A novel shortfiber composite in which the microscopic advanced fiber reinforcements are coated with radially aligned carbon nanotubes (CNTs) is analyzed in this study. A shear-lag model is developed to analyze the load transferred to such coated fibers from the aligned-CNT reinforced matrix in a hybrid composite application. It is found that if the carbon fibers are coated with radially aligned CNTs, then the axial load transferred to the fiber is reduced due to stiffening of the matrix by the CNTs. Importantly, it is shown that at low loading of CNTs in the polymer matrix, there is a significant reduction in the maximum interfacial shear stress, e.g., at 1% CNTs, there is an ~25 % reduction in this maximum stress. Further, the modification in the load sharing between the fiber and the matrix plateaus at ~2% CNT matrix loading, indicating a small but critical window for engineering the interface in this manner. Effects of the variation of the aspect ratio of the fiber, CNT volume fraction and the application of radial load on the load transferred to such CNT coated fibers are also investigated.

**KEYWORDS:** Nanocomposites, Composite Structures
1. Introduction

The identification of carbon nanotubes (CNTs) has stimulated extensive research devoted to the prediction of their elastic properties through experiments and theoretical modeling. Of interest is to determine elastic properties of the CNTs as an input to models that predict composite behaviour. Early work by Treacy et al.\textsuperscript{2} experimentally determined that CNTs have Young’s modulus in the terapascal (TPa) range. Li and Chou\textsuperscript{3} linked structural and molecular mechanics (MM) approaches to compute elastic properties of CNTs. Sears and Batra\textsuperscript{4} used three MM potentials to simulate axial and torsional deformations of a CNT assuming that the tube can be regarded as a hollow cylinder of mean diameter equal to that of the CNT and determined the wall thickness, Young’s modulus and Poisson’s ratio of the CNT. Shen and Li\textsuperscript{5} assumed that a CNT should be modeled as a transversely isotropic material with the axis of transverse isotropy coincident with the centroidal axis of the tube. They determined values of the five elastic constants by using a MM potential and an energy equivalence principle. Batra and Sears\textsuperscript{6} proposed that the axis of transverse isotropy of a CNT is a radial line rather than the centroidal axis of the tube and found that Young’s modulus in the radial direction equals about 1/4th of that in the axial direction. Wu et al.\textsuperscript{7} developed an atomistic based finite deformation shell theory for single-walled CNT and found its stiffness in tension, bending and torsion.

A great deal of research has also been carried out on the prediction of effective elastic properties of CNT-reinforced composites. For example, Thostensen and Chou\textsuperscript{8} have estimated the elastic moduli of CNT-reinforced composite through micromechanical analysis. Gao and Li\textsuperscript{9} derived a shear lag model of CNT reinforced
polymer composites by replacing the CNT with an equivalent solid fiber. Song and Yoon\textsuperscript{10} numerically estimated the effective elastic properties of CNT-reinforced polymer based composites. Siedel and Lagoudas\textsuperscript{11} carried out a micromechanical analysis to estimate the effective properties of CNT-reinforced composites. Guzman de Villoria and Miravete developed a model to estimate the effect of the CNTs dispersion in composites matrix by micromechanical analysis\textsuperscript{12}. Jiang et al.\textsuperscript{13} derived a continuum based model to study the effect of CNT/matrix interface on the macroscopic properties of CNT-reinforced composites. Odegard et al.\textsuperscript{14, 15} have modeled CNT-reinforced composite to estimate effective elastic moduli using an equivalent-continuum modeling method that connects computational chemistry and solid mechanics models. To avoid the long times of simulation of materials at nanoscale level, Yamakov and Glaessgen\textsuperscript{16} have linked continuum mechanics in with atomic-level simulations, in one case to study the fracture tip of several metals. Zhang and He\textsuperscript{17} theoretically investigated the viscoelastic behavior of CNT-reinforced composites developing a three-phase shear-lag model. Most recently, Ray and Batra\textsuperscript{18} carried out a micromechanical analysis to estimate the effective elastic and piezoelectric properties of CNT and piezoelectric fiber reinforced hybrid composite. Several review articles have appeared that summarize the various advances in these two-phase (CNTs plus a matrix) nanocomposites\textsuperscript{19-21}.

Here we analyze a new hybrid composite composed of micron-scale diameter advanced fibers with \textit{in situ} grown radially aligned CNTs and a polymer matrix. Growth of aligned CNTs on advanced fibers (see examples in Figure 1) have been investigated by several groups\textsuperscript{22-26} and recently, bulk composites have been realized using aligned ‘fuzzy’ fibers\textsuperscript{27-29}. The objective of this work is to investigate the load transferred to a
carbon fiber from the matrix in the case where the micron-scale fuzzy fiber is discontinuous (see Figure 2). A closed-form shear lag model is developed for such investigation, incorporating a micromechanics model for predicting the radially-orthotropic properties of the aligned-CNT reinforced polymer matrix. Such composites can be described as a hybrid nano-engineered composite, where the polymer matrix is reinforced with radially aligned-CNT resulting in an aligned-CNT nanocomposite matrix that surrounds the micron-scale advanced fiber (see Figure 3). Such nano-engineered composites can be fabricated using capillarity-driven wetting of the aligned CNTs by advanced polymers\textsuperscript{29, 30}.

