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Citation: Levin, Michael, and Steven G. Johnson. "Is the Electrostatic Force Between a Point Charge and a Neutral Metallic Object Always Attractive?" *American Journal of Physics* 79.8 (2011): 843. Web. 26 June 2012. © 2011 American Association of Physics Teachers

As Published: <http://dx.doi.org/10.1119/1.3595554>

Publisher: American Association of Physics Teachers (AAPT)

Persistent URL: <http://hdl.handle.net/1721.1/71213>

Version: Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

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Is the electrostatic force between a point charge and a neutral metallic object always attractive?

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(Dated: July 14, 2010)

We give an example of a geometry in which the electrostatic force between a point charge and a neutral metallic object is repulsive. The example consists of a point charge centered above a thin metallic hemisphere, positioned concave up. We show that this geometry has a repulsive regime using both a simple analytical argument and an exact calculation for an analogous two-dimensional geometry. Analogues of this geometry-induced repulsion can appear in many other contexts, including Casimir systems.

I. INTRODUCTION

A classic problem in electrostatics is to compute the force between a point charge and a perfectly conducting, neutral metallic sphere [Fig. 1(a)]. The problem can be easily solved using the method of images. One finds that the force on the point charge can be computed by summing the forces exerted by two image charges—one located at the center of the sphere, carrying like charge, and one closer to the surface, carrying opposite charge.

A simple corollary of this calculation is that the force is always attractive, since the oppositely charged image charge is always closer to the point charge than its partner. This makes sense intuitively, since one expects that a positively charged point charge will induce negative charges on the part of the sphere that is closest to it, and positive charges on the part that is further away. In fact, from this point of view, it is natural to wonder if this phenomena is more general and the force is attractive for *any* geometry, not just a sphere.

This question is the main subject of this paper. In some sense, one can think of this as an attempt to strengthen Earnshaw’s theorem: recall that Earnshaw’s theorem and its generalizations^{1,2} tell us that a point charge can never be trapped in a stable equilibrium via electrostatic interactions with a metallic object. Here, we ask whether one can go further in the case where the metallic object is neutral, and show the force is *always attractive*.

To make the question precise, we need to define what we mean by an “attractive” force. To this end, it is useful to make the additional assumption that the charge and metallic object lie on opposite sides of a plane, say, the $z = 0$ plane, with the charge in the upper half space $\{z > 0\}$ and the metal object in the lower half space $\{z < 0\}$ [Fig. 1(b)]. Then, by an “attractive” force we mean a force \mathbf{F} on the charge with $F_z < 0$.

Given this definition, it is not hard to show that the force is attractive in a number of cases. The first case is if the point charge is very close to the surface of the metal object. In this case, the problem reduces to the standard system of a charged particle and an infinite metal plate, which clearly has an attractive force. The second

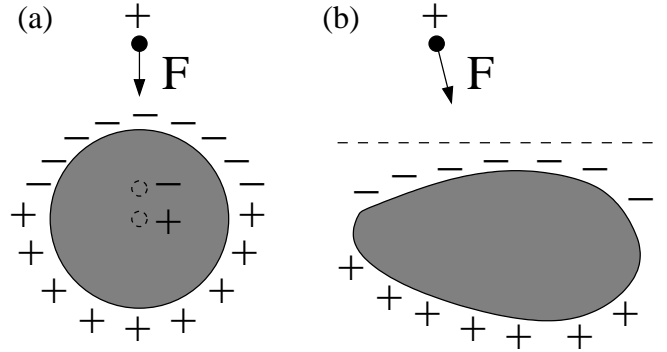


FIG. 1: (a) Electrostatics of a point charge interacting with a neutral metallic sphere. Using image charges (dotted circles) it is easy to see that the force \mathbf{F} on the point charge is attractive. (b) In this paper, we ask whether the force is attractive for *any* shape of metallic object. More precisely, if the point charge and the object lie on opposite sides of the $z = 0$ plane (dotted line), with the charge in $\{z > 0\}$ and the object in $\{z < 0\}$, is F_z always negative?

