Rényi entropy, mutual information, and fluctuation properties of Fermi liquids

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We compute the leading contribution to the ground-state Rényi entropy S_α for a region of linear size L in a Fermi liquid in d dimensions. The result contains a universal boundary law violating term simply related to the more familiar entanglement entropy. We also obtain a universal crossover function that smoothly interpolates between the zero-temperature result and the ordinary thermal Rényi entropy of a Fermi liquid. Formulas for the entanglement entropy of more complicated regions, including nonconvex and disconnected regions, are obtained from the conformal field theory formulation of Fermi surface dynamics. These results permit an evaluation of the quantum mutual information between different regions in a Fermi liquid. We also study the number fluctuations in a Fermi liquid. Taken together, these results give a reasonably complete characterization of the low-energy quantum information content of Fermi liquids.

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I. INTRODUCTION

Recently, motivated by studies of entanglement in quantum systems, there has been a profitable exchange of ideas between the quantum information and condensed matter communities. Much of this exchange has focused on the physics of entanglement entropy, defined as the von Neumann entropy of the reduced density matrix of a subsystem. Entanglement entropy is the basic quantity that provides a characterization of the entanglement properties of many-body quantum states.[1,2] Insights gained from the study of entanglement led to a new class of variational many-body states collectively known as tensor network states.[3–5] In addition, entanglement entropy can characterize interesting topological[6,7] and critical phases of matter. Conformal field theories in $1+1$ dimensions are partially characterized by the leading universal term in their entanglement entropy.[1,8,9] On the other hand, most phases in more than one spatial dimension have an entanglement entropy that scales as the boundary of the spatial region considered,[2,10,11] although this result is not formally proven. This boundary term is nonuniversal and typically forces us to study subleading terms in the entanglement entropy to find universal quantities characteristic of a particular phase.

There exists a class of systems in higher dimensions that violate the boundary law for the entanglement entropy. This class includes free fermions,[12–18] Fermi liquids,[19] Weyl fermions in a magnetic field,[19] and more exotic systems including critical Fermi surfaces and Bose metals.[18,20] The apparent unifying theme is the existence of many gapless one-dimensional degrees of freedom in all these systems. Fermi liquids with a codimension one Fermi surface have long been known to be equivalent to a set of nearly decoupled one-dimensional gapless modes.[21–23] The system of free Weyl fermions in a background magnetic field is also explicitly $1+1$ dimensional due to the presence of zero modes propagating along the magnetic field lines. Even critical Fermi surfaces behave thermodynamically like a collection of $1+1$ dimensional critical fields with critical exponent $z_f \neq 1$.[24] In all these cases we may think of one-dimensional theories patched together to form a higher dimensional theory, but this picture is most concrete for the case of Fermi liquids.

In this paper we use the one-dimensional framework to compute the Rényi entropies, defined below, for free fermions and interacting fermions in a Fermi liquid state. These quantities are interesting because they give in principle complete knowledge of the spectrum of the reduced density matrix, although we will only compute them in the low-energy limit. The qualifier "in principle" is included because only with the full set of exact Rényi entropies can we compute the spectrum, but in the field theory approach used here we have access only to the low-energy universal parts of some of the Rényi entropies. We also derive the form of the entanglement entropy of more general regions (nonconvex or even disjoint) using the same one-dimensional formulation. This permits a calculation of the mutual information between two distant regions in a Fermi liquid. All of these information theoretic quantities turn out to provide direct access to the geometry of the interacting Fermi surface. They are universal in the sense that they depend only on the geometry of the interacting Fermi surface and not on any other details of the Fermi liquid fixed point. Additionally, we compute the fermion number fluctuations in a region of size L in a Fermi liquid and give general finite temperature crossover forms for both the entropy and the number fluctuations. Together these results provide a very complete characterization of the quantum information theoretic content of Fermi liquids in terms of the geometry of low-energy excitations.

II. ONE-DIMENSIONAL FRAMEWORK

Here we review the one-dimensional framework used in Ref. 18 to compute the entanglement entropy. The low-energy physics of a system of free fermions is given in terms of the geometry of the free Fermi surface. For example, the heat capacity of such a free fermion system is simply proportional to the density of states times temperature, and the density of states may be written as an integral of $1/v_F$ over the Fermi surface. Each patch of the Fermi surface contributes to physical quantities like a chiral $1+1$ dimensional free fermion.

