SOME CONDITIONS OF MACRO-ECONOMIC STABILITY OF MULTIREGIONAL MODELS

by

Ranko Bon

Dipl. Ing. Arh.,
Belgrade University
1969

M.C.P., Harvard University
1972

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF
PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF
TECHNOLOGY
April, 1975

Signature of Author ....
Department of Urban Studies and
Planning, April 10, 1975

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee

ABSTRACT OF THESIS:

SOME CONDITIONS OF MACRO-ECONOMIC STABILITY OF MULTIREGIONAL MODELS

Ranko Bon

Submitted to the Department of Urban Studies and Planning on April 10, 1975 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

This investigation was a part of a comparative study of the three multiregional input-output (MRIO) models: column coefficient, row coefficient, and gravity coefficient.

The objectives of this research were twofold: (a) to examine the causes underlying negative values in the inverse generated by the row coefficient model, as well as negative projections generated by the model; and (b) to explain why the column coefficient model did not present any of the above problems.

In the investigation of these problems several theorems concerning positive and non-negative matrices associated with Leontief's input-output model were employed and extended to multiregional input-output models.

The results of this research provide: (a) construction rules for the regional trade matrix which ensure that the projections generated by MRIO models will be non-negative; (b) on the basis of the above construction rules, a test of regional trade and regional technology data that ensures non-negative projections for well-constructed MRIO models; and (c) an explanation of the malfunction of the row coefficient model, which concentrates on the violation of the above construction rules.

Thesis Supervisor: Karen R. Polenske
Title: Associate Professor of Urban and Regional Studies

TABLE OF CONTENTS

INTRODUCTION[1]

This investigation was a part of a comparative study of the three multiregional input-output (MRIO) models: column coefficient, row coefficient, and gravity coefficient.[2]

The objectives of this research were twofold: (a) to examine the causes underlying negative values in the inverse generated by the row coefficient model, as well as negative projections generated by the model; and (b) to explain why the column coefficient model did not present any of the above problems.

The first chapter provides a brief introduction to the two MRIO models. In the first section of the second chapter, several theorems concerning the required properties of the technical coefficient matrix that ensure the generation of non-negative inverse and non-negative projections of Leontief's input-output model are employed and extended to MRIO models. A new theorem

---

1 The author wishes to thank Karen Polenske, Aaron Fleisher, Nathaniel Ng, Mark Schuster, and Malte Möhr for numerous comments and suggestions that helped shape this work. The research reported in this paper is financed with funds from Contract No. DOT-OS-30104, University Research Program, U.S. Department of Transportation. The author takes full responsibility for the conclusions.

2 For a comparative analysis of the column coefficient and gravity coefficient models see Fencl and Ng [5].

concerning the required properties of the regional trade coefficient matrix that ensure the generation of non-negative inverses and non-negative projections in MRIO models in general is provided. In the following two sections of the second chapter, the results concerning MRIO models in general, which were derived in the first section, are applied in the analysis of the two MRIO models. The objective of these sections is to determine whether the two models satisfy the conditions that ensure non-negative inverses and projections. The last section of the second chapter provides an economic interpretation of the relationship between the column coefficient and row coefficient models.

The results of this research provide: (a) construction rules for the regional trade coefficient matrix that ensure that the projections generated by MRIO models will be non-negative; (b) on the basis of the above construction rules, a test of regional technology and regional trade data that ensures non-negative projections for well-constructed MRIO models; and (c) an explanation of the malfunction of the row coefficient model, which concetrates on the violation of the above construction rules.

Finally, the policy implications of this investigation extend the conclusions of Hawkins and Simon [7] from the single-region economy to the multiregional economy: if the production system is internally consistent, it will be consistent with <u>any</u> schedule of consumption goods, the latter representing a set of policy variables.

MULTIREGIONAL INPUT-OUTPUT MODELS

Multiregional input-output models are essentially conventional input-output models modified to incorporate interregional trade.[3] These models are founded on one basic economic principle: the total output of an industry is equal to the sum of intermediate demands by various industries (including the industry itself) and demands by final users of the industry's products.

Mathematically, this relationship can be expressed as a set of linear equations:

$$x_i = \sum_{j=1}^{m} a_{ij} x_j + y_i \qquad \text{(all } i\text{)}, \qquad (1)$$

where

a_{ij} = a technical coefficient representing the

---

[3] The reader who is not familiar with multiregional input-output models is advised to refer to Yan [16] for detailed analysis of national input-output models and to Miernyk [9] for an introduction to regional input-output models. More advanced material on the models can be found in Polenske [11; 12].

amount of input of commodity i required by industry j to produce one unit of output of commodity j;

x_i = total supply of commodity i;

x_j = total production of commodity j;

y_i = final demand of commodity i;

i, j = 1,...,m.

Assuming no trade between regions, an input-output model for m industries and n regions can be represented by the following set of linear equations:

$$x_i^{og} = \sum_{j=1}^{m} a_{ij}^{g} x_j^{go} + y_i^{g} \qquad \text{(all i)}, \qquad (2)$$

where

a_{ij}^g = a technical coefficient representing the amount of input of commodity i required by industry j located in region g to produce one unit of output of commodity j;

x_i^{og} = total supply of commodity i in region g;

x_j^{go} = total production of commodity j in region g;

y_i^g = final demand of commodity i in region g;

i, j = 1,...,m;

g = 1,...,n.

If equation (2) is to be used to describe a multi-regional model, it must be further modified to account for the commodities traded between regions. The following two sections will describe the column coefficient and row coefficient models, respectively, since each of the two models utilizes a different accounting scheme for interregional trade.[4]

Column Coefficient Model[5]

Interregional trade is described in the column coefficient model by means of the following relationship:

$$x_i^{gh} = c_i^{gh} x_i^{oh} \qquad \text{(all i)}, \qquad (3)$$

where

x_i^{gh} = amount of commodity i produced in region g that is shipped to region h;

x_i^{oh} = total amount of commodity i consumed in region h;

---

4 The reader who desires a more detailed description of the accounting frameworks should refer to Polenske [12].

5 This is the version as first described by Chenery and Clark [2] and Moses [10].

c_i^{gh} = a trade parameter, indicating the fraction of total consumption of commodity i in region h that is produced in and shipped from region g;

i = 1,...,m;

g,h = 1,...,n.

Equations (2) and (3) are combined to obtain the following set of linear equations (in matrix notation):

$$X = C(AX + Y), \tag{4}$$

where

X = nm·1 vector of regional outputs, x_i^{go}, arranged as a column vector with m outputs for each of the n regions;

C = nm·nm diagonal block matrix of regional trade coefficients, $c_i^{gh} = x_i^{gh}/x_i^{oh}$, where $\sum_g c_i^{gh} = 1$, with each of the diagonals of the n·n submatrices C_i containing the coefficients for m traded commodities and all off-diagonal elements equal to zero;

A = nm·nm block diagonal matrix of regional technical coefficients, $a_{ij}^h = x_{ij}^h/x_{oj}^h$, where $\sum_i a_{ij}^h < 1$, with each of the n submatrices A^h along the principal diagonal containing the m·m coefficient matrix derived from each of the n regional input-output tables, and the elements in all blocks off the principal diagonal equal to zero;

Y = nm·1 vector of regional final demands, y_i^h, arranged as a column vector with m elements representing the amount of commodity i purchased by final users in each of the n regions.