case is if the point charge is very far from the metallic object—say at position $(0, 0, z)$ where z is large. To see that the force is attractive in this case, recall Thomson’s theorem³: the induced charges in a metallic object always arrange themselves to minimize the total electrostatic energy of the system. A corollary of this is that the electrostatic energy of a system composed of a metallic object and a charge is always lower than the energy of the charge in vacuum. Letting $U(z)$ denote the electrostatic energy when the charge is at position $(0, 0, z)$, we conclude that $U(z) \leq U(\infty)$ so that $F_z = -dU/dz$ must be negative (i.e. attractive) for large z . In addition, one can show that the force is attractive at any distance, in the case where the metal object is replaced by a dielectric material with a dielectric constant $\epsilon = 1 + \delta$, $0 < \delta \ll 1$. One way to see this is to note that, to lowest order in δ , the electrostatic interaction between the charge and the object can be decomposed into a sum of independent interactions with infinitesimal patches of dielectric material. One can then check that each patch gives rise to an attractive interaction, so that the total interaction

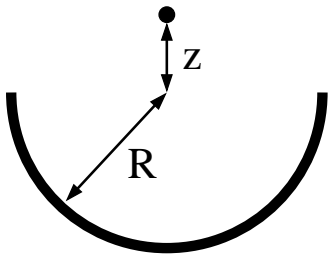


FIG. 2: Example of a geometry in which a neutral metallic object repels a point charge: a point charge centered above a thin metallic hemisphere (side view).

is necessarily attractive. As a final example, one can show that the force is attractive if the metallic object is grounded rather than neutral: in that case, a positively charged point charge will only induce negative charges on the metallic object, leading to an attractive force.

Given all of this evidence, one might think that the force is always attractive. Surprisingly, this is *not* the case. In this paper, we give a simple example of a geometry in which a neutral metallic object *repels* a point charge. We establish repulsion using both a simple analytical argument and an exact calculation for an analogous two-dimensional (2d) geometry. In accordance with Earnshaw's theorem and its generalizations,^{1,2} our geometry does not yield any stable equilibria. However, the fact that one can have repulsion at all is surprising, and we show that analogues of this unusual geometric effect exist in several other contexts, including Casimir systems.

This paper is organized as follows. In section II, we describe the counterexample geometry and show that it has a repulsive regime. In section III, we investigate the origin of the repulsion and in section IV we present an exact solution of an analogous 2d geometry. Finally, in section V, we discuss generalizations and analogues of this unusual geometric effect.

II. EXAMPLE OF A REPULSIVE GEOMETRY

The geometry that gives a repulsive force consists of a thin metallic hemisphere of radius R , centered at the origin and positioned in the lower half space $\{z < 0\}$, together with a point charge at some position $(0, 0, z)$ on the positive z axis (Fig. 2). Note that this system is cylindrically symmetric about the z axis, so the force that the hemisphere exerts on the charge necessarily points in the z direction. When the charge is far from the hemisphere—that is, $z \gg R$ —the force is necessarily attractive by the general argument described above. We now show that the force changes sign and becomes repulsive when the charge approaches the $z = 0$ plane.

Surprisingly, we can establish the existence of a repulsive regime without any calculation at all if we assume an idealized geometry where the hemisphere is infinitely

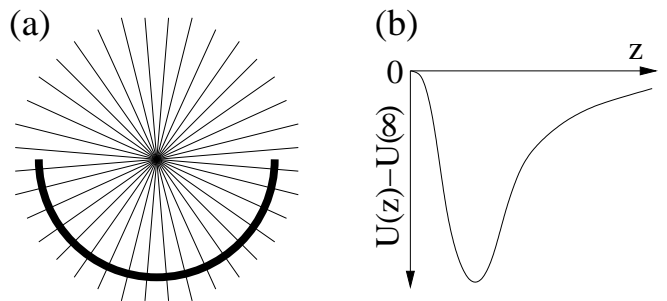


FIG. 3: Argument that a thin metallic hemisphere repels a point charge. (a) At $z = 0$, the vacuum electric field lines of the point charge are already perpendicular to the hemisphere (side view), so the electric field is unaffected by the presence of the hemisphere. (b) Schematic charge-hemisphere interaction energy $U(z) - U(\infty)$: zero at $z = 0$ and at $z \rightarrow \infty$, and attractive for $z \gg R$, so there must be repulsion for small positive z .

thinly thin. The basic idea is to consider the case where the point charge is at the origin, $z = 0$. Notice that when the point charge is at this special point, the vacuum electric field lines of the point charge are all perpendicular to the hemisphere [Fig. 3(a)]. This means that the *vacuum* electric field solves the relevant boundary value problem. Since the electrostatic energy U of the system is proportional to the volume integral of E^2 ,

$$U = \frac{1}{8\pi} \int E^2 d^3\mathbf{x} \quad (1)$$

we conclude that the energy of the system is the same as the energy of a point charge in vacuum.⁸ In other words, the electrostatic energy at $z = 0$ is identical to the energy at infinite separation: $U(z = 0) = U(z = \infty)$. The existence of a repulsive regime now follows since the energy U must vary non-monotonically between $z = 0$ and $z = \infty$ and hence must be repulsive at some intermediate points [Fig. 3(b)].