Including interactions is possible at the free Fermi fixed point when $d > 1$ because phase-space restrictions reduce

the effects of interactions to certain forward scattering terms labeled by an infinite set of Landau parameters. Each Landau parameter corresponds to a single exactly marginal deformation of the free Fermi fixed point which preserves all scaling dimensions. These forward scattering interactions preserve the fermion number on each patch of the Fermi surface, and the Fermi liquid has a very large symmetry group: $U(1)^{\infty}$.[25,26] This one-dimensional framework permits computation of the usual physical observables of Fermi liquids, observables such as heat capacity and compressibility. In addition, it provides simple access to many of the anomalous entanglement and fluctuation properties of Fermi liquids.

The entanglement entropy can be generalized to the Rényi entropy defined as

$$S_{\alpha} = \frac{1}{1-\alpha} \ln\left[\mathrm{tr}\left(\rho_L^{\alpha}\right)\right]. \tag{1}$$

The von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{tr}(\rho_L \ln \rho_L)$ is recovered from S_{α} in the limit $\alpha \to 1$. Fermionic systems without a codimension one Fermi surface and most bosonic systems have a Rényi entropy which scales with the boundary L^{d-1} of the region considered. This leading scaling behavior is sensitive to the cutoff of the low-energy effective theory and does not define a universal low-energy observable. Other subleading terms in the entanglement entropy may provide universal numbers characterizing different phases, but these terms are in general hard to calculate and interpret.

Fluctuations of conserved quantities also behave similarly to the entanglement entropy.[13,19,27,28] Consider a system of fermions with a conserved fermion number obtained by integrating the density $n(x)$ over the entire system. Given access to only a subregion of linear size L we may ask about the observable $N_L = \int_{x \in L^d} n(x)$. The average $\langle N_L \rangle = \mathrm{tr}(\rho_L N_L)$ is simply the average density times the volume of the subregion for a translation invariant system. The fluctuations $\Delta N_L^2 = \langle (N_L - \langle N_L \rangle)^2 \rangle = \mathrm{tr}[\rho_L (N_L - \langle N_L \rangle)^2]$ are generically nonzero and typically scale as the boundary of the subregion for gapped systems. Note that the presence of fluctuations in fermion number within a subregion is not in conflict with a fixed total fermion number for the entire system as we always study small subsystems of a larger system. However, there are exceptions to this scaling; for example, it receives a logarithmic correction, like the entanglement entropy, for critical one-dimensional systems. Symmetry-breaking states have fluctuations in the broken conserved quantity that scale as the volume of the subregion; these fluctuations are due to the zero mode of the order parameter. Like the entanglement entropy of Fermi liquids, the number fluctuations in a metal with codimension one Fermi surface are anomalously large. One finds that $\Delta N_A^2 \sim L^{d-1} \ln L$ with a multiplicative logarithmic correction.[13,19,29,30] We will later determine the precise form of this leading logarithmic term.

These results are not restricted to zero temperature; indeed, there are universal crossover functions that determine how the entanglement entropy and ground-state number fluctuations go over into the corresponding thermal quantities as the temperature is raised.[8,19,31] For example, consider a circular region in a two-dimensional spherically symmetric Fermi liquid. The reduced density matrix for this region displays crossover behavior at finite temperature from the zero-temperature anomalous entanglement regime to the finite-temperature thermal regime. This crossover is captured by a universal scaling function

$$S(L,T) = \frac{1}{2\pi}\frac{1}{12}\int_k \int_x dA_k dA_x |n_x \cdot n_k| \times \ln\left[\frac{\beta v_F}{\pi \epsilon} \sinh\left(\frac{\pi L_{\mathrm{eff}}}{\beta v_F}\right)\right], \tag{2}$$

where L_{eff} is the chordal distance across the circle starting at x and parallel to n_k. n_x and n_k are unit normal vectors to the real space boundary and Fermi surface respectively. For example, for a circular we have $L_{\mathrm{eff}} = 2L\cos\theta$ where L is the radius and $n_x \cdot n_k = \cos\theta$. This crossover function depends only on the geometry of the Fermi surface and the real-space region; the physical Fermi velocity v_F appears simply as a unit conversion. It is the only trace of interactions in this remarkably universal formula. We will discuss and generalize this formula later in the paper. A similar formula exists for the number fluctuations which we will derive below.

