

Inventory Basics¹

Supply chains consist of material and information flows. One of the primary material flows is the movement of physical products between stages in the supply chain. This note discusses two simple inventory models that are the building blocks for more complicated models developed later in the course.

General Inventory Issues

Model assumptions

- Single item at a single stage in the supply chain
- Stationary demand process
- Reliable replenishment process

Policy questions

- When to replenish?
- How much to replenish?

Parameters

- Demand characterization
- Replenishment lead-time (how long does it take to get re-supplied?)
- Reorder interval (are orders placed at certain times?)
- Costs
 - Holding costs – costs associated with the holding of inventory. This includes capital, obsolescence, and handling costs.
 - Order costs – costs associated with placing orders
- Fill rate – the level of service provided to the customer. In this note, we use fill rates in lieu of penalty costs.

¹ Copyright © 2001 by Sean P. Willems, Boston University. This note is developed from lecture notes of Stephen C. Graves. No part of this publication may be reproduced without permission.

Continuous Review versus Periodic Review Policies

This section provides an overview of the two models discussed in this note.

Continuous review

- The inventory position, equal to the on-hand and on-order inventory level, is continuously monitored. When the inventory position drops below R units (the reorder point), order Q units (order quantity).

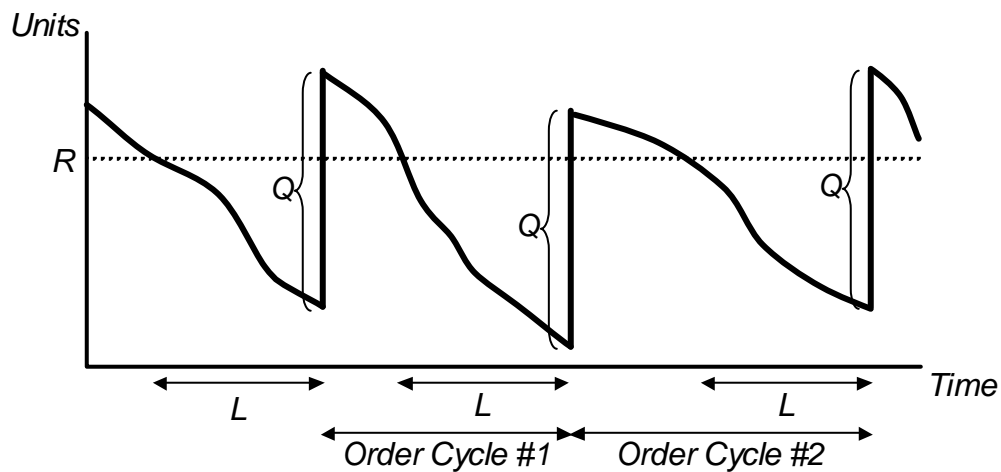


Figure 1: Continuous review policy (simulated values)

- In a continuous review system, the time between orders varies but the amount ordered is fixed.
 - We can see this point graphically in Figure 1: order cycles #1 and #2 are of different lengths but each order contains exactly Q units.
- The reorder point needs to be chosen so that sufficient inventory is available to cover the demand over the replenishment lead-time (L).
- This model is most appropriate for:
 - A items – high value items
 - fixed order sizes dictated by supplier (or manufacturing)

Periodic review

- The inventory position is monitored at periodic intervals of length r ; orders are placed at fixed intervals of length r .
- Base stock policies are a special case of periodic review policies. In base stock policies, after an interval of length r has elapsed, the order quantity is set equal to the amount consumed during the interval.
 - In this note, we restrict ourselves to base stock policies.