In fact, we can go a bit further and argue that there is a repulsive regime when z is small and positive. Indeed, recall the inequality $U(z) \leq U(\infty)$ derived in the introduction. Since $U(0) = U(\infty)$, we have the inequality $U(z) \leq U(0)$ implying that $F_z = -dU/dz$ is positive (repulsive) for small positive z .

As the force is attractive for large z and repulsive for small z , the simplest consistent scenario is that the electrostatic energy $U(z) - U(\infty)$ is zero at $z = 0$, decreases to negative values for small $z > 0$ then increases to zero for large z , as depicted in Fig. 3(b). We confirm this scenario in section IV with an exact solution of an analogous 2d system.

Note that the point of minimum U is an equilibrium position, stable under perturbations in the z direction. By Earnshaw's theorem and its generalizations^{1,2}, this equilibrium point must be unstable to lateral (xy) perturbations. More generally, using the exact 2d solution (12), one can show that the point charge is unstable to

(xy) perturbations at *all* points on the z axis.

So far, we have focused on an idealized geometry where the hemisphere is infinitesimally thin. Now suppose that the hemisphere has a finite thickness t . In this case, it is no longer true that $U(0) = U(\infty)$ and hence the above argument cannot be applied directly. However, as long as t/R is small, the repulsive regime must persist since the electrostatic energy curve [Fig. 3(b)] can only shift by a small amount from the $t = 0$ case. To make this more quantitative, note that the main effect of the finite thickness is to expel the electric field from a finite volume $V = t \cdot 2\pi R^2$. As a result, we have (to lowest order in t)

$$U(0) - U(\infty) = -\frac{1}{8\pi} E^2 V = -\frac{q^2 t}{4R^2} \quad (2)$$

where q is the charge carried by the point charge. Comparing this with the minimum value of U , which is of order $U_{min} - U(\infty) \sim -q^2/R$ at $t = 0$, we see that $U_{min} < U(0)$ for small t/R , so the repulsive regime must persist in the presence of small, finite thickness. On the other hand, when t/R becomes sufficiently large, the repulsive regime disappears completely, as we explain in the next section.

III. GEOMETRIC ORIGIN OF THE REPULSION

The general argument described above proves that there must be a repulsive regime, but it does not tell us what causes the repulsion. To address this question, it is useful to consider the induced charges on the hemisphere when the charge is at some point $(0, 0, z)$ on the positive z axis. In general, there will be charges on both sides of the hemisphere, but in the limit where the hemisphere is very thin, we can make the approximation of combining the charges on the two sides into a single surface charge density σ . Assuming that the point charge is positive, we expect this total charge density to be of the form shown in Fig. 4(a), with σ positive in the center of the hemisphere and negative near the boundary. We would like to understand the force that these induced charges exert on the point charge. Clearly the negative charges are closer to the point charge than the positive charges, so they exert a stronger force on it. Naively, one might expect this to lead to a net attractive force. However, the key point is that the angle between the force direction and the z axis is smaller for the positive charges than the negative charges, so even though they are further away, they can potentially exert more force in the z direction, depending on the position of the point charge.

More precisely, the z component of the force that a charge on the hemisphere exerts on the point charge is proportional to $\cos\theta/r^2$ where r is the distance to the charge, and θ is the angle with respect to the z axis [Fig. 4(a)]. The positive charges have a larger r , but also a larger $\cos\theta$ than the negative charges; the competition between these two geometrical effects determines the sign

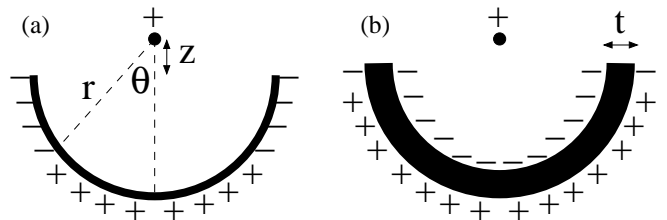


FIG. 4: (a) Schematic charge density σ induced by a point charge on an infinitesimally thin hemisphere (side view). The force that an induced charge on the hemisphere exerts on the point charge in the z direction is proportional to $\cos\theta/r^2$. The positive charges have a larger r than the negative charges, but also a larger $\cos\theta$. The latter effect dominates and leads to a repulsive force for small, positive z . (b) Induced charge density on a hemisphere with finite thickness t . The displacement between the two surface densities makes an attractive contribution to the force and destroys the repulsive regime when t is large.