2. Shear lag model

A schematic sketch of the cylindrical representative volume element (RVE) of the composite analyzed here is shown in Figure 3. The cylindrical coordinate system ($r$, $\theta$ and $x$) is considered in such a way that the axis of the RVE coincides with the $x$-axis while the CNTs are aligned along the $r$-direction. The model is derived by dividing the RVE into three zones. The portion of the RVE in the zone $-L_f \leq x \leq L_f$ consists of a discontinuous micro-scale advanced fiber (carbon is considered here) reinforcement on which radially aligned CNTs have been grown. When this resulting fuzzy fiber is embedded in a polymer material, the CNT forest is filled with the polymer creating a nano-reinforced polymer matrix, what many have called a polymer nanocomposite (PNC)\textsuperscript{31, 32}. Thus, the radially aligned CNTs reinforce the polymer matrix and the portion of the RVE in the zone $-L_f \leq x \leq L_f$ can be viewed as a hybrid composite comprised of the carbon fiber reinforcement embedded in the CNT-reinforced polymer.
matrix composite phase. The radius and the length of the carbon fiber are denoted by $a$ and $2L_f$, respectively. The inner and outer radii of the CNT-reinforced matrix phase are $a$ and $R$, respectively. The portions of the RVE in the zones $-L \leq x \leq -L_f$ and $L_f \leq x \leq L$ are treated in the model as an imaginary fiber and the matrix phase, both composed of the polymer material. The radius of the imaginary fiber is also denoted by $a$ while the inner and outer radii of the matrix phase are also represented by $a$ and $R$, respectively. Thus, the shear lag model developed for the zone $-L_f \leq x \leq L_f$ can be applied to derive the shear lag models for the zones $-L \leq x \leq -L_f$ and $L_f \leq x \leq L$.

In what follows, the shear lag model for the zone $-L_f \leq x \leq L_f$ is first derived. A tensile stress $\sigma_0$ is applied to the RVE along $x$ direction at $x = \pm L$ while the RVE is subjected to a radial normal stress $q_0$ at $r = R$. In order to derive this shear lag model, the effective properties of the aligned CNT-reinforced matrix phase are needed. This PNC matrix phase has transverse isotropy in a radial coordinate system due to the CNT alignment and isotropic nature of the polymer. This is a slight approximation because the grown CNTs have reduced volume fraction as they grow radially, but volume fraction may be considered constant over the small (microns) CNT lengths considered. Micromechanics is used to calculate properties in this region as they have not been determined experimentally to date. A micromechanics model by Ray and Batra$^{18}$ is used to calculate the effective elastic constants for a forest of aligned single-walled CNTs (properties from Ref.$^5$ given in Table 1) embedded in a polymer (results summarized in Table 2). These results are used as an input to the shear-lag model.

Returning to the shear-lag model, the governing equations for the different
phases of this RVE concerning equilibrium along x direction are given by

\[
\frac{\partial \sigma^i_x}{\partial x} + \frac{1}{r} \frac{\partial (r \sigma^i_{xr})}{\partial r} = 0, \quad i = f \text{ and } m
\]

(1)

while the relevant constitutive relations are

\[
\sigma^i_x = C^i_{11} \varepsilon^i_x + C^i_{12} \varepsilon^i_\theta + C^i_{13} \varepsilon^i_r \quad \text{and} \quad \sigma^i_{xr} = C^i_{55} \varepsilon^i_{xr}; \quad i = f \text{ and } m
\]

(2)

In Eqs. (3) and (4), superscripts f and m denote, respectively, the carbon fiber and the CNT-reinforced PNC matrix. For the i-th constituent phase, \(\sigma^i_x\) and \(\sigma^i_{xr}\) represent the normal stresses in the x and r, directions, respectively; \(\varepsilon^i_x\), \(\varepsilon^i_\theta\) and \(\varepsilon^i_r\) are the normal strains along x, \(\theta\) and r, directions, respectively; \(\sigma^i_{xr}\) is the transverse shear stress, \(\varepsilon^i_{xr}\) is the transverse shear strain and \(C^i_{ij}\) are the elastic constants. It should be noted here that the principal material coordinates 1, 2, 3 axes are also considered to be coincident with the problem coordinate axes x, \(\theta\), r, respectively. Hence, the conventional subscripts are used to write the elastic constants appearing in Eq. (2). The strain-displacement relations for an axisymmetric problem relevant to this RVE are

\[
\varepsilon^i_x = \frac{\partial u^i}{\partial x}, \quad \varepsilon^i_\theta = \frac{w^i}{r}, \quad \varepsilon^i_r = \frac{\partial w^i}{\partial r} \quad \text{and} \quad \varepsilon^i_{xr} = \frac{\partial u^i}{\partial r} + \frac{\partial w^i}{\partial x}; \quad i = f \text{ and } m
\]