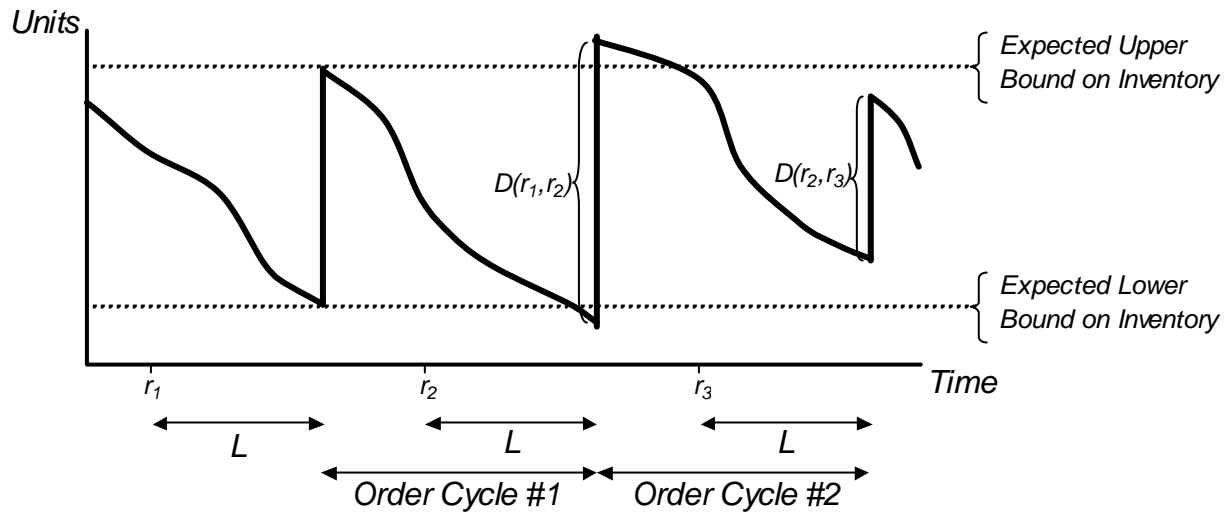


Figure 2: Periodic review policy (simulated values)

- In a periodic review system, the time between orders is fixed but the amount ordered varies.
 - We can see this graphically in Figure 2: order cycles #1 and #2 are the same length but the amount ordered in each interval is different.
- This model is most appropriate for:
 - fixed reorder intervals dictated by supplier (or by logistics department)
 - items that have a joint dependency. When two items have to be coordinated, then it may be beneficial to have them replenished at fixed times.
 - items with small order volume. In this case, it may be uneconomical to order the item in isolation.

- low value items (C items). The low cost of these items may make the monitoring cost of a continuous system prohibitively expensive.

Mathematical Derivation of Expected Inventory Levels²

For each inventory policy, we characterize the expected on-hand inventory in an order cycle (defined as the time between two successive order arrivals). The expected inventory level will be a function of the policy's parameters and decision variable. We then characterize good choices for each policy's decision variable.

Continuous review

Assumptions

- We assume that demands in non-overlapping time intervals are independent. Let $D(a, b)$ denote demand over the time interval from $t = a$ to $t = b$; we assume that $D(a, b)$ has expectation $(b - a)\mu$, and variance $(b - a)\sigma^2$, where μ and σ^2 are the mean and variance of demand per unit time.
 - For example, these demand assumptions are consistent with a daily demand process that has a mean μ and variance σ^2 . $D(a, b)$ then characterizes the demand process over any interval of interest; say a few days, a week, a month or a year. This characterization is necessary since the order cycle is typically more than one day.
- We suppose that we continuously review the inventory; we reorder when the inventory (*on hand and on order*) reaches the order (or reorder) point R .
- When we reorder, we order an amount equal to the order quantity Q .
- The replenishment lead-time is a known constant L .

² This analysis is approximate, where the key approximation is to treat backorders as negative inventory when determining the expected inventory level.

Inventory Dynamics

- Let $I(t)$ denote the inventory at time t . At time of reorder (say time zero), the inventory position $I(0) = R$ by definition of the policy
- At the time just prior to the replenishment, the on-hand inventory level $I(L)^-$ is