of the force. If the point charge is very close to the origin, $z \ll R$, then the trigonometric factor $\cos\theta$ wins out: r is virtually the same for the positive and negative charges ($r \approx R$), but $\cos\theta$ is much larger for the positive charges. The result is a repulsive force. On the other hand, if the point charge is very far away, $z \gg R$, then $\cos\theta$ is virtually the same for the positive and negative charges (i.e. $\cos\theta = 1 + O(R^2/z^2)$) while the $1/r^2$ factor is larger for the negative charges (i.e. larger by a factor of size $1 + O(R/z)$). The result is an attractive force.

This picture also explains why the repulsive regime disappears when the thickness t becomes comparable to R . Indeed, once t/R is appreciable, we can no longer make the approximation of combining the charges on the two sides of the hemisphere into a single charge density. Instead, we need to treat the two surface charge densities separately. While the sum of the two charge densities has the form shown in Fig. 4(a), we expect that the charges on the inner surface are primarily negative, while the charges on the outer surface are primarily positive, as depicted in Fig. 4(b). The finite displacement between the two surfaces makes an attractive contribution to the total force, since the negative charges are closer to the point charge than the positive charges. This effect can overwhelm the $\cos\theta$ trigonometric factor when t/R is sufficiently large, destroying the repulsive regime completely.

IV. EXACT SOLUTION IN TWO DIMENSIONS

In this section, we consider a two-dimensional (2d) analogue of the repulsive geometry, and solve the associated electrostatics problem exactly. (The three-dimensional problem can also be solved exactly, though the calculation is more involved⁴). The 2d geometry consists of a metal semicircle of radius R , which we denote by S_R , together with a point charge. In analogy with the

three-dimensional (3d) case, we take the semicircle to be centered at the origin and positioned in the lower half plane, that is, $S_R = \{(x_1, x_2) : |x_1|^2 + |x_2|^2 = R^2, x_2 \leq 0\}$, and the point charge to be at position $\mathbf{y} = (0, z)$ with z positive.

We now compute the 2d electrostatic interaction between the point charge and the metal semicircle assuming that the point charge carries charge q . Our starting point is the 2d boundary value problem defined by

$$\nabla^2 \phi_{\mathbf{y}}(\mathbf{x}) = -2\pi q \cdot \delta(\mathbf{x} - \mathbf{y}) \quad (3)$$

with the boundary conditions

$$\begin{aligned} \phi_{\mathbf{y}}(\mathbf{x}) &= \text{const. for } \mathbf{x} \in \partial S_R \\ \phi_{\mathbf{y}}(\mathbf{x}) + q \log |\mathbf{x}| &= 0 \text{ for } \mathbf{x} \rightarrow \infty \\ \int_{\partial S_R} \mathbf{n} \cdot \nabla \phi_{\mathbf{y}}(\mathbf{x}) d\mathbf{x} &= 0 \end{aligned} \quad (4)$$

Here, the first equation imposes the boundary condition that the semicircle is an equipotential surface, while the third equation imposes the condition that the semicircle is electrically neutral. The force that the metallic object exerts on the charge is given by

$$\mathbf{F}(\mathbf{y}) = -q \nabla \tilde{\phi}_{\mathbf{y}}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}} \quad (5)$$

where

$$\tilde{\phi}_{\mathbf{y}}(\mathbf{x}) = \phi_{\mathbf{y}}(\mathbf{x}) + q \log |\mathbf{x} - \mathbf{y}| \quad (6)$$

is the potential created by the induced charges on the metal object. The electrostatic energy of the system, $U(\mathbf{y})$, is given by

$$\begin{aligned} U(\mathbf{y}) - U(\infty) &= - \int_{\infty}^{\mathbf{y}} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} \\ &= \frac{q}{2} \tilde{\phi}_{\mathbf{x}}(\mathbf{x})|_{\infty}^{\mathbf{y}} \\ &= \frac{q}{2} \tilde{\phi}_{\mathbf{y}}(\mathbf{y}) \end{aligned} \quad (7)$$

Here the second equality follows from the fact that $\tilde{\phi}_{\mathbf{x}}(\mathbf{y}) = \tilde{\phi}_{\mathbf{y}}(\mathbf{x})$ so that $\frac{q}{2} \nabla_{\mathbf{x}} \tilde{\phi}_{\mathbf{x}}(\mathbf{x}) = -\mathbf{F}(\mathbf{x})$.