(3)
in which $u^i$ and $w^i$ represent the axial and radial displacements at any point of the $i$-th phase along $x$ and $r$, directions, respectively. The traction boundary conditions are given by

$$
\sigma_r^m \bigg|_{r=R} = q_0 \quad \text{and} \quad \sigma_{xr}^m \bigg|_{r=R} = 0
$$

(4)

and the continuity conditions are

$$
\sigma_r^f \bigg|_{r=a} = \sigma_r^m \bigg|_{r=a}, \quad \sigma_{xr}^f \bigg|_{r=a} = \sigma_{xr}^m \bigg|_{r=a} = \tau_i, \quad u^f \bigg|_{r=a} = u^m \bigg|_{r=a} \quad \text{and} \quad w^f \bigg|_{r=a} = w^m \bigg|_{r=a}
$$

(5)

Where, $\tau_i$ is the transverse shear stress at the interface between the carbon fiber and the PNC matrix phase. The average axial stresses in the different phases are defined as

$$
\bar{\sigma}_x^f = \frac{1}{\pi a^2} \int_{0}^{a} \sigma_x^f 2\pi r dr, \quad \text{and} \quad \bar{\sigma}_x^m = \frac{1}{\pi (R^2 - a^2)} \int_{a}^{R} \sigma_x^m 2\pi r dr
$$

(6)

Now, making use of Eqs. (1) and (4) to (6), it can be derived that

$$
\frac{\partial \bar{\sigma}_x^f}{\partial x} = -\frac{2}{a} \tau_i \quad \text{and} \quad \frac{\partial \bar{\sigma}_x^m}{\partial x} = \frac{2a}{R^2 - a^2} \tau_i
$$

(7)
Since the radial dimension of this RVE is very small, it is reasonable to assume that the gradient of \( \sigma^m_x \) with respect to the axial coordinate (\( x \)) is independent of the radial coordinate (\( r \)). Thus let us assume that
\[
\frac{\partial \sigma^m_x}{\partial x} = \phi(x)
\]  
(8)

Integrating the governing equation (1) for the PNC matrix phase from \( r \) to \( R \), it can be shown that
\[
\phi(x) = -\frac{2r}{R^2 - r^2} \sigma^m_{xr}
\]  
(9)

Substituting \( \sigma^m_{xr} \bigg|_{r=a} = \tau_i \) in Eq. (9), the transverse shear stress in the PNC matrix can be expressed in terms of the interface shear stress \( \tau_i \) as follows:
\[
\sigma^m_{xr} = \left( \frac{R^2}{r} - r \right) \frac{a}{R^2 - a^2} \tau_i
\]  
(10)

Also, since the RVE is an axisymmetric problem, it may further be assumed that the gradient \( \frac{\partial w^i}{\partial x} \) of radial displacements with respect to \( x \)-direction is negligible and so, from the constitutive relation (2) between \( \sigma^m_{xr} \) and \( \epsilon^m_{xr} \) one can write,
\[
\frac{\partial u^m}{\partial x} = \frac{1}{C_{55}}\sigma^m_{rx}
\]

(11)

Solving Eq. (11), the axial displacement of the matrix phase along \( x \) direction can be derived as follows:

\[
u^m = u^f_a + A_1 \tau_i
\]

(12)

in which \( u^f_a = u^f \bigg|_{r=a} \) and \( A_2 = \frac{a}{C_{55}(R^2 - a^2)} \left\{ R^2 \ln \frac{r}{a} - \frac{1}{2} (r^2 - a^2) \right\} \)

(13)

The radial displacements in the two phases can be assumed as in\(^{33}\):

\[
w^f = A_f r \quad \text{and} \quad w^m = A_m r + \frac{B_m}{r}
\]

(14)

where \( A_f, A_m \) and \( B_m \) are unknown constants. Invoking the continuity conditions for radial displacement at the interface \( (r = a) \) it can be found that:

\[
w^m = \frac{a^2}{r} A_f + (r - \frac{a^2}{r}) A_m
\]

(15)
Invoking the continuity condition \( \sigma^f_r\big|_{r=R} = \sigma^m_r\big|_{r=R} \) and satisfying the boundary condition \( \sigma^m_r\big|_{r=R} = q_0 \), the following equations for solving \( A_f \) and \( A_m \) are derived:

\[
\begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
    A_f \\
    A_m
\end{bmatrix}
= \begin{bmatrix}
    C_{13}^m - C_{12}^f \\
    C_{13}^m
\end{bmatrix}
\begin{bmatrix}
    \sigma^f_x \\
    C_{11}^f
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    -1
\end{bmatrix} q_0
+ \begin{bmatrix}
    0 \\
    A_2 C_{13}^m
\end{bmatrix}
\frac{\partial \tau_i}{\partial x}
\]