III. RÉNYI ENTROPY

We turn first to the calculation of the Rényi entropy of the Fermi liquid. As always, we only compute the leading term in the Rényi entropy due to low-energy modes. Like the entanglement entropy, the Rényi entropy is a cutoff-sensitive quantity. It typically satisfies a boundary law, but we will find that Fermi liquids have a universal boundary law violating piece tied to the geometry of the Fermi surface. In fact, it turns out that the Rényi entropy is trivially related to the entanglement entropy because of a similar simple relationship in one-dimensional conformal field theories. The Rényi entropy is of interest for several reasons. It is often easier to compute, both in field theory and numerically, for example, by quantum Monte Carlo.[32] It can also be used to characterize phases of matter, for example in topological phases, conformal field theories, and holographic theories.[8,33,34] From a quantum information perspective, the notion of single-copy entanglement,[35] given by the limit $\alpha \to \infty$ of S_{α}, determines the maximal deterministically distillable entanglement from the bipartition of a single copy of a quantum state. The entanglement entropy only gives the distillable entanglement in the limit of many copies of the quantum state. We find that the single-copy entanglement is, for large L, simply one half the entanglement entropy, similar to the one-dimensional conformal result.[36] However, we emphasize again that we are computing only the leading term in the entropy from low-energy modes. We will now use the one-dimensional framework to compute the Rényi entropy of a Fermi liquid.

Restricting first to the noninteracting case, each patch on the Fermi surface is equivalent to the chiral half of a one-dimensional free fermion. The Rényi entropy S_{α} of such a one-dimensional chiral fermion is known from conformal field theory.[37] In fact, it is simply proportional to the entanglement entropy with an α-dependent prefactor. The precise result is

$$S_{\alpha\,(1+1)} = \frac{1}{2}\left(1 + \frac{1}{\alpha}\right)\frac{c_L + c_R}{6} \ln\left(\frac{L}{\epsilon}\right), \tag{3}$$

where L is the length of the one-dimensional interval and ϵ is a short-distance cutoff.[37] Returning to the Fermi surface prob-

lem, each patch contributes such a term to the entanglement entropy with $c_L = 1$ and $c_R = 0$. The resulting Rényi entropy of the Fermi liquid is remarkably simple; the universal part controlled by the low-energy theory is simply

$$S_\alpha \sim \frac{1}{2}\left(1+\frac{1}{\alpha}\right) S_1, \tag{4}$$

where again S_1 is the usual entanglement entropy in two dimensions given by

$$S_1 = S = \frac{1}{2\pi}\frac{1}{12}\int_k\int_x dA_k dA_x |n_x \cdot n_k| \ln\left(\frac{L}{\epsilon}\right). \tag{5}$$

Note that as we said above, the Rényi entropy can be used to compute the full spectrum of the density matrix as in Ref. 37. Since our Rényi entropy is formally identical as a function of α to the one-dimensional result, the results of Ref. 37 also immediately apply here modulo the issue of subleading corrections.

This is not the end of the story. Using the finite-temperature crossover form for the von Neumann entropy of the reduced density matrix we obtain a finite-temperature crossover form for the Rényi entropy. The result for two dimensions is

$$S_\alpha = \frac{1}{2}\left(1+\frac{1}{\alpha}\right) S(L,T), \tag{6}$$

and in any spatial dimension with a codimension one Fermi surface

$$S_\alpha = \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\frac{1}{(2\pi)^{d-1}}\frac{1}{12}\int_k\int_x dA_k dA_x |n_x \cdot n_k| \times \ln\left[\frac{\beta v_F}{\pi\epsilon}\sinh\left(\frac{\pi L_{\text{eff}}}{\beta v_F}\right)\right]. \tag{7}$$

Note that since the Fermi surface system has a cutoff, this formula cannot truly be valid for all α. For example, the $\alpha \to 0$ limit of the Rényi entropy always produces the Schmidt rank of the reduced density matrix under study. However, the $\alpha \to 0$ limit of the conformal field theory result diverges corresponding to the statement that a theory which is truly conformal to arbitrarily high energy must have an infinite number of local degrees of freedom. At finite temperature we must keep $T/\alpha < T_F$ or $\alpha > T/T_F$ in order not to probe high-energy physics, and similarly, at zero temperature we must keep $\alpha > 1/(k_F L)$ to avoid the influence of nonuniversal physics away from the Fermi surface.