In the implementation of the column coefficient model, specified by equation (4), X is the unknown and is eliminated from the right-hand side of the equation as follows:

$$X = CAX + CY,$$

$$X - CAX = CY,$$

$$(I - CA)X = CY,$$

$$X = (I - CA)^{-1}CY, \tag{5}$$

or

$$X = (C^{-1} - A)^{-1}Y.^6 \tag{6}$$

---

6 It should be noted that in this formulation it is implied that $|C| \neq 0$, since (5) and (6) are equivalent only under this condition. This means that, among other things, C cannot have zero columns or zero rows. In economic terms,

To calculate the regional outputs, X, from equations (5) or (6), matrices A and C and the vector Y must first be obtained.

Row Coefficient Model

Since there are many similarities between the column coefficient and row coefficient models, the latter having been concieved as the "mirror image" of the former, the row coefficient model will be described in less detail.

Interregional trade is described in the row coefficient model by means of the following relationship:

$$x_i^{gh} = r_i^{gh} x_i^{go} \qquad \text{(all i),} \qquad (7)$$

where

x_i^{gh} = amount of commodity i produced in region g that is shipped to region h;

x_i^{go} = total amount of commodity i produced in region g;

r_i^{gh} = a trade parameter, indicating the fraction of

---

it is implied that if there is an industry i in the economy, then commodity i must be both produced and consumed in region g. Consequently, this formulation may be of restricted applicability in regional analysis. More precisely, it is contingent upon the level of aggregation of the data employed. It should be added, however, that this problem has not appeared so far in the work with the model, even though this formulation is typically used in empirical work. Therefore, this implicit assumption is likely to be reasonable for highly aggregated data.

total production of commodity i in region g that is shipped to region h;

$i = 1,\ldots,m;$

$g,h = 1,\ldots,n.$

Equations (2) and (7) are combined to obtain the following set of linear equations (in matrix notation):

$$R'X = AX + Y, \qquad (8)$$

where

X = nm·1 vector of regional outputs;

R'= transpose of R, where R is an nm·nm diagonal block matrix of regional trade coefficients, $r_i^{gh} = x_i^{gh}/x_i^{go}$, where $\sum_h r_i^{gh} = 1$, with each of the diagonals of the n·n submatrices R_i containing the coefficients for m traded commodities and all off-diagonal elements equal to zero;

A = nm·nm block diagonal matrix of regional technical coefficients;

Y = nm·1 vector of regional final demands.

In the implementation of the row coefficient model, specified by equation (8), X is the unknown and is eliminated from the right-hand side of the equation as follows:

$$(R' - A)X = Y,$$

$$X = (R' - A)^{-1}Y, \qquad (9)$$

or

$$X = \left[I - (R')^{-1}A\right]^{-1}(R')^{-1}Y.^{7} \qquad (10)$$

To calculate the regional outputs, X, from equations (9) or (10), matrices A and R' and the vector Y must first be obtained.

MACRO-ECONOMIC STABILITY OF
MULTIREGIONAL INPUT-OUTPUT MODELS

A real n-square matrix $A = \|a_{ij}\|$ is called __positive__ (__non-negative__) if $a_{ij} > 0$ ($a_{ij} \geq 0$) for i, j = 1,...,n. If A is positive (non-negative), it is denoted by $A > 0$ ($A \geq 0$).

The properties of positive matrices were first investigated by Perron, and then amplified and generalized for non-negative matrices by Frobenius. Wielandt provided considerably more simple proofs for the results of

---

[7] It should be noted that in this formulation it is implied that $|R'| \neq 0$, since (9) and (10) are equivalent only under this condition (see footnote 6).

Frobenius. Positive and non-negative square matrices have played an important role in the probabilistic theory of Markov chains, as well as in the more recent study of linear models in economics, and particularly in connection with the input-output model of Leontief. The matrices of interest in this study were first noted by Minkowski.[8]

In the first section of this chapter the problem under investigation is first rigorously stated. Second, several well-known theorems concerning the required properties of the technical coefficient matrix that ensure the generation of non-negative projections of Leontief's input-output model are summarized and stated without proof. Third, these theorems are applied to the multiregional input-output models. And fourth, a new theorem, concerning the required properties of the regional trade coefficient matrix that ensure the generation of non-negative projections in MRIO models in general, is proved.

In the following two sections of this chapter the results derived in the first section are applied in the analysis of the column coefficient and row coefficient

---

8 For an historical outline of the underlying concepts, the basic theorems on positive and non-negative matrices, and an extensive bibliography, see Bellman [1, Ch.16, pp.286-315].

models, respectively, with the objective of determining whether the two models satisfy the conditions that ensure non-negative projections. A formal argument is presented demonstrating that the mathematical properties of the column coefficient model are compatible with the above conditions, while the opposite is true in the case of the row coefficient model.

The last section of this chapter provides an economic interpretation of the formal argument concerning the structure of the two models developed in the preceding two sections. It is argued that the present formulation of the row coefficient model is not a consistent "mirror image" of the column coefficient model, as it was intended to be.

Multiregional Input-Output Models

Consider the general formulation of an MRIO model that corresponds to equations (5) and (10) for the column coefficient and row coefficient models, respectively:

$$X = (I - \Theta A)^{-1} \Theta Y, \tag{11}$$

where

X = vector of regional outputs;

Θ = diagonal block regional trade coefficient matrix;

A = block diagonal regional technical coefficient matrix;

Y = vector of regional final demands; $y_i^h \geq 0$ for $i = 1,\ldots,m$ and $h = 1,\ldots,n$.

It is assumed that Θ, A, and Y are independent, and that $\left| I - \Theta A \right| \neq 0$.[9]

To be economically meaningful all the elements of X must be (a) positive for indecomposable ΘA, and (b) non-negative for decomposable ΘA.[10] This will be ensured if

---

9 Matrices Θ, A, and ΘA for an n-region, m-industry economy can be found in Appendix A.

10 An n-square matrix A $(n > 1)$ is said to be indecomposable if for no permutation matrix T does

$$A_T = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square. Otherwise A is decomposable. If $A_{12} = 0$, A is completely decomposable. Terms irreducable and reducable are often used instead of indecomposable and decomposable. A permutation matrix is obtained by permuting the columns of an identity matrix. TAT^{-1} is obtained by performing the same permutation on the rows and on the columns of A. These concepts can be economically interpreted as follows: If n industries are connected by two-way links directly or indirectly, the system is indecomposable. If k $(k \leq n)$

$$(I - \Theta A)^{-1} \qquad (12)$$

is (a) positive for indecomposable ΘA, and (b) non-negative for decomposable ΘA. If so much as one negative element appears in (12), then there is at least one ΘY that will lead to economically meaningless negative outputs. Suppose the (g,h) element of (12) is negative; then if ΘY is a vector with very small elements except for a large $\sum_h \theta_i^{gh} y_i^h$, the element x_i^g of X will be negative. Indeed, since (12) represents both direct and indirect interindustry requirements of a productive system, negative values in (12) cannot be meaningfully interpreted in economic terms.