(16)

where,

\[
A_{11} = C_{11}^f + C_{12}^f - \frac{2(C_{12}^f)^2}{C_{11}^f} + C_{33}^m - C_{23}^m, \quad A_{12} = -2C_{33}^m, \quad A_{21} = (C_{33}^m - C_{23}^m) \frac{a^2}{R^2} + \frac{2C_{13}^m C_{12}^f}{C_{11}^f},
\]

\[
A_{22} = C_{23}^m + C_{33}^m + (C_{33}^m - C_{23}^m) \frac{a^2}{R^2}, \quad A_2 = A_1\big|_{r=R}.
\]

From Eq. (16), the solutions of \( A_f \) and \( A_m \) can be expressed as:

\[
A_f = L_{11} \sigma^f_x + L_{12} q + L_{13} \frac{\partial \tau_i}{\partial x} \quad \text{and} \quad A_m = L_{21} \sigma^f_x + L_{22} q + L_{23} \frac{\partial \tau_i}{\partial x}
\]

(17)

The expressions of the coefficients \( L_{ij} \) are evident from Eq. (16) and are not shown here for the sake of brevity. The equilibrium of force along the axial \( x \) direction yields
\[
\pi R^2 \sigma_0 = \int_0^a \sigma \frac{r}{2} 2\pi r dr + \int_a^R \sigma_m^m 2\pi r dr
\]

(18)

Using Eqs. (6), (17) and the constitutive relations, Eq. (18) can be reduced to

\[
\frac{\partial \tau_i}{\partial x} = \frac{R^2 \sigma_0}{L_3} + \frac{L_i}{L_3} q_0 - \frac{L_2}{L_3} \sigma_x^f
\]

(19)

where,

\[
L_1 = B_1 L_{12} - B_2 L_{22}, \quad L_2 = a^2 + \frac{C_{11}^{m}}{C_{11}^{f}} (R^2 - a^2) - B_1 L_{11} + B_2 L_{21}, \quad L_3 = A_3 C_{11}^{m} - B_1 L_{13} + B_2 L_{23},
\]

\[
A_3 = \int_a^R [2A_1 r dr, \quad B_1 = \int_a^R \left[ \frac{2C_{12}^{f} C_{11}^{m}}{C_{11}^{f}} r - (C_{12}^{m} - C_{13}^{m}) \frac{a^2}{r} \right] dr,
\]

and \[B_2 = \int_a^R \left[ (C_{12}^{m} + C_{13}^{m}) r - (C_{12}^{m} - C_{13}^{m}) \frac{a^2}{r} \right] dr.
\]

Substitution of Eq. (7) into Eq. (19) yields the governing equation for the average axial stress in the carbon fiber coated with radially aligned CNTs as follows:

\[
\frac{\partial^2 \sigma_x^f}{\partial x^2} - \alpha_x^f \sigma_x^f = -\eta \sigma_0 - \mu q_0
\]
where,

\[ \alpha = \frac{2L_2}{aL_3}, \quad \eta = \frac{2R^2}{aL_3} \quad \text{and} \quad \mu = \frac{2L_1}{aL_3} \]

Following the above procedure, the governing equation for the average axial stress \( \overline{\sigma}^{pf}_x \) in the imaginary fiber made of the polymer material lying in the zones \(-L \leq x \leq -L_f\) and \(L_f \leq x \leq L\) can be written as

\[ \frac{\partial^2 \overline{\sigma}_x^{pf}}{\partial x^2} - \alpha \frac{\partial \overline{\sigma}_x^{pf}}{\partial x} = -\eta \sigma_0 - \mu q_0 \]

In the above equation, the expressions for \( \alpha \), \( \eta \) and \( \mu \) are similar to those of \( \alpha^2 \), \( \eta \) and \( \mu \), respectively. But these are to be derived by considering \( C_{ij}^f = C_{ij}^m = C_{ij}^p \). Solutions of Eqs. (21) and (22) are given by:

\[ \overline{\sigma}_x^{f} = c_1 e^{\alpha x} + c_2 e^{-\alpha x} + \frac{\eta}{\alpha^2} \sigma + \frac{\mu}{\alpha^2} q \]

\[ \overline{\sigma}_x^{pf} = c_3 e^{\alpha x} + c_4 e^{-\alpha x} + \frac{\eta}{\alpha_1^2} \sigma_0 + \frac{\mu}{\alpha_1^2} q_0 \]
in which \( c_1, c_2, c_3 \) and \( c_4 \) are the constants of integrations to be evaluated from the following end conditions:

\[
\bar{\sigma}^p_x = \sigma_0 \text{ at } x = \pm L \quad \text{and} \quad \bar{\sigma}^f_x = \bar{\sigma}^p_x \text{ at } x = \pm L_f
\]