Now at last restoring the interactions, the complete finite-temperature crossover function remains correct even in the presence of interactions because of the nature of the Fermi liquid fixed point (more properly, fixed manifold). The counting of low-energy degrees of freedom remains the same. There are two possible modifications of this formula when including interactions. First, the interactions may change the geometry of the low-energy Fermi surface. Spherical symmetry can prevent such a modification, but in a solid-state metal the crystal lattice breaks the rotational symmetry. Nevertheless, the interacting Fermi surface in these less symmetric systems continues to control the Rényi entropy. Second, the Fermi velocity must be replaced by the renormalized physical Fermi velocity.

There are several checks of this formula. First, it correctly reproduces the one-dimensional result that must occur for noninteracting systems with nested Fermi surfaces. Second, it reproduces the finite-temperature Rényi entropy of the free Fermi gas; this follows by a direct computation. Third, it reproduces the thermal Rényi entropy of an interacting Fermi liquid. At finite temperature the Rényi entropy is

$$S_\alpha = \frac{1}{1-\alpha}\ln\{\text{tr}[\rho(T)^\alpha]\} = \frac{1}{1-\alpha}\ln\left(\frac{Z(\alpha\beta)}{Z(\beta)^\alpha}\right), \tag{8}$$

where $Z(\beta)$ is the partition function. The partition function of the Fermi liquid is $-\ln Z \sim T$ for small T compared to E_F, and this permits us to write $Z(\alpha\beta) = Z(\beta)^{1/\alpha}$. The final result is that the Rényi entropy for an interacting Fermi liquid is the same universal function of α times the thermal entropy as predicted by the formula above.

IV. ENTANGLEMENT ENTROPY OF COMPLEX REGIONS

Having described in detail the leading behavior of the Rényi entropy for a single convex region, we now turn to the problem of describing the entanglement entropy for more complicated subregion geometries. Perhaps the most important motivation for this study is the calculation of the quantum mutual information $I(A,B)$ between two regions A and B. The mutual information is

$$I(A,B) = S(A) + S(B) - S(AB), \tag{9}$$

and this definition requires knowledge of the entanglement entropy of disjoint regions to compute. We will see that the entropy of disjoint regions can be computed in a manner similar to that of convex regions via a more general formula that can handle arbitrary region geometry. The basic strategy remains the same: Express all quantities in terms of sums over one-dimensional modes on the Fermi surface. Because these one-dimensional modes are conformal and, in fact, essentially free fermions, much can be calculated. In particular, the main information we need is the entanglement entropy of a one-dimensional conformal field theory when the one-dimensional subregion consists of multiple regions.

This quantity, the entanglement entropy of multiple regions in a conformal field theory, is not a simple quantity to compute in general. Despite some early claims in the literature (see Ref. 8) it is not as universal as the single-region entanglement entropy which depends only on the central charge of the conformal field theory. The multi-interval entanglement entropy depends on the entire operator content of the theory as it requires evaluating higher point correlation functions of twist fields.[38–41] However, the relative simplicity of the Fermi liquid is a boon: The multiple interval result for a one-dimensional free fermion is known.[8]

Let us briefly recall the one-dimensional result. We consider a subsystem consisting of several disjoint intervals labeled $[a_i,b_i]$ $i = 1,\ldots,m$ with m the number of intervals. As we said before, the entanglement entropy of such a subregion is in general complicated, but for free fermions the answer is given by the original result in Ref. 8. This follows from standard formulas in Refs. 39 and 42 after identification of the twist fields as vertex operators of a free boson (see Refs. 39 and 43

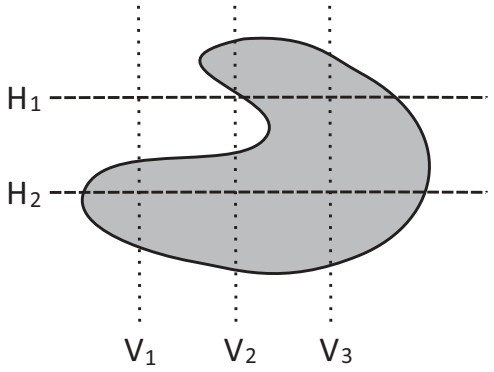

FIG. 1. Geometry of the entanglement entropy for a nonconvex real-space region. The shaded gray region surrounded by the thick black line represents a particular nonconvex real-space region. Dotted lines labeled V_i $(i = 1,2,3)$ represent particular one-dimensional cuts experienced by a patch of the Fermi surface with vertical Fermi velocity. Similarly, dashed lines labeled H_i $(i = 1,2)$ represent particular one-dimensional cuts experienced by a patch of the Fermi surface with horizontal Fermi velocity. Notice how the vertically moving fermion switches from an effectively single-interval geometry in V_1 to a two-interval geometry in V_2 and back to a single interval in V_3. On the other hand, H_1 and H_2 are both effectively single-interval geometries.