The problem under investigation can therefore be stated as follows:

I. What are the necessary and sufficient conditions on Θ that ensure that (12) is (a) positive for indecomposable ΘA, and (b) non-negative for decomposable ΘA, given that A has the following properties:

---

industries are connected by one-way links, the system is decomposable. The system is completely decomposable if there are no links between two or more groups of industries. These separable groups can then be analysed separately. For discussion of the economic significance of these concepts see Dorfman, Samuelson, and Solow [4, pp.254-255].

$$0 \leq a_{ij} < 1 \qquad \text{(all } i,j\text{)},^{11} \qquad (13)$$

and

$$\sum_{i=1}^{n} a_{ij} < 1 \qquad \text{(all } j\text{)}. \qquad (14)$$

II. Which MRIO models satisfy the conditions on Θ to be derived under I.

Several theorems concerning a particular class of positive and non-negative matrices arise in connection with the solution of the system of linear equations of the form:

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + y_i \qquad \text{(all } i\text{)}, \qquad (1)$$

which is associated with Leontief's input-output model. These will now be summarized and stated without proof.[12]

---

[11] It should be noted that throughout this chapter, whenever a symbol has no superscripts, the subscripts denote only the position of an element within a matrix; for instance, a_{ij} does not mean "the amount of input of commodity i required by industry j to produce one unit of output of commodity j," as was the case in the preceding chapter, but simply "an element in ith row and jth column of matrix A."

[12] For proofs see Bellman [1, Ch.16, pp.286-315, and especially p.298], Debreu and Herstein [3], Hadley

THEOREM I: $(I - A)^{-1}$ will be (a) positive if $A > 0$ and condition (14) is satisfied, implying that A is indecomposable, or if conditions (13) and (14) are satisfied and A is indecomposable,[13] and (b) non-negative if conditions (13) and (14) are satisfied and A is decomposable.

This conclusion, mutatis mutandis, applies to multiregional models as well.

[6, pp.118-119], Hawkins and Simon [7], Marcus and Minc [8, Part II, Ch.5, pp.121-133], Rogers [14, Ch.7, pp.405-438, and especially Section 7.3.2, pp.418-420], and Solow [15].

13 There is an equivalent theorem by Hawkins and Simon [7] that, although contained in the above theorem, provides an important economic interpretation of the phenomenon under investigation. Hawkins and Simon show that a necessary and sufficient condition on A that ensures that all the elements of $(I - A)^{-1}$ are positive is that all the principal minors of (I - A) are positive, given that A is indecomposable. Furthermore, it is a corollary of this theorem that a necessary and sufficient condition that all the elements of X satisfying $(I - A)^{-1}$ be positive for any Y is that all the principal minors of (I - A) are positive. Hawkins and Simon [7, p.248] provide the following economic interpretation of their theorem and corollary:

> From the corollary, we see that if the production equations are internally consistent in permitting the production of some fixed schedule of consumption goods, then these consumption goods can be obtained in any desired proportion from this production system. Hence the system will be consistent with any schedule of consumption goods.

COROLLARY I: $(I - \Theta A)^{-1}$ will be (a) positive if conditions (15a) and (16) below are satisfied, implying that ΘA is indecomposable, or if conditions (15b) and (16) are satisfied and ΘA is indecomposable, and (b) non-negative if conditions (15b) and (16) are satisfied and ΘA is decomposable:

$$0 < d_{ij} < 1 \qquad \text{(all } i,j), \qquad (15a)$$

or

$$0 \leq d_{ij} < 1 \qquad \text{(all } i,j), \qquad (15b)$$

and

> The condition that all principal minors must be positive means, in economic terms, that the group of industries corresponding to each minor must be capable of supplying more than its own needs for the group of products produced by this group of industries.... For example, if the principal minor involving the *i*th and *j*th commodities is negative, this means that the quantity of the *i*th commodity required to produce one unit of the *j*th commodity is greater than the quantity of the *i*th commodity that can be produced with an input of one unit of the *j*th commodity. Under these circumstances, the production of these two commodities could not be continued, for they would exhaust each other in their joint production.

For the discussion of Hawkins-Simon conditions the reader is advised to refer to Dorfman, Samuelson, and Solow [4, especially pages 215, 254-257, and 500], and Solow [15].

It should be noted that the conditions of Theorem I above are often referred to as Hawkins-Simon conditions, even though their original result has subsequently been considerably improved and sharpened.

$$\sum_{i=1}^{n} d_{ij} < 1 \qquad \text{(all } j\text{)}, \tag{16}$$

where

$$d_{ij} = \sum_{k=1}^{n} \theta_{ik} a_{kj}.^{14} \tag{17}$$

The properties of Θ that satisfy conditions on ΘA still remain to be established.

THEOREM II: When conditions (13) and (14) are satisfied, conditions (15a) or (15b) and (16) for indecomposable ΘA, and conditions (15b) and (16) for decomposable ΘA will be satisfied if the following necessary and sufficient conditions on Θ are satisfied:

---

14 It should be noted that due to the construction of Θ and A all the sums d_{ij} will have only one term, all other terms being equal to zero (see Appendix A). It should also be noted that all the elements of ΘA will be positive if all the elements along the diagonals of all the blocks of Θ are positive, and if all the elements in the blocks on the principal diagonal of A are positive. In other words, ΘA may be indecomposable regardless of the fact that both Θ and A are completely decomposable (note that the diagonal block matrix Θ can be transformed into a block diagonal matrix by regrouping rows and corresponding columns with the same pattern of elements in blocks along the principal diagonal). Whether ΘA will be indecomposable or decomposable will depend upon the particular economic system under investigation.

$$0 \leqslant \theta_{ik} \leqslant 1 \qquad \text{(all } i,k\text{)}, \tag{18}$$

and

$$\sum_{i=1}^{n} \theta_{ik} \leqslant 1 \qquad \text{(all } k\text{)}. \tag{19}$$

The proof will be provided in two parts: (a) <u>sufficiency</u>: it will be assumed that (18) and (19) are satisfied, and (b) <u>necessity</u>: it will be assumed that (18) or (19) or both are not satisfied.

<u>Sufficiency</u>. Condition (16) will be examined first. From (16) and (17) it follows that

$$\sum_{i=1}^{n} d_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj}, \tag{20}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} \theta_{ik} a_{kj},$$

$$= \sum_{k=1}^{n} \left[a_{kj} \sum_{i=1}^{n} \theta_{ik} \right] .$$

Let

$$\sum_{i=1}^{n} \theta_{ik} = 1 \qquad \text{(all } k\text{)};$$

since

$$\sum_{k=1}^{n} a_{kj} < 1 \qquad \text{(all } j\text{)},$$

it follows that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj} < 1.$$

Condition (16) is therefore satisfied, given conditions (18) and (19). Now conditions (15a) and (15b) will be examined. First, it follows by implication of the above result that $d_{ij} < 1$ since (16) is satisfied. Second, $d_{ij} \geqslant 0$ follows from (13) and (18), ensuring that all the elements of A and Θ, respectively, are non-negative. The possibility of $d_{ij} > 0$ has already been shown in footnote 14 above: all the elements of ΘA will be positive if all the elements along the diagonals of all the blocks of Θ are positive, and if all the elements in the blocks on the principal diagonal of A are positive. Conditions (15a) and (15b) are threfore satisfied as well.