(25)

Utilizing the end conditions given by (25) in Eqs. (23) and (24), the final solutions for \( \bar{\sigma}^p_x \) and \( \bar{\sigma}^f_x \) are obtained as follows:

\[
\bar{\sigma}^p_x = \frac{\cosh(\alpha x)}{\cosh(\alpha L)} \left\{ \frac{1}{\alpha^2} \left[ \frac{\eta}{\alpha_1^2} + \frac{\mu q_0}{\alpha_1^2 \sigma_0} \right] + \frac{\bar{\eta}}{\alpha^2} + \frac{\mu}{\alpha_1^2} \frac{q_0}{\sigma_0} \right\} \sigma_0
\]

(26)

\[
\bar{\sigma}^f_x = \frac{\cosh(\alpha L_f)}{\cosh(\alpha L)} \left\{ \frac{1}{\alpha^2} \left[ \frac{\eta}{\alpha_1^2} + \frac{\mu q_0}{\alpha_1^2 \sigma_0} \right] + \frac{\bar{\eta}}{\alpha^2} + \frac{\mu}{\alpha_1^2} \frac{q_0}{\sigma_0} \right\} \sigma_0
\]

(27)

In the case that the fiber and matrix are isotropic, and with \( q_0 = 0 \), the above model reduces to that presented by Gao and Li\(^9\) for a CNT reinforced polymer composite. Finally, substitution of Eq. (27) into Eq. (7) yields the expression for the interface shear stress as follows:

\[
\tau_1 = -\frac{a}{2} \left[ \frac{\alpha \sinh(\alpha x)}{\cosh(\alpha L_f)} \left\{ \frac{1}{\alpha^2} \left[ \frac{\eta}{\alpha_1^2} + \frac{\mu q_0}{\alpha_1^2 \sigma_0} \right] + \frac{\bar{\eta}}{\alpha^2} + \frac{\mu}{\alpha_1^2} \frac{q_0}{\sigma_0} \right\} \right] \sigma_0
\]

(28)
3. Results and Discussion

The elastic coefficients of arm chair type CNTs with respect to the coordinate system considered here are obtained from Shen and Li\textsuperscript{5} which are listed in Table 1. The polymer material and the carbon fiber are elastically isotropic. The isotropic elastic coefficients ($C^p_{ij}$) of the polymer material\textsuperscript{18} and the elastic constants ($C^f_{ij}$) of the high modulus M40 carbon fiber\textsuperscript{34} needed for computing the numerical results are as given by

\begin{align*}
C^p_{11} &= 5.3 \text{GPa}, \quad C^p_{12} = 3.1 \text{GPa}, \quad C^f_{11} = 373.89 \text{GPa} \quad \text{and} \quad C^f_{12} = 6.5 \text{GPa}.
\end{align*}

A discussion on the effective properties of the PNC matrix is now in order. Recently, Ray and Batra\textsuperscript{18} derived a micromechanics model to predict the effective properties of CNT and piezoelectric fiber reinforced hybrid composite. In the absence of piezoelectric fibers this micromechanics model is reduced to a model which predicts the effective elastic properties of the transversely isotropic PNC matrix with radially aligned CNTs considered here and is given by:

\begin{align*}
[C^m] &= [C_1][V_3]^{-1} + [C_2][V_4]^{-1}
\end{align*}

(29)

The various matrices appearing in (29) are presented in the Appendix. At a particular value of CNT volume fraction ($V_{\text{CNT}} = 1.0\%$), the effective values of the elastic constants $C^m_{ij}$ of the PNC matrix predicted from Eq. (29) are presented in Table 2 for different types of armchair CNTs. Also, for $V_{\text{CNT}} = 1.0\%$, the elastic constants $C^m_{ij}$ of the PNC
matrix with (10, 10) CNTs presented in Table 2 yield the values of the Young’s modulus \(E_r^m\) in the radial direction and the Poisson’s ratio \(\nu_{rr}^m\) as 14.34 GPa and 0.369, respectively. The values of the same are also predicted identically from simple rule of mixtures validating the micromechanics model given by Eq. (29). Thus Eq. (29) can be used to compute the effective elastic constants \(C_{ij}^m\) of the PNC matrix for evaluating the numerical results.