$$I(L)^- = R - D(0,L) \quad (1)$$

and the expectation of $I(L)^-$ is

$$E[I(L)^-] = R - L\mu \quad (2)$$

- At the time just after the replenishment, the inventory level $I(L)^+$ is

$$I(L)^+ = R - D(0,L) + Q \quad (3)$$

and the expectation of $I(L)^+$ is

$$E[I(L)^+] = R - L\mu + Q \quad (4)$$

- These inventory dynamics are illustrated graphically in the following figure

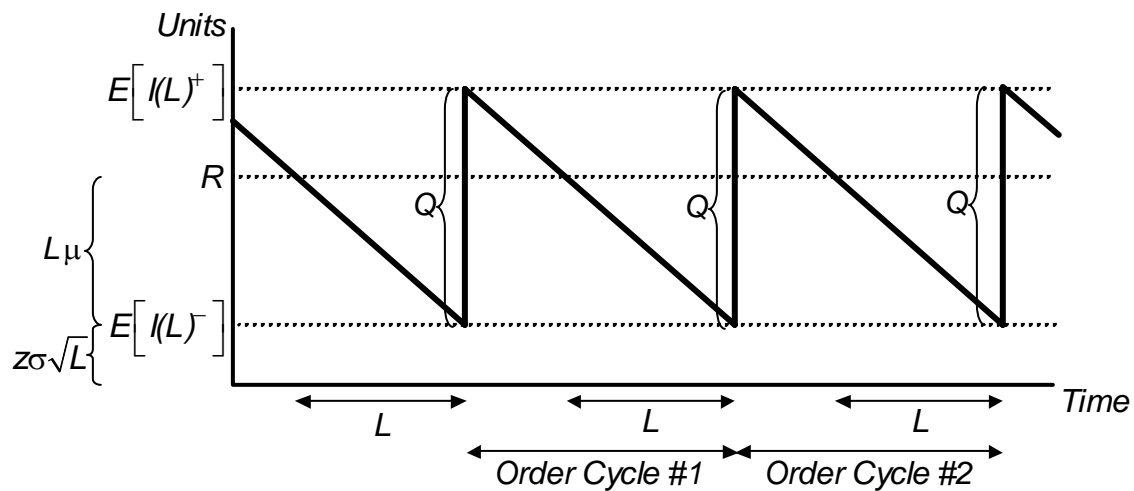


Figure 3: Expected behavior of continuous review system

- As this is true for every replenishment cycle, we can approximate the expected inventory level by the average of the expectations for the low and high points³, given by (2) and (4):

$$\text{Expected inventory level} = R - L\mu + Q/2 \quad (5)$$

- We typically set R to cover the demand over the lead-time with high probability. The goal is to ensure that $I(L)^-$, from equation (1), is nonnegative with high probability. Assuming normally distributed demand, a common approach is to set R as

$$R = L\mu + z\sigma\sqrt{L} \quad (6)$$

That is, we set R equal to the mean demand over the lead-time, plus some number (z = safety factor) of standard deviations of lead-time demand.

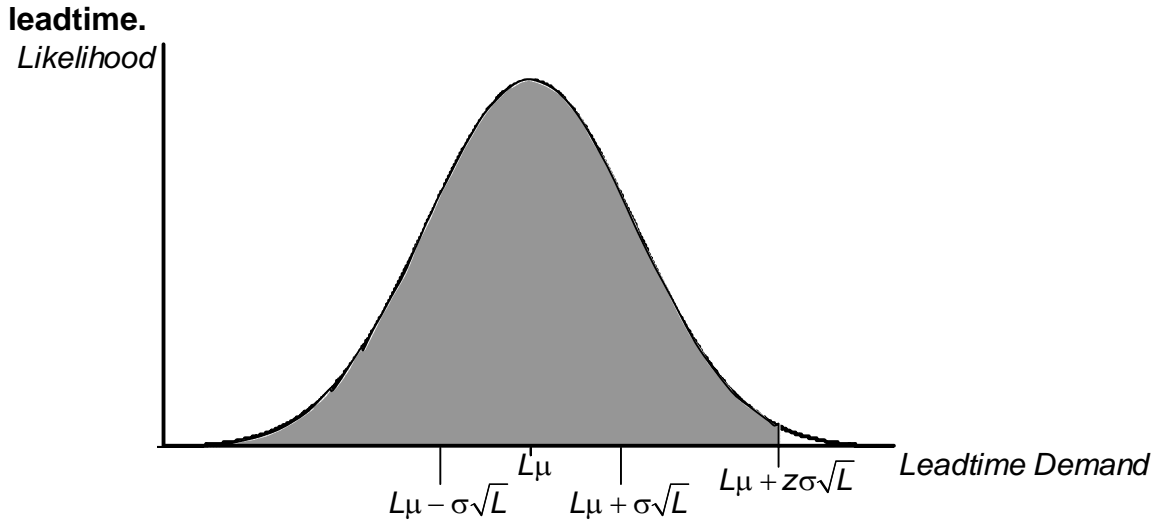
- Substituting (6) into (5) we have

$$\begin{aligned} \text{Expected inventory level} &= \frac{Q}{2} + z\sigma\sqrt{L} \\ &= \text{cycle stock} + \text{safety stock} \end{aligned} \quad (7)$$

Digression: How to choose z ?

- The figure below represents the probability mass function of the random variable $D(0,L)$, the demand over the

³ It may be helpful to look at Figure 3 and think back to high school geometry. A cycle's inventory area consists of a triangle sitting on top of a rectangle. The triangle's area is $(\frac{1}{2})(\text{base})(\text{height})$ with the height equal to $E[I(L)^+] - I(L)^-$ and the base equal to the cycle length. The rectangle's area equals $(E[I(L)^-])(\text{cycle length})$. Adding the two areas together yields $(\frac{1}{2})(E[I(L)^+] + E[I(L)^-])(\text{cycle length})$. The time horizon and stationarity assumption allow us to ignore cycle length.