<u>Necessity</u>. Now suppose that either (18) or (19) or both are not satisfied.

(i) Suppose $\theta_{ik} < 0$ (all i,k); in this case neither (15a) nor (15b) will be satisfied since

$$\sum_{k=1}^{n} \theta_{ik} a_{kj} < 0.$$

If so much as one element of Θ is smaller than zero, then there is at least one A that will lead to the violation of conditions (15a) and (15b). Suppose the (g,h) element of Θ is negative; then if column vector a_m of A has very small elements except for a large a_{hm}, the element d_{gm} of ΘA will be negative.

(ii) Suppose $\theta_{ik} > 1$ (all i,k); using the same method of proof as was used in the treatment of sufficient conditions above, it can be shown that in this case (15a), (15b), and (16) will not be satisfied since

$$\sum_{i=1}^{n} \theta_{ik} > 1 \qquad \text{(all k)},$$

and consequently

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj} > 1$$

for some values of a_{kj}, $0 \leqslant a_{kj} < 1$. If so much as one element of Θ is greater than one, there is at least one A that will lead to the violation of conditions (15a), (15b), and (16). Suppose the (g,h) element of Θ is greater than one; then if column vector a_m of A has very small elements except for a large a_{hm}, the element d_{gm} of ΘA will be greater than one.

(iii) Suppose

$$\sum_{i=1}^{n} \theta_{ik} > 1 \qquad \text{(all k);}$$

in this case (15a), (15b), and (16) will not be satisfied for the reasons discussed in (ii) above.

Therefore, when conditions (13), (14), (18), and (19) are satisfied, the elements of (12) will be (a) <u>positive</u> when ΘA is indecomposable, and (b) <u>non-negative</u> when ΘA is decomposable. Furthermore, given that all the elements of ΘY are non-negative, all the elements of X will be (a) <u>positive</u> when ΘA is indecomposable, and (b) <u>non-negative</u> when ΘA is decomposable.

<u>Column Coefficient Model</u>

Consider the formulation of the column coefficient model that corresponds to the general formulation of an MRIO model specified by equation (11):

$$X = (I - CA)^{-1}CY, \qquad (5)$$

where $C = \|c_{ij}\|$ has the following properties:

$$0 \leqslant c_{ij} \leqslant 1 \qquad \text{(all } i, j\text{)}, \tag{21}$$

and

$$\sum_{i=1}^{n} c_{ij} = 1 \qquad \text{(all } j\text{)}.^{15} \tag{22}$$

Since (5) and (11) are equivalent, and since (21) and (22) satisfy (18) and (19), it follows that the column coefficient MRIO model satisfies the conditions on C that ensure that

$$(I - CA)^{-1} \tag{23}$$

is (a) positive for indecomposable CA, and (b) non-negative for decomposable CA. Consequently, given that all the elements of CY are non-negative, all the elements of X are (a) positive for indecomposable CA and (b) non-negative for decomposable CA. In other words, the column coefficient MRIO model is structurally correct.[16]

---

[15] It is interesting to note that these properties are shared by Markov matrices, associated with finite Markov chains, and also that the research of the properties of positive and non-negative matrices started in connection with these matrices, and was only later extended in connection with linear economic models. For an historical outline of the underlying concepts and an extensive bibliography, see Bellman [1, Ch.14, pp.263-280].

[16] Moses [10] briefly argued that the column coefficient model is consistent, although he did not provide a

<u>Row Coefficient Model</u>

Consider the formulation of the row coefficient model that corresponds to the general formulation of an MRIO model specified by equation (11):

$$X = \left[I - (R')^{-1} A \right]^{-1} (R')^{-1} Y, \qquad (10)$$

where $R' = \left\| \bar{r}_{ij} \right\|$ has the following properties:

$$0 \leqslant \bar{r}_{ij} \leqslant 1 \qquad \text{(all } i, j), \qquad (24)$$

and

$$\sum_{i=1}^{n} \bar{r}_{ij} = 1 \qquad \text{(all } j).^{17} \qquad (25)$$

The properties of $(R')^{-1} = \left\| \bar{r}^{*}_{ij} \right\|$ will be examined next. Assuming that conditions (13) and (14) are satisfied,

$$\left[I - (R')^{-1} A \right]^{-1} \qquad (26)$$

will be (a) <u>positive</u> if conditions (27) and (28) below are

---

rigorous proof of his argument. Also, his argument is incomplete since it does not take into consideration the distinctions between positive and non-negative matrices, and indecomposable and decomposable matrices.

[17] See footnote 15.

satisfied and $(R')^{-1}A$ is indecomposable, and (b) <u>non-negative</u> if conditions (27) and (28) are satisfied and $(R')^{-1}A$ is decomposable:

$$0 \leqslant \bar{r}^{*}_{ij} \leqslant 1 \qquad \text{(all } i,j\text{)}, \qquad (27)$$

and

$$\sum_{i=1}^{n} \bar{r}^{*}_{ij} \leqslant 1 \qquad \text{(all } j\text{)}. \qquad (28)$$

It will now be shown that elements of $(R')^{-1}$ do not satisfy conditions (27) and (28).

Given properties (24) and (25) of R', the absolute value of the dominant characteristic root[18] of R' is equal

---

18 Several important problems in regional analysis require the solution of the following set of simultaneous linear equations:

$$Ax = \lambda x,$$

where A is a given square matrix, x is a column vector of unknowns, and λ is an unknown scalar. To solve for λ and x, the above equation system may be expressed as the following system of homogeneous equations:

$$(\lambda I - A)x = 0,$$

where I is the identity matrix of the same order as A. Since this system of equations has a non-trivial solution only if the determinant of the <u>characteristic matrix</u> $(\lambda I - A)$ is equal to zero:

$$|\lambda I - A| = 0.$$

to one (Bellman [1, p.270]). Now the characteristic root of a matrix and its inverse are inverses of each other (Rogers [14, pp.410-411]):

$$\beta_i = \frac{1}{\lambda_i},$$

where λ_i is a characteristic root of R', and β_i is a characteristic root of $(R')^{-1}$. Consequently, since there is a characteristic root of R' the absolute value of which is smaller than one, then the absolute value of the corresponding characteristic root of $(R')^{-1}$ will be greater than one. Indeed, the absolute value of the dominant characteristic root of R' will correspond to the absolute value of the smallest characteristic root of $(R')^{-1}$, and will be equal to one. Therefore, the elements of $(R')^{-1}$ will take both negative values and values greater than one. It

---

This equation is called the <u>characteristic equation</u> of matrix A and may be expanded in powers of λ; the roots of this equation, that is, the values of λ that satisfy it, are called <u>characteristic roots</u> of A. Characteristic roots are often referred to as eigenvalues, characteristic values, proper values, and latent roots. The largest characteristic root is called a <u>dominant characteristic root</u>.