For presenting the results, following nondimensional parameters are adopted:

\[
\sigma^* = \frac{\sigma_f}{\sigma_0} \text{ and } \tau^* = \frac{10\tau_1}{\sigma_0}
\]

(31)

Unless otherwise mentioned, the values of the geometrical parameters of the RVE are taken as:

\[
a = 3.5 \mu m, \quad R = a + 10 \mu m, \quad L_f / a = 10 \text{ and } L / L_f = 1.2
\]

Arm chair type (10, 10) CNTs are used to compute the numerical results for the axial stress and the interface shear stress, unless specifically varied. In order to validate the model derived in the previous section, first the normalized average axial stress in the fiber without coated with CNTs and the interface shear stress are compared with those obtained by an existing model\(^9\) as shown in Figure 4. For this comparison, the fiber is an arm chair (10, 10) CNT as considered in Ref.\(^9\). It may be noted that the good
agreement between the two sets of results have been obtained verifying the present model. The marginal differences observed may be attributed to the fact that the model in Ref.\textsuperscript{9} did not consider the radial deformation, whereas in the present model radial deformations have been taken into account. Next, results are computed for carbon fibers coated with radially aligned CNTs.

The variations of the axial normal stress in the short carbon fiber and the transverse shear stress at the interface between the fiber and the PNC matrix along the length of the fiber are shown in Figure 5. It may be observed that the carbon fiber coated with radially aligned-CNTs shares less load than the fiber without coated with CNTs. This is attributed simply to the radial and axial stiffening of the polymer matrix by the CNTs. Note that the axial Young’s moduli of the PNC matrix with 1% (10, 10) CNT and the polymer are $E_x^m = 3.41 \text{GPa}$ and $E_p^p = 3.01 \text{GPa}$, respectively while the radial Young’s moduli are $E_r^m = 14.34 \text{GPa}$ and $E_p^r = 3.01 \text{GPa}$, respectively. The CNTs create a radially orthotropic PNC matrix and an increase in the CNT volume fraction increases both the axial and radial modulii. Importantly, the maximum interfacial shear stress is reduced in the case of CNTs reinforcing the matrix. A critical parameter in the design of polymer matrices for composites is the ratio of this maximum stress to the strength of the matrix. Load sharing improves this ratio with the presence of only 1% CNTs by 23%, and it is also expected that the strength of the interface should increase due to the CNTs as well, further improving the effect. The axial load transferred to the carbon fiber and the interfacial shear stress decreases with the increase in the radial stiffness of the PNC matrix as shown in Figure 6 and Figure 7, respectively. Also, compared in Figure 6 and Figure 7 is the case where the matrix remains isotropic but the value of its Young’s
modulus is increased to that of $E_r^m$ of the PNC with $V_{CNT} = 1.0\%$ (Poisson’s ratio is assumed as 0.33). It may be observed from these figures that isotropically stiffening the matrix causes (as expected) a significant increase in load carried by the matrix and a reduction in the interfacial maximum shear stress beyond what is seen at 1% radially-aligned PNC.

In Figure 8, it may be seen that if the value of $(R/a)$ decreases then the load transfer from the matrix to the fiber coated with radially aligned CNTs significantly decreases as expected due to the larger proportion of overall load carried by the enhanced-stiffness matrix relative to the fiber. Beyond a few percent volume fraction of CNTs in the matrix, overall load sharing is not significantly affected as evidenced both in Figure 8 and Figure 9. In Figure 9 the effect of CNT volume fraction on the critical length ($L_{crit}$) of the fiber is presented. Here, $L_{crit}$ is measured from the center of the fiber and determined based on the situation when $\sigma^* = 98\%$ of the maximum value of $\sigma^*$. It may be noted from this figure that as the CNT volume fraction increases, the critical length of the fiber decreases rapidly in the region $V_{CNT} < 1.0\%$ and then decreases monotonically. For a particular value of volume fraction of the carbon fiber, the critical length increases if the aspect ratio of the fiber increases, while for a particular value of aspect ratio of the fiber this critical length decreases with the increase in the volume fraction of the carbon fiber. In Figure 10 a critical value of $V_{CNT}$ is shown to exist beyond which the radial orthotropy of the PNC matrix does not appreciably alter load sharing capability of the fiber. After $\sim 2\%$ $V_{CNT}$, there is little change in the load sharing between the PNC matrix and the fiber.

Variations of maximum values of the axial stress in the carbon fiber coated with
radially aligned CNTs ($V_{\text{CNT}} = 1.0\%$) and the interface shear stress with the aspect ratio of the fiber are presented in Figure 11. The maximum value of the axial load shared by the fiber increases sharply with the increase in the value of the aspect ratio as long as $L_f/a < 12$. For $L_f/a > 20$, the axial load sharing capability of the fiber becomes independent of the variation of the aspect ratio of the fiber. In case of interface shear stress, its maximum value also increases rapidly with the increase in the value of the aspect ratio of the fiber till $L_f/a < 8$. The maximum value of $\tau^*$ becomes saturated for $L_f/a > 10$.