for additional discussion of this point). The result is

$$S_{1+1} = \frac{c_L + c_R}{6}\left(\sum_{ij}\ln|a_i - b_j| - \sum_{i<j}\ln|a_i - a_j| - \sum_{i<j}\ln|b_i - b_j|\right), \tag{10}$$

with the implicit presence of cutoffs in the logarithms understood. This formula reduces to the usual result in one dimension in the case of a single interval, but we emphasize again that it is not valid in a general conformal field theory. Note that when computing the mutual information, the number of logarithms being added and subtracted is equal giving a result independent of the cutoff.

How do we use this formula to compute the entanglement entropy of a Fermi liquid in an arbitrary geometry? The procedure is illustrated in Fig. 1 and goes as follows:

(i) We choose a point k on the Fermi surface; this point defines a direction in space via the local Fermi velocity $v(k)$.

(ii) We choose a point x on the real-space boundary and draw a line through this point with direction determined by the Fermi velocity from step one.

(iii) We determine the intersections of this line with the boundary of the real-space region; these intersection points are grouped into effectively one-dimensional intervals. This is possible since such a line will generically hit the real-space region an even number of times.

We see that the effective one-dimensional intervals $[a_i,b_i]$ are all given by one-dimensional cuts along the line $x(s) = x + n_k s$ through the region of interest.

The contribution to the entropy of such a configuration is given by the one-dimensional result above Eq. (10). Furthermore, to each logarithm in the one-dimensional result we

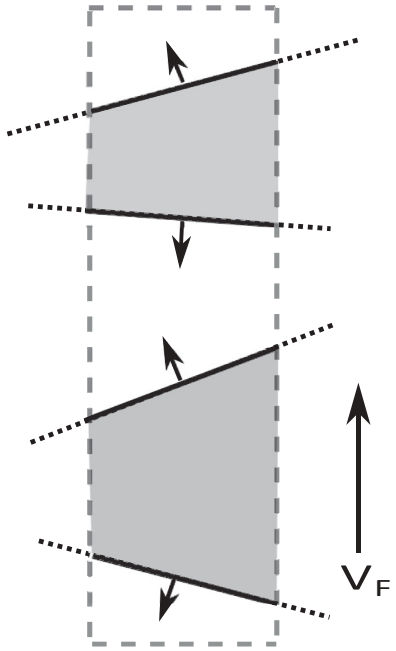

FIG. 2. Multiple-interval geometry and flux factors. A sequence of intervals similar to the situation shown in Fig. 1 V_2 with the local Fermi velocity vertical. The dashed gray line encloses the real-space boundary segments of interest; the segments themselves are the black lines which continue into dotted lines outside the dashed gray enclosing line. The dark gray shaded regions are interior regions of the real-space region, while arrows at the surfaces of these regions indicate local normals to the real-space boundary.

append geometrical "flux factors" $|n_x \cdot n_k|$ times a differential area element dA_x where n_k is the direction determined by the Fermi velocity and n_x is a real-space normal vector. As illustrated in Fig. 2, the flux factors and differential area elements satisfy $|n_x \cdot n_k|dA_x = |n_y \cdot n_k|dA_y$ so the same flux factor applies to all the logarithmic factors. Finally, the one-dimensional contribution must be divided by the number of intersection points to avoid overcounting and integrated over all such points on the real-space boundary and the Fermi surface. In terms of the one-dimensional result $S_{1+1}(x,k)$ and the number of intersection points $n_{\rm int}(x,k)$ (both functions of x and k and defined above), the final result is

$$S = \frac{1}{(2\pi)^{d-1}}\int dA_x dA_k |n_x \cdot n_k| \frac{S_{1+1}(x,k)}{n_{\rm int}(x,k)}. \tag{11}$$

For example, if we consider a single convex region then $n_{\rm int} = 2$ and $S_{1+1} = \ln[L_{\rm eff}(x,k)/\epsilon]/6$ and we obtain the usual Widom formula.