It should be noted that when a matrix is non-negative or positive, and either its row sums or column sums are smaller than one, its characteristic roots are all positive, and its dominant characteristic root is smaller than one.

follows that $(R')^{-1}$ does not satisfy conditions (27) and (28) which correspond to conditions (18) and (19) for MRIO models in general, that is, that $(R')^{-1}A$ does not satisfy conditions (15a) or (15b) and (16). The row coefficient MRIO model is, therefore, structurally incorrect.[19]

---

[19] It can be shown that this conclusion holds even when the assumption that R' is non-singular, made explicitly in footnote 7, is dropped. Suppose $|R'| = 0$. The least restrictive formulation of the row coefficitn model, that is,

$$X = (R' - A)^{-1}Y, \qquad (9)$$

will be considered in this case. Now a matrix will have <u>positive</u> inverse if it is indecomposable and all its elements on the principal diagonal are positive while all the off-diagonal elements are <u>negative</u>; a matrix will have <u>non-negative</u> inverse if it is decomposable and all its elements on the principal diagonal are positive while the off-diagonal elements are <u>non-positive</u> (Debreu and Herstein [3, pp.602-603]). Given the properties of R' and A it is obvious that there is nothing in the structure of the row coefficient model that prevents the elements on the principal diagonal of R' from being smaller than the corresponding elements of A. In other words, the elements on the principal diagonal of (R' - A) may be non-positive. Furthermore, the off-diagonal elements of (R' - A) can be positive, negative, and equal to zero. Consequently, the assumption that R' is singular does not modify the above conclusions. (In the next section of this chapter it will be demonstrated that a certain number of the off-diagonal elements of (R' - A) will be positive by structural necessity. It follows that the row coefficient model will generate negative inverses even when the elements on the principal diagonal of (R' - A) are positive.)

The Relationship Between the Column Coefficient and Row Coefficient Models

The objective of this section is to provide an economic interpretation of the formal argument presented in the preceding sections, and to re-examine the structure of the two models in more detail in light of this interpretation.

The column coefficient and row coefficient models will be developed following Chenery and Clark [2] in order to trace the economic reasoning underlying the two models. For simplicity, Chenery and Clark consider a 2-region, n-industry model that can be easily extended to any number of regions. For each industry i, there is a set of accounting relations describing the flows between the two regions, as shown in Table 1.

Table 1. INTERREGIONAL ACCOUNTS FOR INDUSTRY i

To / From	Consuming region g	Consuming region h	Production in region
Producing region g	X_i^{gg}	X_i^{gh}	X_i^g
Producing region h	X_i^{hg}	X_i^{hh}	X_i^h
Supply in region	Z_i^g	Z_i^h	

From the above table, it follows that the production of industry i in region g can be defined as:

$$X_i^g = X_i^{gg} + X_i^{gh}, \qquad (29)$$

while the supply of industry i in region g can be defined as:

$$Z_i^g = X_i^{gg} + X_i^{hg}. \qquad (30)$$

The set of input-output balance equations,

$$Z_i^g = \sum_{j=1}^{n} a_{ij}^g X_j^g + Y_i^g \qquad \text{(all i)}, \qquad (1a)$$

cannot be solved since there are 2n equations and 6n variables: 2n autonomous demands, 2n production levels, and 2n import levels. In order to solve this set of equations for given final demands, therefore, an assumption about either <u>supply</u> or <u>production</u> must be made. An assumption concerning supply sources will first be made, leading to the column coefficient model: imports are a fixed fraction of the total supply of each commodity. (Chenery and Clark [2] call these proportions "supply coefficients.") These coefficients are defined as:

$$X_i^{gh} = c_i^{gh} Z_i^h. \tag{31}$$

As Chenery and Clark [2, p.67] point out, "the supply coefficient therefore extends the idea of a given marginal propensity to import each commodity to any number of regions."

This fixed-supply assumption makes it possible to express the total production of industry i in region g as a function of the total demands in all regions:

$$X_i^g = c_i^{gg} Z_i^g + c_i^{gh} Z_i^h \qquad \text{(all i).} \tag{32}$$

It is now possible to solve for the production levels corresponding to given final demands in all regions by substituting from the set of equations (1a) into (32) and collecting terms:

$$X_i^g = \left[\sum_{j=1}^{n} c_i^{gg} a_{ij}^g X_j^g + \sum_{j=1}^{n} c_i^{gh} a_{ij}^h X_j^h \right] + \left[c_i^{gg} Y_i^g + c_i^{gh} Y_i^h \right] \qquad \text{(all i).} \tag{33}$$

In other words, the total production of industry i in region g is equal to the amounts of commodity i used for further production in both regions plus the shipments to both regions for final demand.

Now an assumption concerning production will be made, leading to the row coefficient model: the exports are a fixed fraction of the total production of each commodity. These proportions may be called "production coefficients," and are defined as:

$$X_i^{gh} = r_i^{gh} X_i^g. \qquad (34)$$

This fixed-production assumption makes it possible to express the total supply of a given commodity as a function of the total production in all regions:

$$Z_i^g = r_i^{gg} X_i^g + r_i^{hg} X_i^h \qquad \text{(all i).} \qquad (35)$$

By substituting from the set of equations (1a) into (35) and collecting terms the following set of equations is obtained:

$$X^g = 1/r_i^{gg} \left[\sum_{j=1}^{n} a_{ij}^g X_j^g + Y_i^g \right] - 1/r_i^{gg} \left[r_i^{hg} X_i^h \right] \qquad \text{(all i).}^{20} \qquad (36)$$

20 It should be noted that if there are k regions, there may be at most (k - 1) negative terms in each equation (36).

That is, the total production of industry i in region g is equal to the amount used for further production in region g plus the shipments to the final demand in region g minus the amount exported to region h.

The clue to the understanding of the problems encountered in the testing of the row coefficient model lies in the interpretation of these negative terms. It is important to emphasize that the economic interpretation of equations (34) and (35) is straightforward, although the occurrence of the implied pattern of trade in actual economies is implausible (except, perhaps, for a certain class of commodities). Difficulties arise when equations (1a) and (35) are combined. As Richardson [13, pp.66-67] points out,

> The main feature of the row coefficient model, that the proportion of the output of industry i in region r [g in the text above] sold to region s [h in the text above] remains constant irrespective of changes in the level of demand in any of the regions, is theoretically implausible, and infringes the Walrasian assumptions of input-output models that output changes are generated only by shifts in demand and price changes by shifts in supply.

The conflict between these two economic principles is expressed by the fact that the technical and trade coefficients in the column coefficient model represent inputs and imports, respectively, while they represent inputs and exports in the row coefficient model.

In other words, in the case of the column coefficient model the output of an industry is determined only by the demand for its products, while in the case of the row coefficient model the output is determined by the demand for its products and also by some characteristics of the technology employed in the process of production (wherefrom the notion of "production coefficients"). More precisely, in the latter case it is implied that demand changes are determined by changes in output. Equation (36) is therefore self-contradictory.