The effect of application of radial load on the load transferred to the fiber is presented in Figure 12. If the applied radial load is compressive, then the maximum values of the axial normal stress in the fiber and the interface shear stress are higher than those without the application of radial load ($q_0 = 0$), and vice versa. The variations of axial normal stress in the carbon fiber and the interface shear stress along its length are presented in Figure 13 and Figure 14, respectively, for different arm chair type CNTs in Table 1. It may be observed from these figures that for a particular value of $V_{\text{CNT}}$, as the diameter of CNT increases, both the axial normal load transferred to the fiber and the interface shear stress increase. This may be attributed to the fact that as the diameter of CNT increases, elastic coefficients of CNT decreases (see Table 1) which results in the decrease in the values of the effective elastic properties of the CNT-reinforced PNC matrix. Overall, the type of CNT has a small effect on the composite fiber load sharing relative to varying volume fraction of the CNTs in the PNC matrix.
4. Conclusions

In this paper, load sharing in a shortfiber composite where the matrix is reinforced with radially-aligned CNTs has been analyzed. The fiber reinforcement of the composite is a discontinuous carbon fiber coated with radially aligned CNTs. A shear lag model considering radial and axial deformations of the different phases of the RVE has been developed to analyze the axial load transferred to this carbon fiber. Since the radially aligned CNTs grown on the carbon fiber reinforce the polymer matrix, the effective elastic properties of the resulting CNT-reinforced PNC matrix are modified. Hence, if the fiber is coated with CNTs, the axial load transferred to the carbon fiber and the shear stress at the interface between the fiber and the PNC matrix decrease, and the CNT-reinforced matrix carries more of this load. If the volume fraction of CNTs increases, both the axial load transferred to the fiber and the interface shear stress decrease, including importantly the maximum shear stress at the fiber-matrix interface. The critical length of the carbon fiber varies little with CNT volume fraction beyond a few percent. For a particular value of CNT volume fraction, compressive radial load applied to the RVE increases the axial load transferred to the fiber and the interface shear stress. Future work should consider load-transfer in randomly-oriented shortfiber composites, and also load transfer around a broken fiber in continuous filament composites, in addition to load sharing in the presence of applied shear stress.
Appendix

The various matrices appearing in Eq. (29) are given by

\[
[C_1] = v_n \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
C_{13}^n & C_{23}^n & C_{33}^n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
[C_2] = \begin{bmatrix}
C_{11}^p & C_{12}^p & C_{12}^p & 0 & 0 & 0 \\
C_{12}^p & C_{11}^p & C_{12}^p & 0 & 0 & 0 \\
v_p C_{12}^p & v_p C_{12}^p & v_p C_{11}^p & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^p & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^p & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^p \\
\end{bmatrix},
\]

\[
v_n = \frac{R^2}{R^2 - a^2} V_{CNT}, \quad v_p = 1 - v_n, \quad [V_3] = [V_1] + [V_2] [C_4]^{-1} [C_3], \quad [V_4] = [V_2] + [V_1] [C_3]^{-1} [C_4],
\]

\[
[C_3] = \begin{bmatrix}
C_{11}^n & C_{12}^n & C_{13}^n & 0 & 0 & 0 \\
C_{12}^n & C_{22}^n & C_{23}^n & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^n & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55}^n & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^n \\
\end{bmatrix},
\]

\[
[C_4] = \begin{bmatrix}
C_{11}^p & C_{12}^p & C_{12}^p & 0 & 0 & 0 \\
C_{12}^p & C_{11}^p & C_{12}^p & 0 & 0 & 0 \\
v_p C_{12}^p & v_p C_{12}^p & v_p C_{11}^p & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^p & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^p & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^p \\
\end{bmatrix},
\]

\[
[V_1] = \begin{bmatrix}
v_n & 0 & 0 & 0 & 0 & 0 \\
0 & v_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_n & 0 & 0 \\
0 & 0 & 0 & 0 & v_n & 0 \\
0 & 0 & 0 & 0 & 0 & v_n \\
\end{bmatrix},
\]

\[
[V_2] = \begin{bmatrix}
v_p & 0 & 0 & 0 & 0 & 0 \\
0 & v_p & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & v_p & 0 & 0 \\
0 & 0 & 0 & 0 & v_p & 0 \\
0 & 0 & 0 & 0 & 0 & v_p \\
\end{bmatrix}
\]
In the above matrices, $C_{ij}^n$ and $C_{ij}^p$ are the elastic coefficients of the CNT and the polymer material, respectively. The volume fractions of the polymer and the CNT with respect to the volume of the PNC are represented by $\nu_p$ and $\nu_n$ while $V_{CNT}$ denotes the volume fraction of CNTs with respect to the volume of the RVE.
References


R. B. Mathur, S. Chatterjee, B. P. Singh, Growth of carbon nanotubes on carbon fibre substrates to produce hybrid/phenolic composites with improved mechanical properties, *Composite Science and Technology* 2008, 68, 1608-1615.


Table 1. Material properties of CNTs (Ref. 5). The 3-axis is aligned with the long axis of the CNT.