As noted earlier, $D(L)$ has a mean $L\mu$ and variance $L\sigma^2$.

- z corresponds to the number of standard deviations of protection the safety stock will cover.
 - If the realized demand over the lead-time is less than R , then a stock-out will not occur.
 - Typical choices for z :

z value	Prob. of no stock-out	Expected Backorders per cycle
1.645	0.95	.021 * σ * \sqrt{L}
2	0.98	.0085* σ * \sqrt{L}
3	0.999	.0004* σ * \sqrt{L}

Periodic Review

Assumptions

- We assume that demands in non-overlapping time intervals are independent. Let $D(a, b)$ denote demand over the time interval from $t = a$ to $t = b$; we assume that $D(a, b)$ has expectation $(b - a)\mu$, and variance $(b - a)\sigma^2$, where μ and σ^2 are the mean and variance of demand per unit time.
 - For example, these demand assumptions are consistent with a daily demand process that has a mean μ and variance σ^2 . $D(a, b)$

then characterizes the demand process over any interval of interest; say a few days, a week, a month or a year. This characterization is necessary since the order interval is typically more than one day.

- We suppose that we place a replenishment order on a regular cycle, say at times $t = r, 2r, 3r, \dots$ where we call r the review period.
- The replenishment lead-time is a known constant L ; thus we receive replenishments at times $t = r+L, 2r+L, 3r+L, \dots$
- At each replenishment epoch, we order an amount equal to the demand since the last replenishment epoch. That is, at time $t = r$, we order $D(0, r)$; at time $t = 2r$, we order $D(r, 2r)$; at time $t = 3r$, we order $D(2r, 3r)$, etc. These orders are received into inventory at times $t = r+L, 2r+L, 3r+L, \dots$
- In effect, this is just how a "pull" system operates in discrete time.

Inventory Dynamics

- Define $I(t)$ to be on-hand inventory at time t ; $I(t)$ equals some starting inventory, call it $B = I(0)$, minus demand from time 0 to t , plus order replenishments received up to time t :

$$I(t) = B - D(0, t) + \text{order replenishments} \quad (8)$$

- Consider $t = kr + L$ for some integer k ; then the inventory at time $t = kr + L$, just before receipt of the order is denoted $I(kr + L)^-$ and given by:

$$\begin{aligned} I(kr + L)^- &= B - D(0, kr + L) + D(0, r) + \dots + D((k-2)r, (k-1)r) \\ &= B - D((k-1)r, kr + L) \end{aligned} \quad (9)$$

and the expectation of $I(kr + L)^-$ is

$$E\left[I(kr + L)^-\right] = B - (r + L)\mu \quad (10)$$

- By assumption, at time $t = kr + L$, we will receive the order placed at time $t = kr$ for the amount $D((k-1)r, kr)$, which raises the inventory to:

$$I(kr + L)^+ = B - D(kr, kr + L) \quad (11)$$

and the expectation of $I(kr + L)^+$ is

$$E\left[I(kr + L)^+\right] = B - L\mu \quad (12)$$

- Equation (9) gives the "low" point for inventory in a replenishment cycle, while equation (11) gives the "high" point.
- These inventory dynamics are illustrated graphically in the following figure