In order to shed some additional light on the relationship between the economic and mathematical reasons for the failure of the row coefficient model, the structure of the matrices $(C^{-1} - A)$ and $(I - CA)$ for the column coefficient model, and matrices $(R' - A)$ and $\left[I - (R')^{-1}A\right]$ for the row coefficient model will now be re-examined in more detail. These matrices must have positive elements on the principal diagonal and negative (non-positive) off-diagonal elements if their inverses are to be positive (non-negative) (see footnote 19). For simplicity, a 2-region, 2-industry economy will be considered.[21]

---

[21] Matrices C, C^{-1}, CA, R', $(R')^{-1}$, and $(R')^{-1}A$ in terms of interregional trade flows for a 2-region, 2-industry economy can be found in Appendix B.

If it is assumed that intraregional trade in each

Consider matrix $(C^{-1} - A)$. Elements on the principal diagonal are positive, because the elements on the principal diagonal of C^{-1} are positive and greater than one, while the elements on the principal diagonal of A are positive and smaller than one. Off-diagonal elements of $(C^{-1} - A)$ are non-positive, because the off-diagonal elements of C^{-1} are non-positive, while the off-diagonal elements of A are positive or non-negative. Therefore, matrix $(C^{-1} - A)$ satisfies the conditions that ensure positive or non-negative inverses.

Consider matrix (I - CA). Elements on the principal diagonal are positive, because the elements on the principal diagonal of CA are positive and smaller than one. Off-diagonal elements of (I - CA) are negative or non-positive, because the off-diagonal elements of CA are positive or non-negative. It follows that matrix (I - CA) also satisfies the above conditions.

---

commodity is greater than interregional trade in that commodity, it follows that all the elements on the principal diagonal of C^{-1} and $(R')^{-1}$ are positive, while all the off-diagonal elements are negative or non-positive. It should be added that this assumption is corroborated by empirical evidence. Finally, if the above assumption holds, it also follows that the elements on the principal diagonal of C^{-1} and $(R')^{-1}$ are greater than or equal to one.

Now consider matrix $(R' - A)$. Elements on the principal diagonal are not always positive, because the elements on the principal diagonal of R' are smaller than or equal to one, while the elements on the principal diagonal of A are positive and smaller than one. Off-diagonal elements of $(R' - A)$ can be positive, negative, or equal to zero, because the off-diagonal elements of both R' and A are positive or non-negative. It should also be noted that a certain number of the off-diagonal elements of $(R' - A)$ will always be positive; these elements represent exports. In other words, matrix $(R' - A)$ does not generally satisfy the conditions that ensure positive or non-negative inverses.

Finally, consider matrix $\left[I - (R')^{-1}A\right]$. Again, the elements on the principal diagonal are not always positive, because the elements on the principal diagonal of $(R')^{-1}$ are greater than one, which means that the elements on the principal diagonal of $(R')^{-1}A$ may be greater than one. Off-diagonal elements of $\left[I - (R')^{-1}A\right]$ can be positive, negative, and equal to zero, because the off-diagonal elements of $(R')^{-1}A$ can be positive, negative, and equal to zero. Again, it should be noted that a certain number of the off-diagonal elements of $\left[I - (R')^{-1}A\right]$ will always be positive; these elements represent exports, as was the case with matrix (R' - A). Consequently, matrix $\left[I - (R')^{-1}A\right]$ also fails to satisfy the above conditions.

It can be concluded that the structures of the column coefficient and row coefficient models are not fully symmetrical, as they were intended to be. The mathematical properties of the row coefficient model demand that the technical coefficient matrix be redefined to represent <u>outputs</u>, and not <u>inputs</u>, if the row coefficient model is indeed to be the "mirror image" of the column coefficient model, and also be internally consistent. One of the objectives of future research will therefore be to examine the economic implications of this requirement.

CONCLUSIONS

The conclusions of the evaluation of the two MRIO models derived above are consistent with the empirical evidence accumulated over a decade of testing. The column coefficient model always generates an inverse with all the elements larger than zero (positive), as well as positive projections. The row coefficient model always generates an inverse with a large proportion of elements smaller than zero (negative). Also, the row coefficient model frequently generates negative projections.

Three conclusions can be drawn from the above discussion:

(a) All multiregional input-output models of the general formulation given by equation (11) must be constructed in accordance with construction rules (13), (14), (18), and (19), that ensure that (12) will be (a) positive if ΘA is indecomposable, and (b) non-negative if ΘA is decomposable. Furthermore, given that all the elements of the final demand vector, Y, are non-negative, all the elements of the regional output vector, X, will be (a) positive if (12) is positive, and (b) non-negative if (12) is non-negative. The policy implications of this conclusion were already mentioned in the introduction: if a productive system is internally consistent, any schedule of regional final demands (policy variables) can be produced.

(b) Regional trade and technology data (matrices Θ and A) for well-constructed multiregional input-output models can be tested using the conditions discussed in (a) above. The consistency of the data with conditions (13), (14), (18), and (19) ensures that (12) will be (a) positive if ΘA is indecomposable, and (b) non-negative if ΘA is decomposable. Furthermore, given that all the elements of the final demand vector, Y, are non-negative, all the elements of the regional output vector, X, will be (a) positive if (12) is positive, and (b) non-negative if (12) is non-negative.

(c) Unlike the structure of the column coefficient model, the very structure of the row coefficient model violates the conditions (18) and (19). Now that the structure of the MRIO models is better understood, the research will proceed toward restructuring of the row coefficient model in accordance with the construction rules discussed in this work. The objective of this research will be to construct a multiregional input-output model that represents a consistent "mirror image" of the column coefficient model, which the present formulation of the row coefficient model is not.

APPENDIX A

MATRICES Θ, A, AND ΘA FOR AN n-REGION, m-INDUSTRY ECONOMY

$$
\Theta = \begin{bmatrix}
\begin{bmatrix} \theta_1^{11} & & \bigcirc \\ & \theta_2^{11} & \\ & & \ddots \\ \bigcirc & & \theta_m^{11} \end{bmatrix} &
\begin{bmatrix} \theta_1^{12} & & \bigcirc \\ & \theta_2^{12} & \\ & & \ddots \\ \bigcirc & & \theta_m^{12} \end{bmatrix} & \cdots &
\begin{bmatrix} \theta_1^{1n} & & \bigcirc \\ & \theta_2^{1n} & \\ & & \ddots \\ \bigcirc & & \theta_m^{1n} \end{bmatrix} \\
\begin{bmatrix} \theta_1^{21} & & \bigcirc \\ & \theta_2^{21} & \\ & & \ddots \\ \bigcirc & & \theta_m^{21} \end{bmatrix} &
\begin{bmatrix} \theta_1^{22} & & \bigcirc \\ & \theta_2^{22} & \\ & & \ddots \\ \bigcirc & & \theta_m^{22} \end{bmatrix} & \cdots &
\begin{bmatrix} \theta_1^{2n} & & \bigcirc \\ & \theta_2^{2n} & \\ & & \ddots \\ \bigcirc & & \theta_m^{2n} \end{bmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{bmatrix} \theta_1^{n1} & & \bigcirc \\ & \theta_2^{n1} & \\ & & \ddots \\ \bigcirc & & \theta_m^{n1} \end{bmatrix} &
\begin{bmatrix} \theta_1^{n2} & & \bigcirc \\ & \theta_2^{n2} & \\ & & \ddots \\ \bigcirc & & \theta_m^{n2} \end{bmatrix} & \cdots &
\begin{bmatrix} \theta_1^{nn} & & \bigcirc \\ & \theta_2^{nn} & \\ & & \ddots \\ \bigcirc & & \theta_m^{nn} \end{bmatrix}
\end{bmatrix}
$$