This formula is well defined but complex. There are a number of simple checks and calculations that can be performed. For two or more well-separated convex regions, the formula above reduces to a sum of terms corresponding to the entanglement entropy of each region taken separately. This follows already from the one-dimensional result when the intervals are well separated, and we will compute the leading correction below. For free fermions with a nested Fermi surface one immediately recovers the one-dimensional result as all the flux factors become trivial. As an example, consider fermions in 2d with dispersion relation $\epsilon(k_x,k_y) = -2t_y \cos k_y$ and

let the real-space region consist of two long slabs, slab 1: $0 < x < L$ and $0 < y < w$, and slab 2: $0 < x < L$ and $w + x < y < 2w + x$. x is the separation between the slabs and we assume $L \gg w,x$. All Fermi velocities point in the y direction, so the interval construction above is trivial and we find a one-dimensional contribution of

$$2\ln\left(\frac{w}{\epsilon}\right) + \ln\left(\frac{2w+x}{\epsilon}\right) + \ln\left(\frac{x}{\epsilon}\right) - 2\ln\left(\frac{w+x}{\epsilon}\right). \tag{12}$$

The full entropy is thus

$$S \sim k_F L\left[2\ln\left(\frac{w}{\epsilon}\right) - \ln\left(\frac{(w+x)^2}{x(2w+x)}\right)\right], \tag{13}$$

which we have written as a sum of two terms, the entropy of the two slabs separately minus the mutual information.

We can also perform a similar computation for a spherical Fermi surface. Consider the same two-slab geometry but with a Fermi velocity that that can take any direction. Let the angle of the Fermi velocity with the y axis be θ which we restrict to $[0,\pi/2]$ (all other angles follow from the results for this interval). Following our procedure above, the one-dimensional intervals are $[0, w/\cos\theta]$ and $[(w+x)/\cos\theta, (2w+x)/\cos\theta]$. We again assume L is very large and neglect the small part of the Fermi surface near $\theta = \pi/2$ where these intervals diverge (or more properly where they get bigger than L and cease to make sense). Note that the $\ln[1/(\theta - \pi/2)]$ singularity is weak and integrable. The one-dimensional entropy formula is quite similar; indeed the extra factors of $\cos\theta$ do not affect the leading logarithmic part of the entropy. Only the flux factor $|n_x \cdot n_k| = \cos\theta$ enters so that the integration $\int_{\theta=0}^{\theta=\pi/2} dA_k |n_x \cdot n_k|$ over the spherical Fermi surface ultimately gives k_F, the projected length.

At finite temperature, all the logarithms $\ln L$ should be replaced by crossover functions of the form $\ln[\sinh(\pi LT/v)]$ leading to a simple form at high temperature that is extensive in the total real-space region size. This one-dimensional crossover form is well known for single intervals in one-dimensional conformal field theory, but it is nontrivial for the multiple interval case. It comes from evaluating higher point correlations functions of twist fields on a Euclidean torus in the free fermion conformal field theory. As before, the simple form depends crucially on the fact that we are dealing with free fermions; this result does not hold in a general conformal field theory.

The absence of Landau parameters is reasonable given the fact that the entanglement entropy (at least for a single interval) does not depend on interaction strength even for a Luttinger liquid in one dimension. We cannot completely rule out the possibility that there is a prefactor f in the thermal-to-entanglement crossover function which is a function only of LT: $f = f(LT)$ with the limits $f(x \to \infty) \to 1$ and $f(x \to 0) \to$ constant, but such a prefactor seems unlikely. Given the absence of Landau parameters at high temperature and the large $U(1)^\infty$ symmetry, we believe the Fermi liquid entanglement entropy is universal as we have described.

Let us perform one final computation of the mutual information between two distant spheres in any dimension. Take a d-dimensional spherically symmetric Fermi liquid, and consider a real-space region consisting of two spheres A_1 and A_2 of radius R separated by a center-to-center distance ℓ. A very interesting quantity is the mutual information between the two spheres defined as $I(R,\ell) = S(A_1) + S(A_2) - S(A_1A_2)$. This quantity is quite powerful as it bounds normalized connected correlation functions between operators localized in the two spheres. We now evaluate this quantity for the two-sphere geometry in the limit $\ell \gg R$ thus bounding the long-distance decay of correlations in the Fermi liquid. The full integral is quite complex, but we can obtain the dependence on distance, which is what we really care about, with a simple estimate.