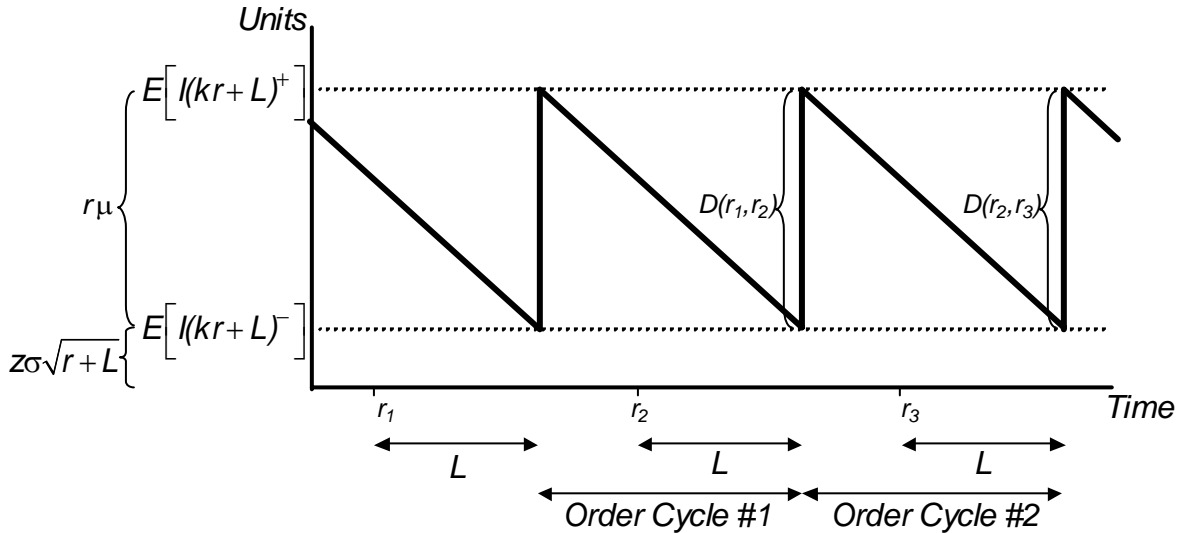


Figure 4: Expected behavior of periodic review system

- As this is true for every replenishment cycle, we can approximate the expected inventory level by the average of the expectations for the low and high points, given by (10) and (12):

$$\text{Expected inventory level} = B - L\mu - \frac{r\mu}{2} \quad (13)$$

- To determine how big B should be, we set B so that the probability of stock-out is small; that is, we want to have $I(t) > 0$ with high probability. We can assure this by assuring that it is true for the "low" inventory point given by (9). Thus, we need to set B so that with high probability

$$B > D((k-1)r, kr+L) .$$

The right hand side of the inequality is a random variable with mean $(r+L)\mu$, and variance $(r+L)\sigma^2$; for normally distributed demand, we could then set

$$B = (r+L)\mu + z\sigma\sqrt{r+L} \quad (14)$$

where z is the safety factor, equal to the number of standard deviations of protection chosen. We term B to be the base stock.

- By substituting (14) into (13), we have that

$$\begin{aligned}
 \text{Expected inventory level} &= (r+L)\mu + z\sigma\sqrt{r+L} - L\mu - \frac{r\mu}{2} \\
 &= \frac{r\mu}{2} + z\sigma\sqrt{r+L} \\
 &= \text{cycle stock} + \text{safety stock}
 \end{aligned}
 \tag{15}$$

Conclusions

Summary of differentiating characteristics

Characteristic	Continuous Review	Periodic Review
Decision variable	Reorder point (R) Order quantity (Q) ⁴	Reorder interval (r) Base stock (B)
Reorder interval	Variable	Fixed
Order quantity	Fixed	Variable
Expected inventory level	$\frac{Q}{2} + z\sigma\sqrt{L}$	$\frac{r\mu}{2} + z\sigma\sqrt{r+L}$
Candidate products	High value	Coordinated parts

Extensions

- The following issues will not be addressed today, they can be incorporated into these models:
 - Constraints on order sizes
 - Price discounts
 - Multiple sourcing options
 - Stochastic lead-times
 - Stochastic replenishment yields

Stylized Example - Water bottle manufacturing

Problem Statement

- Consider a manufacturer of plastic water bottles. The sole raw material is plastic pellets. The firm's daily demand is normally distributed with a

⁴ In this note, I do not address the optimal choice of Q. I have essentially assumed that it is either a fixed amount or determined using EOQ.

mean of 200 bottles/day and a variance of 120 bottles²/day². The firm takes an inventory count and places orders on the first and third Monday of every month. The plastic supplier delivers an order within one week of receipt. The manufacturer is conservative when it comes to inventory (i.e., they hold a lot). Their justification is that the cost is minimal and it is unlikely to become obsolete since all they make are plastic bottles.

- a. What is the inventory policy?
- b. What are the parameters of interest?
- c. What is the expected inventory?

Solution

- Since orders are placed every two weeks, this is a periodic review model. Assuming 20 working days in a month, the reorder interval (r) is 10 days and the replenishment lead-time is 5 days. If the manufacturer chooses a fill rate of 98%, then $z = 2$. This results in the following inventory figures:

$$\text{Cycle stock} = \frac{r\mu}{2} = \frac{(10 \text{ days})(200 \text{ units/day})}{2} = 1000 \text{ units}$$

$$\text{Safety stock} = z\sigma\sqrt{r+L} = 2(10.95 \text{ units/day})\sqrt{10 \text{ days} + 5 \text{ days}} = 84.9 \text{ units}$$