Our strategy will be to solve the boundary value problem (3) using a conformal mapping, obtain $\phi_{\mathbf{y}}$, and then compute the energy (7). To this end, let us view our 2d system as the complex plane \mathbb{C} , and use complex coordinates $u = x_1 + ix_2$, $v = y_1 + iy_2$ in place of \mathbf{x}, \mathbf{y} . One can check that the analytic function

$$h(u) = \frac{iR + u + i\sqrt{R^2 - u^2}}{2} \quad (8)$$

defines a conformal map from the region outside the semicircle, $\mathbb{C} \setminus S_R$ to the region outside the disk D of radius $R/\sqrt{2}$ centered at the origin, $\mathbb{C} \setminus D$.

The boundary value problem for a metallic disk can be easily solved using image charges. The potential for this geometry is given by

$$\phi_v^D(u) = -q \log |u - v| + q \log \left| u - \frac{R^2}{2\bar{v}} \right| - q \log |u| \quad (9)$$

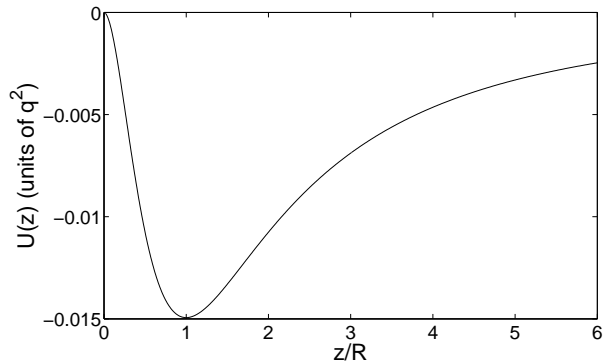


FIG. 5: Exact 2d electrostatic interaction energy $U(z)$ for charge-semicircle geometry. In analogy with the 3d case, the metallic semicircle is centered at the origin and positioned in the lower half plane, while the point charge is at position $\mathbf{y} = (0, z)$ with z positive.

It follows that the potential for the semicircle geometry is

$$\begin{aligned} \phi_v(u) &= \phi_{h(v)}^D(h(u)) \\ &= -q \log |h(u) - h(v)| + q \log \left| h(u) - \frac{R^2}{2h(\bar{v})} \right| \\ &\quad - q \log |h(u)| \end{aligned} \quad (10)$$

so that

$$\tilde{\phi}_v(v) = -q \log \left| \frac{dh}{dv} \right| + q \log \left(1 - \frac{R^2}{2|h(v)|^2} \right) \quad (11)$$

Substituting in the expression for h , we derive

$$\tilde{\phi}_v(v) = q \log \left(\frac{2 - \frac{4R^2}{|iR+v+i\sqrt{R^2-v^2}|^2}}{\left| 1 - i \frac{v}{\sqrt{R^2-v^2}} \right|} \right) \quad (12)$$

Specializing to the case where v is on the positive imaginary axis, $v = iz$, so that $\mathbf{y} = (0, z)$, and using the convention that $U(\infty) = 0$, we obtain the electrostatic energy:

$$U(z) = \frac{q^2}{2} \log \left(\frac{2 - \frac{4R^2}{(R+z+\sqrt{R^2+z^2})^2}}{1 + \frac{z}{\sqrt{R^2+z^2}}} \right) \quad (13)$$

A plot of $U(z)$ is shown in Fig. 5. We can see from the figure (or from a little algebra) that the force $F_z = -\frac{dU}{dz}$ is repulsive for $z < R$ and attractive for $z > R$, with F_z vanishing at $z = R$. Using (12), one can check that the equilibrium at $z = R$ is unstable to perturbations away from the symmetry axis, as required by Earnshaw's theorem and its generalizations.^{1,2} More generally, this instability persists for all z , not just $z = R$.