Because we subtract the entropy of each region separately, the only contribution to the mutual information comes from configurations that involve both spheres. As shown in Fig. 2, each pair consisting of a Fermi surface point and a point on the real-space boundary only contributes if the line drawn from the real-space point parallel to the local Fermi velocity intersects both spheres (see again the procedure for forming one-dimensional intervals above). The fraction of pairs of points satisfying this constraint vanishes in the limit $\ell/R \to \infty$ which is the statement that the mutual information vanishes for infinite sphere separation. Our task is to estimate how this fraction vanishes in said limit. Let one sphere sit at $z = 0$ and the other sit at $z = \ell$; both spheres sit at $x_i = 0$ for $i = 1 \ldots d-1$ with $x_d = z$. Since we have assumed spherical symmetry, the Fermi surface is sphere of radius k_F in momentum space with k_F fixed by density. Now the Fermi surface points with $\vec{v}_F = \pm v_F \hat{z}$ always gives lines connecting the two spheres, but as $\ell/R \to \infty$, only a small neighborhood around these points continues to generate lines connecting the two spheres as shown in Fig. 3. We can estimate the linear size δk of this neighborhood as $\delta k \sim k_F(\frac{\ell}{R})$. For $\ell/R \to \infty$, the difference of logarithms appearing in the mutual information nearly cancels with a residual $(R/\ell)^2$ falloff. Putting everything together, we find that roughly $(R\delta k)^{d-1}$ patches on the Fermi surface each contribute $(R/\ell)^2$ to the mutual information for a total mutual information going like

$$I(\ell,R) \sim (Rk_F)^{d-1}\left(\frac{R}{\ell}\right)^{d+1}. \tag{14}$$

It is known that the mutual information provides a bound on connected correlation functions.[10] The precise statement is that the square of the connected correlation function of any two operators, normalized by the operator norms, is bounded by the mutual information. Thus we learn that connected correlation functions of local operators in a Fermi liquid must decay at least as $x^{-(d+1)/2}$ as $x \to \infty$. Calculations are trivially possible for free fermions, and one can directly verify that all two-point functions of bosonic composite operators do indeed decay fast enough to satisfy the bound. The bound is saturated by the two-point function of fermions which decays as $x^{-(d+1)/2}$ times an oscillating function of $k_F x$. This is not completely trivial since the fermion creation operator is not strictly speaking a local operator, i.e., in the Jordan-Wigner representation, but this mild nonlocality does not spoil the utility of the mutual information. The exact coefficient is calculable by performing the geometrical integrals outlined above. On the other hand, nested Fermi surfaces give a mutual information that only

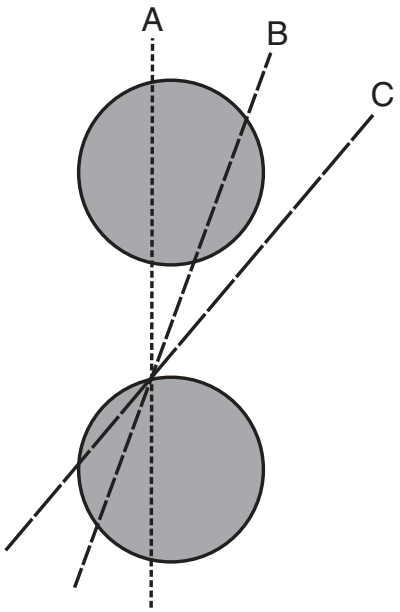

FIG. 3. Geometry of the mutual information for disconnected real-space regions. The three one-dimensional cuts labeled A, B, and C represent three different patches on the Fermi surface for different angles of the Fermi velocity. Cut A comes from a mode with vertical Fermi velocity; it always cuts both spheres and hence always contributes to the mutual information in this geometry. Cut B also connects both spheres, at least for some choices of real-space boundary point. However, cut C never intersects both spheres simultaneously and hence does not contribute to the mutual information. For the sphere geometry shown here, there is always a maximum angle from vertical such that cuts beyond that angle never intersect both spheres. As described in the text, this maximum angle approaches zero as the spheres are taken far apart relative to their size.

decays as $1/\ell^2$ indicating the possible presence of slowly decaying correlations. Indeed, this bound is again saturated by the fermion two-point function which decays as $1/x$ for a nested Fermi surface with x parallel to the nesting vector.