<table>
<thead>
<tr>
<th>CNT Type</th>
<th>$C_{11}^n$ (GPa)</th>
<th>$C_{22}^n$ (GPa)</th>
<th>$C_{33}^n$ (GPa)</th>
<th>$C_{12}^n$ (GPa)</th>
<th>$C_{13}^n$ (GPa)</th>
<th>$C_{44}^n$ (GPa)</th>
<th>$C_{55}^n$ (GPa)</th>
<th>$C_{66}^n$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,5)</td>
<td>668</td>
<td>668</td>
<td>2143</td>
<td>404</td>
<td>184</td>
<td>791</td>
<td>791</td>
<td>132</td>
</tr>
<tr>
<td>(10,10)</td>
<td>288</td>
<td>288</td>
<td>1088</td>
<td>254</td>
<td>87.7</td>
<td>442</td>
<td>442</td>
<td>17</td>
</tr>
<tr>
<td>(20,20)</td>
<td>138</td>
<td>138</td>
<td>545</td>
<td>134</td>
<td>43.5</td>
<td>227</td>
<td>227</td>
<td>2</td>
</tr>
<tr>
<td>(50,50)</td>
<td>55.1</td>
<td>55.1</td>
<td>218</td>
<td>54.9</td>
<td>17.5</td>
<td>92</td>
<td>92</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Table 2. Material properties of PNC with different arm chair type CNTs

\( (V_{\text{CNT}} = 1.0\%) \).

<table>
<thead>
<tr>
<th>PNC with CNT Type</th>
<th>( C_{11}^m ) (GPa)</th>
<th>( C_{22}^m ) (GPa)</th>
<th>( C_{33}^m ) (GPa)</th>
<th>( C_{12}^m ) (GPa)</th>
<th>( C_{13}^m ) (GPa)</th>
<th>( C_{44}^m ) (GPa)</th>
<th>( C_{55}^m ) (GPa)</th>
<th>( C_{66}^m ) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,5)</td>
<td>5.357</td>
<td>5.357</td>
<td>27.565</td>
<td>3.133</td>
<td>3.115</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
</tr>
<tr>
<td>(10,10)</td>
<td>5.356</td>
<td>5.356</td>
<td>16.628</td>
<td>3.134</td>
<td>3.114</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
</tr>
<tr>
<td>(20,20)</td>
<td>5.349</td>
<td>5.349</td>
<td>10.956</td>
<td>3.139</td>
<td>3.114</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
</tr>
<tr>
<td>(50,50)</td>
<td>5.235</td>
<td>5.235</td>
<td>7.546</td>
<td>3.248</td>
<td>3.111</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
</tr>
</tbody>
</table>
List of figures

Figure 1. Fibers coated with in situ grown radially aligned CNTs. Fuzzy alumina (left) and carbon fiber (right).
Figure 2. Model nano-engineered composite with representative volume element (RVE) indicated at the center.
Figure 3. RVE of the composite containing a fiber reinforcement coated with radially-aligned CNTs.
Figure 4. Model validation by comparison to Ref.⁹ for the case of an isotropic matrix phase (no CNTs) around the fiber. \( L_f / a = 10 \)
Figure 5. Variation of normalized axial normal stress in the carbon fiber and the interface shear stress along the fiber length ($q_0 = 0$).
Figure 6. Variation of axial normal stress in the carbon fiber along its length

\( (q_0 = 0) \)
Figure 7. Variation of transverse shear stress at the interface between the matrix and the carbon fiber along the length of the fiber for different CNT volume fraction and considering isotropic stiffening of the matrix ($q_0 = 0$).
Figure 8. Load transfer in the carbon fiber for different values of $R/a$.

$(V_{CNT} = 1.0\%, q_0 = 0)$
Figure 9. Effect of CNT volume fraction on the critical length of a fiber for full load transfer ($q_0 = 0$).
Figure 10. Variation of maximum axial stress and interfacial shear stress on the carbon fiber ($L_f/a = 10$) as a function of CNT volume fraction ($q_0 = 0$).
Figure 11. Variation of maximum values of the axial normal stress and the transverse shear stress at the interface between the matrix and the carbon fiber with the aspect ratio of the fiber ($q_0 = 0$, $V_{CNT} = 1.0\%$).
Figure 12. Variation of maximum values of the axial normal stress and the transverse shear stress at the interface between the matrix and the carbon fiber with the applied radial load ($L_f/a = 10, V_{CNT} = 1.0\%$).
Figure 13. Variation of the axial normal stress in the carbon fiber along its length when the fiber is coated with different arm chair type CNTs ($q_0 = 0$, $V_{\text{CNT}} = 1.0\%$).
Figure 14. Variation of the transverse shear stress at the interface between the matrix and the carbon fiber along its length when the fiber is coated with different arm chair type CNTs ($q_0 = 0$, $V_{\text{CNT}} = 1.0\%$).