As an aside, we note that the vanishing of F_z at $z = R$ can be established without any calculation at all: it follows from a simple geometric argument similar to the one

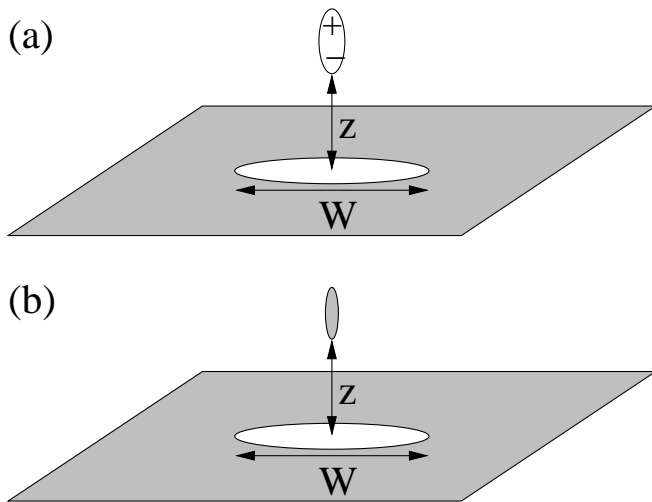


FIG. 6: (a) Example of a geometry in which a neutral metallic object repels an electric dipole: a dipole centered above a thin metallic plate with a hole. (b) Example of a geometry achieving Casimir repulsion: an elongated metal particle centered above a thin metal plate with a hole.

in section III. To see this, consider the z component of the force that an induced charge on the semicircle exerts on the point charge. In analogy with Fig. 4(a), this quantity is proportional to $\cos\theta/r$ where r is the distance to the point charge, and θ is the angle with respect to the z axis. For most locations of the point charge, this geometric factor varies from place to place on the semicircle, so the force that an induced charge exerts on the point charge depends on where it is located. However, when the point charge is at exactly at position $(0, R)$, a little geometry shows that $\cos\theta/r \equiv 1/(2R)$ for *every* point on the semicircle. This means that all the induced charges exert the same force in the z direction. Since the object is neutral, the contributions from the positive and negative induced charges cancel exactly and we conclude that $F_z = 0$.

V. RELATED PHENOMENA

A. A metallic object that repels an electric dipole

In this section, we give an example of another unusual electrostatic geometry: a metallic object that repels an electric dipole. As in the point charge case, this effect is quite counterintuitive. In most cases the interaction between a dipole and a metallic object is attractive: one needs a special geometry to get a repulsive force.

The counterexample geometry consists of a metallic plate with a circular hole of diameter W , located in the $z = 0$ plane and centered at the origin, together with a z -directed dipole at position $(0, 0, z)$ [Fig. 6(a)].

To see that this system has a repulsive regime, we use the same argument as before: we consider the special case

where the dipole is located at the origin, $z = 0$. When the dipole is at this special point, the vacuum dipole field lines are all perpendicular to the metal plate. This means that vacuum electric field solves the relevant boundary value problem. Since the electric field for $z = 0$ is identical to the field in vacuum ($z = \infty$), we conclude that the energy U is also identical: $U(0) = U(\infty)$. As before, this implies that the energy is non-monotonic and hence must be repulsive at some intermediate points. Note that the key property of this geometry is that the metal plate is an equipotential surface for the dipole at $z = 0$, just as the hemisphere was an equipotential surface for a point charge at $z = 0$.

Again one expects the force to be attractive for large z , and repulsive for small z so that the energy $U(z) - U(\infty)$ is of the form shown in Fig. 3(b). One can confirm this picture by exactly solving a 2d analogue of this geometry (the 3d case can also be solved exactly, though the calculation is more involved⁴). The 2d analogue consists of a metal line with a gap of width W located at $\{(x_1, x_2) : x_2 = 0, |x_1| \geq W/2\}$, together with an electric dipole at $(0, z)$, oriented in the z -direction. A conformal mapping approach similar to the one in section IV gives

$$U(z) = -p_z^2 \cdot \frac{2z^2}{(W^2 + 4z^2)^2} \quad (14)$$

(where we are using the convention $U(\infty) = 0$). Taking the derivative with respect to z , one finds that the force is attractive for $z > W/2$ and repulsive for $0 < z < W/2$.

As in the point charge case, one can show that the equilibrium at $z = W/2$ is unstable to perturbations away from the symmetry axis, as required by Earnshaw's theorem and its generalizations.^{1,2} More generally, one can check that this instability persists for all z , not just $z = W/2$.

B. A geometry with a repulsive Casimir force

The Casimir force arises from quantum fluctuations in the electric and magnetic polarization of matter.⁵ It can be regarded as a generalization of the van der Waals force to include retardation effects. Most famously, it gives rise to an attractive interaction between parallel neutral metallic plates in vacuum.

A longstanding question is whether the Casimir force between metallic objects in vacuum is *always* attractive. Using the dipole-metallic object system discussed in the previous section, we can show that this is not the case and construct a simple repulsive geometry for the Casimir force. In the following, we will describe the geometry and briefly explain why it's repulsive and how it's connected to the dipole system. A more detailed discussion can be found in Ref. 6.

