow X/W_1 \rightarrow 0$$

and a nonsplit exact sequence

$$0 \rightarrow W_2 \rightarrow X/W_1 \rightarrow T \rightarrow 0.$$

By Lemma A.2,  $X/W_1$  has nonnegative slopes, as then does  $X$  by Corollary 6.4. Since  $\deg(X) = s - 1$ , we may deduce the claim by the induction hypothesis.  $\square$

**Corollary A.5.** *Let  $W$  be a  $B$ -pair. Then for  $n$  sufficiently large, there exists a short exact sequence  $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$  such that  $X$  is isoclinic of slope  $-n$ .*

*Proof.* Apply Proposition A.4 to  $W \otimes T$ , for  $T$  of rank 1 and degree  $n$ .  $\square$

**Lemma A.6.** *Let  $W$  be a  $B$ -quotient. Then for  $n$  sufficiently large, we can write  $W = (X_1 \hookrightarrow X_2)$  with  $X_1$  isoclinic of slope  $-n$ .*

*Proof.* Put  $W = (W_1 \hookrightarrow W_2)$ . By Corollary A.5, for  $n$  large, we can find a short exact sequence  $0 \rightarrow W_1 \rightarrow X_1 \rightarrow Y \rightarrow 0$  of  $B$ -pairs with  $X_1$  isoclinic of slope  $-n$ . By forming a pushout, we obtain a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

of  $B$ -quotients with  $0 \rightarrow W_2 \rightarrow X_2 \rightarrow Y \rightarrow 0$  exact. In particular,  $X_2$  is a  $B$ -pair, not just a  $B$ -quotient, and  $W \cong (X_1 \hookrightarrow X_2)$ .  $\square$

**Proposition A.7.** *Let  $D_1, D_2$  be  $(\phi, \Gamma)$ -quotients. Then the group  $\text{Ext}(D_2, D_1)$  is the same whether computed in the category of  $(\phi, \Gamma)$ -quotients or in the category of generalized  $(\phi, \Gamma)$ -modules. Consequently, if  $D_2$  is a  $(\phi, \Gamma)$ -module, then the former group is equal to  $H^1(D_2^\vee \otimes D_1)$ .*

*Proof.* Let  $0 \rightarrow D_1 \rightarrow D \rightarrow D_2 \rightarrow 0$  be an extension of generalized  $(\phi, \Gamma)$ -modules. First suppose  $D_2$  is actually a  $(\phi, \Gamma)$ -module. By Lemma A.6, for  $n$  large, we can write  $D_1$  as a quotient  $E_2/E_1$  of  $(\phi, \Gamma)$ -modules with  $E_1$  isoclinic of slope  $-n$ . For  $n$  large, we have  $H^2(D_2^\vee \otimes E_1) = 0$  by Lemma A.3. Since

$$H^1(D_2^\vee \otimes E_2) \rightarrow H^1(D_2^\vee \otimes D_1) \rightarrow H^2(D_2^\vee \otimes E_1) = 0$$

is exact, we can lift the original exact sequence to a sequence  $0 \rightarrow E_2 \rightarrow F \rightarrow D_2 \rightarrow 0$ . Since  $E_2$  and  $D_2$  are  $(\phi, \Gamma)$ -modules, so is  $F$ ; since  $F \rightarrow D$  is surjective,  $D$  is a  $(\phi, \Gamma)$ -quotient.

In the general case, write  $D_2$  as a quotient  $E_2/E_1$  of  $(\phi, \Gamma)$ -modules. By forming a pullback, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & F & \longrightarrow & E_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_1 & \longrightarrow & D & \longrightarrow & D_2 & \longrightarrow & 0 \end{array}$$

of generalized  $(\phi, \Gamma)$ -modules, with  $F \rightarrow D$  surjective. By the previous paragraph,  $F$  is a  $(\phi, \Gamma)$ -quotient, as then is  $D$ . □

**Proposition A.8.** *For any  $B$ -quotient  $W$ , there exists an injection  $W \hookrightarrow X$  with  $H^2(X) = 0$ .*

*Proof.* Let  $T$  be the maximal torsion subobject of  $W$ . By Theorem 9.3,  $H^2(T) = 0$ . By Corollary A.5, for  $n > 0$  large, we can construct an extension  $0 \rightarrow W/T \rightarrow X_0 \rightarrow X_1 \rightarrow 0$  of  $B$ -pairs with  $X_0$  isoclinic of slope  $-n$ . By Lemma A.3,  $H^2(X_0) = 0$ .

In the exact sequence

$$H^1(X_0^\vee \otimes T) \rightarrow H^1((W/T)^\vee \otimes T) \rightarrow H^2(X_1^\vee \otimes T),$$

the last term vanishes by Theorem 9.3 because  $X_1^\vee \otimes T$  is torsion. Hence we can lift the extension  $0 \rightarrow T \rightarrow W \rightarrow W/T \rightarrow 0$  to an extension  $0 \rightarrow T \rightarrow X \rightarrow X_0 \rightarrow 0$ . (For this we need Proposition A.7, to assert that the extension of generalized  $(\phi, \Gamma)$ -modules actually arises from an an extension of  $B$ -quotients.) Now  $W$  injects into  $X$ , and  $0 = H^2(T) \rightarrow H^2(X) \rightarrow H^2(X_0) = 0$  is exact, so  $H^2(X) = 0$  as desired. □

*Proof of Theorem 8.1.* The functor  $H^1$  is weakly effaceable because it coincides with the Ext group by Proposition A.7. (Namely, the class in  $H^1(W)$  corresponding to an extension  $0 \rightarrow W \rightarrow X \rightarrow W_0 \rightarrow 0$  always vanishes in  $H^1(X)$ .) The functor  $H^2$  is effaceable by Proposition A.8. As in [19, §2], the



$H^i$  thus form a universal  $\delta$ -functor; as noted above, this plus Theorem 9.3 proves everything.  $\square$

One can prove some variant statements by the same argument. For one, the  $H^i$  form a universal  $\delta$ -functor on the full category of generalized  $(\phi, \Gamma)$ -modules. For another, the  $H^i$  can be computed by taking derived functors in the ind-category of generalized  $(\phi, \Gamma)$ -modules, i.e., the category of direct limits of generalized  $(\phi, \Gamma)$ -modules.

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