$$A = \begin{bmatrix} \begin{bmatrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1m} \\ a^1_{21} & a^1_{22} & \cdots & a^1_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{m1} & a^1_{m2} & \cdots & a^1_{mm} \end{bmatrix} & & & \bigcirc \\ & \begin{bmatrix} a^2_{11} & a^2_{12} & \cdots & a^2_{1m} \\ a^2_{21} & a^2_{22} & \cdots & a^2_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^2_{m1} & a^2_{m2} & \cdots & a^2_{mm} \end{bmatrix} & & \\ & & \ddots & \\ \bigcirc & & & \begin{bmatrix} a^n_{11} & a^n_{12} & \cdots & a^n_{1m} \\ a^n_{21} & a^n_{22} & \cdots & a^n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{m1} & a^n_{m2} & \cdots & a^n_{mm} \end{bmatrix} \end{bmatrix}$$

$$
\Theta A = \begin{bmatrix}
\theta_1^{11} a_{11}^{1} & \theta_1^{11} a_{12}^{1} & \cdots & \theta_1^{11} a_{1m}^{1} & \theta_1^{12} a_{11}^{2} & \theta_1^{12} a_{12}^{2} & \cdots & \theta_1^{12} a_{1m}^{2} & \cdots & \theta_1^{1n} a_{11}^{n} & \theta_1^{1n} a_{12}^{n} & \cdots & \theta_1^{1n} a_{1m}^{n} \\
\theta_2^{11} a_{21}^{1} & \theta_2^{11} a_{22}^{1} & \cdots & \theta_2^{11} a_{2m}^{1} & \theta_2^{12} a_{21}^{2} & \theta_2^{12} a_{22}^{2} & \cdots & \theta_2^{12} a_{2m}^{2} & \cdots & \theta_2^{1n} a_{21}^{n} & \theta_2^{1n} a_{22}^{n} & \cdots & \theta_2^{1n} a_{2m}^{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{11} a_{m1}^{1} & \theta_m^{11} a_{m2}^{1} & \cdots & \theta_m^{11} a_{mm}^{1} & \theta_m^{12} a_{m1}^{2} & \theta_m^{12} a_{m2}^{2} & \cdots & \theta_m^{12} a_{mm}^{2} & \cdots & \theta_m^{1n} a_{m1}^{n} & \theta_m^{1n} a_{m2}^{n} & \cdots & \theta_m^{1n} a_{mm}^{n} \\
\theta_1^{21} a_{11}^{1} & \theta_1^{21} a_{12}^{1} & \cdots & \theta_1^{21} a_{1m}^{1} & \theta_1^{22} a_{11}^{2} & \theta_1^{22} a_{12}^{2} & \cdots & \theta_1^{22} a_{1m}^{2} & \cdots & \theta_1^{2n} a_{11}^{n} & \theta_1^{2n} a_{12}^{n} & \cdots & \theta_1^{2n} a_{1m}^{n} \\
\theta_2^{21} a_{21}^{1} & \theta_2^{21} a_{22}^{1} & \cdots & \theta_2^{21} a_{2m}^{1} & \theta_2^{22} a_{21}^{2} & \theta_2^{22} a_{22}^{2} & \cdots & \theta_2^{22} a_{2m}^{2} & \cdots & \theta_2^{2n} a_{21}^{n} & \theta_2^{2n} a_{22}^{n} & \cdots & \theta_2^{2n} a_{2m}^{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{21} a_{m1}^{1} & \theta_m^{21} a_{m2}^{1} & \cdots & \theta_m^{21} a_{mm}^{1} & \theta_m^{22} a_{m1}^{2} & \theta_m^{22} a_{m2}^{2} & \cdots & \theta_m^{22} a_{mm}^{2} & \cdots & \theta_m^{2n} a_{m1}^{n} & \theta_m^{2n} a_{m2}^{n} & \cdots & \theta_m^{2n} a_{mm}^{n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots \\
\theta_1^{n1} a_{11}^{1} & \theta_1^{n1} a_{12}^{1} & \cdots & \theta_1^{n1} a_{1m}^{1} & \theta_1^{n2} a_{11}^{2} & \theta_1^{n2} a_{12}^{2} & \cdots & \theta_1^{n2} a_{1m}^{2} & \cdots & \theta_1^{nn} a_{11}^{n} & \theta_1^{nn} a_{12}^{n} & \cdots & \theta_1^{nn} a_{1m}^{n} \\
\theta_2^{n1} a_{21}^{1} & \theta_2^{n1} a_{22}^{1} & \cdots & \theta_2^{n1} a_{2m}^{1} & \theta_2^{n2} a_{21}^{2} & \theta_2^{n2} a_{22}^{2} & \cdots & \theta_2^{n2} a_{2m}^{2} & \cdots & \theta_2^{nn} a_{21}^{n} & \theta_2^{nn} a_{22}^{n} & \cdots & \theta_2^{nn} a_{2m}^{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{n1} a_{m1}^{1} & \theta_m^{n1} a_{m2}^{1} & \cdots & \theta_m^{n1} a_{mm}^{1} & \theta_m^{n2} a_{m1}^{2} & \theta_m^{n2} a_{m2}^{2} & \cdots & \theta_m^{n2} a_{mm}^{2} & \cdots & \theta_m^{nn} a_{m1}^{n} & \theta_m^{nn} a_{m2}^{n} & \cdots & \theta_m^{nn} a_{mm}^{n}
\end{bmatrix}
$$

APPENDIX B

MATRICES C, C^{-1}, CA, R', $(R')^{-1}$, AND $(R')^{-1}A$
IN TERMS OF INTERREGIONAL TRADE FLOWS FOR A
2-REGION, 2-INDUSTRY ECONOMY

$$C = \begin{bmatrix} \frac{x_1^{11}}{x_1^{11} + x_1^{21}} & 0 & \frac{x_1^{12}}{x_1^{12} + x_1^{22}} & 0 \\ 0 & \frac{x_2^{11}}{x_2^{11} + x_2^{21}} & 0 & \frac{x_2^{12}}{x_2^{12} + x_2^{22}} \\ \frac{x_1^{21}}{x_1^{11} + x_1^{21}} & 0 & \frac{x_1^{22}}{x_1^{12} + x_1^{22}} & 0 \\ 0 & \frac{x_2^{21}}{x_2^{11} + x_2^{21}} & 0 & \frac{x_2^{22}}{x_2^{12} + x_2^{22}} \end{bmatrix}$$

Note: Each element x_i^{gh} represents a flow of commodity i from region g to region h.