The repulsive Casimir geometry consists of a metallic plate with a circular hole of diameter W , located in the $z = 0$ plane and centered at the origin, together with an elongated metallic particle at position $(0, 0, z)$, oriented

with the long axis in the z direction [Fig. 6(b)]. Our claim is that this geometry has a repulsive regime in the limit that the particle is infinitesimally small and highly elongated (the limit of an infinitesimal “metallic needle.”)

To see this, note that the Casimir interaction can be thought of as a electromagnetic interaction between zero-point quantum mechanical charge fluctuations on the particle and the associated induced charges on the plate. As the particle is highly elongated and infinitesimally small, the only charge fluctuations are z -directed dipole fluctuations; hence the problem reduces to understanding the classical electromagnetic interaction between these z -directed dipole fluctuations and the plate with a hole.

The argument now proceeds exactly as in the electrostatic case: we consider the special case where the particle is located at the origin, $z = 0$. When the particle is at this special point, its dipole fluctuations do not couple to the plate at all, since the vacuum dipole field lines are already perpendicular to the plate. This is true for not only zero frequency dipole fluctuations (as shown in the previous section), but also for finite frequency fluctuations. Indeed, the decoupling between the dipole fluctuations and the plate is guaranteed by symmetry since the metal plate is symmetric with respect to the $z = 0$ mirror plane, while the dipole fluctuations are antisymmetric. Since the particle and plate do not couple, it follows that the Casimir energy at $z = 0$ is the same as at infinite separation, $U(z = 0) = U(z = \infty)$, so that the energy must vary non-monotonically and hence must be repulsive at some intermediate points.

For $z \gg W$, the hole in the plate can be neglected, and we must have the usual attractive interaction. Therefore we expect the interaction energy to be of the form shown in Fig. 3(b), with a repulsive regime for small z , an attractive regime for large z , and a sign change for at some $z \sim W$. This expectation is confirmed by explicit numerical calculation.⁶

As in the electrostatic examples, the point of minimum U is an unstable equilibrium as the particle is unstable to perturbations away from the symmetry axis. Thus, this geometry does not support stable Casimir levitation. This is consistent with the instability theorem of Ref. 7— an analogue of Earnshaw’s theorem for the Casimir force.

C. Current flow analogues

In this section we construct analogues of these geometric effects involving current flow in a resistive sheet. We show that current flows can behave in very counterintuitive ways in certain geometries. Our starting point is a perfectly homogeneous infinite resistive sheet with conductivity σ . Imagine injecting current I into some point \mathbf{y} and collecting it at the infinitely distant boundary. As long as the material is homogeneous, then the current will flow out from the injection point in a radially symmetric

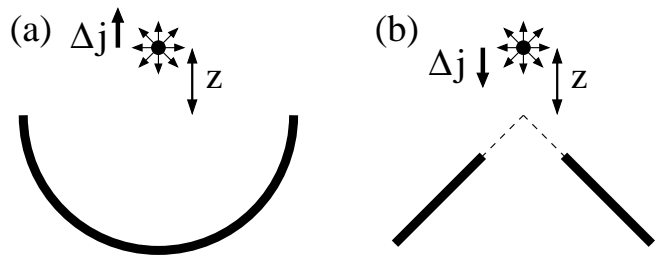


FIG. 7: (a) If current is injected into a homogeneous resistive sheet with conductivity σ , current flows out from the injection point in a radially symmetric way. Surprisingly, reducing the resistivity to 0 in a thin semi-circular region causes an increase, $\Delta\mathbf{j}$, in the current flowing *away* from the semi-circle. (b) Increasing the resistivity to ∞ along two thin line segments intersecting at the origin leads to an increase, $\Delta\mathbf{j}$, in the current flowing *towards* the lines.

way with the current density given by

$$\mathbf{j}(\mathbf{x}) = \frac{I(\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2} \quad (15)$$

Consider what happens if one “shorts out” the sheet, reducing the resistivity to 0 in some region M . This will break the radial symmetry of the problem and change the current flow pattern. Intuitively, one expects that more current will flow in the direction of M . However this need not be the case: we now describe a shape M with the property that shorting out the sheet in M causes current to flow *away* from M .

The counterexample geometry is as follows: one injects current at some point $\mathbf{y} = (0, z)$ in the upper half plane, and one shorts out the sheet along a semicircle centered at the origin and located in the lower half plane [Fig. 7(a)]. When z is small, shorting out the sheet along the circle increases the current flow in the *positive* z direction in the vicinity of \mathbf{y} .

One way to see this is to note that the current flow problem can be exactly mapped onto the original electrostatics problem. Indeed, the current density \mathbf{j} obeys the continuum analogue of Kirchoff’s laws,

$$\begin{aligned} \nabla \cdot \mathbf{j}(\mathbf{x}) &= I \cdot \delta(\mathbf{x} - \mathbf{y}) \\ \nabla \times \left(\frac{\mathbf{j}}{\sigma} \right) &= 0 \end{aligned} \quad (16)$$

with the boundary conditions

$$\begin{aligned} \mathbf{j}(\mathbf{x}) &\perp \partial M \text{ for } \mathbf{x} \in \partial M \\ \mathbf{j}(\mathbf{x}) &= 0 \text{ for } \mathbf{x} \rightarrow \infty \\ \int_{\partial M} \mathbf{n} \cdot \mathbf{j}(\mathbf{x}) d\mathbf{x} &= 0 \end{aligned} \quad (17)$$

(Here, the first boundary condition comes from the vanishing resistivity in the region M , while the third boundary condition comes from current conservation). These equations are identical to the equations obeyed by the

electric field \mathbf{E} in the point charge-metallic object electrostatics problem. But we know that in the charge-semicircle electrostatics problem, the metal semicircle generates a repulsive electric field near the point charge when z is small. Translating this into the current flow language, we conclude that shorting out the semicircle must increase the current flow in the positive z direction, in the vicinity of \mathbf{y} .

It is interesting to consider the opposite question as well: how does the current flow change if we cut a hole in the sheet in some region M , effectively making the resistivity infinite there? Intuitively, one expects that this will decrease the amount of current flowing towards M . Surprisingly, for some shapes of M , this is not the case.

The counterexample geometry for this problem is to inject current at some point $\mathbf{y} = (0, z)$ in the upper half plane and to cut the sheet along two line segments in the lower half plane, which are symmetric with respect to the vertical axis, and which have the property that their extensions pass through the origin [Fig. 7(b)]. When z is small, the effect of making these cuts is to increase the current flow in the *negative* z direction, at least in the vicinity of \mathbf{y} .

To see this, note that in this case, the current density obeys Neumann boundary conditions at ∂M instead of Dirichlet boundary conditions:

$$\begin{aligned} \mathbf{j}(\mathbf{x}) &\parallel \partial M \text{ for } \mathbf{x} \in \partial M \\ \mathbf{j}(\mathbf{x}) &= 0 \text{ for } \mathbf{x} \rightarrow \infty \end{aligned} \quad (18)$$

As a result, this current flow problem maps onto a different kind of electrostatics problem. Instead of the point charge-metallic object problem, the analogue problem in this case involves a point charge and an object with a dielectric constant that is much *smaller* than the surrounding medium. (Such a geometry is unusual, but could in principle be realized by immersing a point charge and an object with a small dielectric constant in a liquid with a large dielectric constant).

While this electrostatics problem is different from the ones we've considered previously, we can analyze it in the same way as before: we note that when $z = 0$, the vacuum field lines of the point charge automatically obey the Neumann boundary conditions (18). This means that the electric field lines at $z = 0$ are the same as in a vacuum, so the electrostatic energy U at $z = 0$ is the same as at infinite separation: $U(z = 0) = U(z = \infty)$. Since the force is repulsive at large z (this follows from general arguments similar to the Dirichlet boundary condition case), we conclude that there is an attractive regime at small z . Translating this into the current flow language, we deduce that cutting the sheet along radial lines increases the current flow in the *negative* z direction in the vicinity of \mathbf{y} , when z is small.

VI. CONCLUSION

In this paper we have shown that, in certain geometries, a neutral metallic object can repel a point charge. We have also shown that analogues of this geometry-induced repulsion can appear in Casimir systems and current flow problems. These examples demonstrate that geometry alone can reverse the sign of electrostatic and Casimir forces, and lead to surprising behavior in many other systems. More generally, we expect that analogues of this effect can appear in almost any physical system governed by Laplace-like equations, from superconductor-magnet systems to (idealized) fluid flow problems.

One direction for future research would be to investigate to what extent these counterexamples are special. For example, are all shapes which repel a point charge similar to the hemisphere geometry discussed here, or are there completely different kinds of geometries with this property? More specifically, is it possible to achieve repulsion with a convex metallic object? One can ask similar questions about Casimir repulsion. There are many open questions here—we have only just begun to understand these counterintuitive geometric effects.

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