$$C^{-1} = \begin{bmatrix} \dfrac{x_1^{22} x_1^{11} + x_1^{22} x_1^{21}}{x_1^{22} x_1^{11} - x_1^{12} x_1^{21}} & 0 & \dfrac{x_1^{12} x_1^{11} + x_1^{12} x_1^{21}}{x_1^{12} x_1^{21} - x_1^{22} x_1^{11}} & 0 \\ 0 & \dfrac{x_2^{22} x_2^{11} + x_2^{22} x_2^{21}}{x_2^{22} x_2^{11} - x_2^{12} x_2^{21}} & 0 & \dfrac{x_2^{12} x_2^{11} + x_2^{12} x_2^{21}}{x_2^{12} x_2^{21} - x_2^{22} x_2^{11}} \\ \dfrac{x_1^{12} x_1^{21} + x_1^{22} x_1^{21}}{x_1^{12} x_1^{21} - x_1^{22} x_1^{11}} & 0 & \dfrac{x_1^{22} x_1^{11} + x_1^{12} x_1^{11}}{x_1^{22} x_1^{11} - x_1^{12} x_1^{21}} & 0 \\ 0 & \dfrac{x_2^{12} x_2^{21} + x_2^{22} x_2^{21}}{x_2^{12} x_2^{21} - x_2^{22} x_2^{11}} & 0 & \dfrac{x_2^{22} x_2^{11} + x_2^{12} x_2^{11}}{x_2^{22} x_2^{11} - x_2^{12} x_2^{21}} \end{bmatrix}$$

$$CA = \begin{bmatrix}
\frac{x_1^{11}}{x_1^{11} + x_1^{21}} \cdot a_{11}^{1} & \frac{x_1^{11}}{x_1^{11} + x_1^{21}} \cdot a_{12}^{1} & \frac{x_1^{12}}{x_1^{12} + x_1^{22}} \cdot a_{11}^{2} & \frac{x_1^{12}}{x_1^{12} + x_1^{22}} \cdot a_{12}^{2} \\
\frac{x_2^{11}}{x_2^{11} + x_2^{21}} \cdot a_{21}^{1} & \frac{x_2^{11}}{x_2^{11} + x_2^{21}} \cdot a_{22}^{1} & \frac{x_2^{12}}{x_2^{12} + x_2^{22}} \cdot a_{21}^{2} & \frac{x_2^{12}}{x_2^{12} + x_2^{22}} \cdot a_{22}^{2} \\
\frac{x_1^{21}}{x_1^{11} + x_1^{21}} \cdot a_{11}^{1} & \frac{x_1^{21}}{x_1^{11} + x_1^{21}} \cdot a_{12}^{1} & \frac{x_1^{22}}{x_1^{12} + x_1^{22}} \cdot a_{11}^{2} & \frac{x_1^{22}}{x_1^{12} + x_1^{22}} \cdot a_{12}^{2} \\
\frac{x_2^{21}}{x_2^{11} + x_2^{21}} \cdot a_{21}^{1} & \frac{x_2^{21}}{x_2^{11} + x_2^{21}} \cdot a_{22}^{1} & \frac{x_2^{22}}{x_2^{12} + x_2^{22}} \cdot a_{21}^{2} & \frac{x_2^{22}}{x_2^{12} + x_2^{22}} \cdot a_{22}^{2}
\end{bmatrix}$$

$$R' = \begin{bmatrix} \frac{x_1^{11}}{x_1^{11} + x_1^{12}} & 0 & \frac{x_1^{21}}{x_1^{21} + x_1^{22}} & 0 \\ 0 & \frac{x_2^{11}}{x_2^{11} + x_2^{12}} & 0 & \frac{x_2^{21}}{x_2^{21} + x_2^{22}} \\ \frac{x_1^{12}}{x_1^{11} + x_1^{12}} & 0 & \frac{x_1^{22}}{x_1^{21} + x_1^{22}} & 0 \\ 0 & \frac{x_2^{12}}{x_2^{11} + x_2^{12}} & 0 & \frac{x_2^{22}}{x_2^{21} + x_2^{22}} \end{bmatrix}$$

Note: Each element x_i^{gh} represents a flow of commodity i from region g to region h.

$$(R')^{-1} = \begin{bmatrix} \dfrac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} & 0 & \dfrac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} & 0 \\ 0 & \dfrac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} & 0 & \dfrac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \\ \dfrac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} & 0 & \dfrac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} & 0 \\ 0 & \dfrac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} & 0 & \dfrac{x_2^{22}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \end{bmatrix}$$

$$\begin{bmatrix}
\dfrac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a^1_{11} & \dfrac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a^1_{12} & \dfrac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a^2_{11} & \dfrac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a^2_{12} \\
\dfrac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a^1_{21} & \dfrac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a^1_{22} & \dfrac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a^2_{21} & \dfrac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a^2_{22} \\
\dfrac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a^1_{11} & \dfrac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a^1_{12} & \dfrac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a^2_{11} & \dfrac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a^2_{12} \\
\dfrac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a^1_{21} & \dfrac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a^1_{22} & \dfrac{x_2^{21}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a^2_{21} & \dfrac{x_2^{22}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a^2_{22}
\end{bmatrix}$$

BIBLIOGRAPHY

1. Bellman, R. Introduction to Matrix Analysis. New York: McGraw-Hill, Inc., 1970.

2. Chenery, H.B. and Clark, P.G. Interindustry Economics. New York: John Wiley & Sons, 1967.

3. Debreu, G. and Herstein, I.M. "Nonnegative Square Matrices." Econometrica, Vol.21 (1953), 597-607.

4. Dorfman, R., Samuelson, P.A., and Solow, R.M. Linear Programming and Economic Analysis. New York: McGraw-Hill, Inc., 1958.

5. Fencl, Z., and Ng, N.K. "Comparison Tests of the Column Coefficient and the Gravity Coefficient Models." DOT Report No.6. Prepared for the University Research Program, U.S. Department of Transportation, April 1974.

6. Hadley, G. Linear Algebra. Reading, Massachusetts: Addison-Wesley, Inc., 1964.

7. Hawkins, D., and Simon, H.A. "Note: Some Conditions of Macro-Economic Stability." Econometrica, Vol.17 (1949), 245-248.

8. Marcus, M., and Minc, H. A Survey of Matrix Theory and Matrix Inequalities. Boston: Allyn and Bacon, Inc., 1964.

9. Miernyk, W.H. The Elements of Input-Output Analysis. New York: Random House, 1965.

10. Moses, L.N. "The Stability of Interregional Trading Patterns and Input-Output Analysis." The American Economic Review, Vol.XLV, No.5 (1955), 803-832.

11. Polenske, K.R. "An Empirical Test of Interregional Input-Output Models: Estimation of 1965 Japanese Production." The American Economic Review, Vol.LX, No.2 (1970), 76-82.

12. ________, "The Implementation of a Multiregional Input-Output Model of the United States." Input-Output Techniques. Edited by A. Brody and A.P. Carter. Amsterdam: North-Holland Publishing Co., 1972, 171-189.

13. Richardson, H.W. Input-Output and Regional Economics. New York: John Wiley & Sons, 1974.

14. Rogers, A. Matrix Methods in Urban and Regional Analysis. San Francisco: Holden-Day, 1971.

15. Solow, R.M. "On the Structure of Linear Models." Econometrica, Vol.20 (1952), 29-46.

16. Yan, Chiou-shuang. Introduction to Input-Output Economics. New York: Holt, Rinehart, and Winston, Inc., 1969.