We have also numerically computed using the correlation matrix method the mutual information between two slabs in a two-dimensional free fermion system with a nearly spherical Fermi surface. The geometrical integrals can be done in the limit $L \gg w$ with the result

$$I \sim \frac{k_F L}{6\pi}\frac{w^2 L}{x^3}, \tag{15}$$

where again w is the slab width, L is the slab length, and x is the separation between the slabs. This expression agrees with the numerical result to within a few percent; additionally, the qualitative $1/x^3$ dependence also agrees with the numerical results.

V. NUMBER FLUCTUATIONS

Finally, we turn briefly to a description of the number fluctuations. Here at last Fermi liquid parameters will make an appearance. Recall that the basic result on the number fluctuations is that they scale as $L^{d-1}\ln L$ in a Fermi gas for a real-space region of linear size L. For free fermions the precise prefactor is known both from operator methods and from the conformal field theory approach advocated here. Unlike the result for entanglement entropy, the number fluctuations do depend on Landau parameters as interactions are turned on. To see this note that the same connected correlation function used to determine the number fluctuations is also related to the compressibility at finite temperature. This compressibility does depend on a Landau parameter in addition to the effective mass through the density of states. Thus we can already say that the number fluctuations $\Delta N_L^2(L,T)$ must depend on the Landau parameters in the limit $T \to \infty$ (always keeping $T \ll E_F$.) In fact, we can say much more because in Fermi liquid theory the interacting density-density correlation function is related to the free correlation function by a series of forward scattering diagrams that can be summed.

Consider a quadratically dispersing band $\epsilon(k) = k^2/2m$. If I_0 is the usual pair bubble (density-density correlator) then the summed series is

$$I = \frac{I_0}{1 - f_0 I_0}, \tag{16}$$

where f_0 is proportional to a Landau parameter. In 2d the pair bubble looks like

$$I_0(\omega,q) = \frac{N(0)}{2\pi}\int d\theta \frac{qv\cos\theta}{i\omega - qv\cos\theta} \tag{17}$$

with $N(0) = m/2\pi$ the density of states and $v = k_F/m$ the Fermi velocity. The static compressibility is proportional to the $\omega \to 0$ limit of I which is

$$I = \frac{-N(0)}{1 + F_0}, \tag{18}$$

where $F_0 = f_0 N(0)$ is the usual Landau parameter. This already shows that the number fluctuations are general sensitive to Landau parameters since these fluctuations are proportional to the compressibility at finite temperature.

At a given temperature we need the equal-time density-density correlator to compute the number fluctuations. At zero temperature we must integrate Eq. (16) over all ω to obtain this correlator and at finite temperature we must sum over all Matsubara frequencies. It is not hard to show at zero temperature that the final result is still proportional to $q = |\vec{q}|$ just as in the noninteracting case (which can be trivially done due to the simple pole), but there is now an F_0-dependent prefactor. Recalling that the number fluctuations in a d-dimensional Fermi gas at zero temperature take the asymptotic form

$$\Delta N_L^2 \sim \frac{1}{(2\pi)^{d-1}}\frac{\ln L}{4\pi^2}\int_x\int_k dA_x dA_k |n_x \cdot n_k|, \tag{19}$$

we see that Fermi liquids will also obey this result modified by an F_0-dependent prefactor which is computable from Eq. (16). More generally, we can formulate the entire finite-temperature crossover function for number fluctuations, but the result is not as simple or as illuminating as the result for entanglement entropy. Nevertheless, an important point is that both quantities continue to scale the same way with L and hence both reflect the physics of many gapless one-dimensional degrees of freedom on the Fermi surface.

VI. DISCUSSION

We have described the calculation of Rényi entropy, mutual information, and number fluctuations in Fermi liquids in any dimension for arbitrary codimension one Fermi surface and subregion geometry. These results give a very complete description of the low-energy structure of quantum information in a Fermi liquid. This basic Fermi liquid state which underlies so many materials is a truly highly entangled quantum phase of matter. Remarkably, the entire structure of quantum information in these systems is controlled by a beautiful interplay between the geometry of the Fermi surface and the structure of one-dimensional conformal field theory.

There is one subtlety which deserves a more careful discussion, namely the issue of singularities on the Fermi surface. The methods described above apply directly to smooth Fermi surfaces, but the formalism is strong enough to handle aspects of singular situations as well. To give a simple example, consider spinless fermions hopping on a square lattice at half filling. The Fermi surface consists of four straight lines running diagonally between the midpoints of the Brillouin zone boundary. There are singularities on the Fermi surface at $(\pm\pi,0)$ and $(0,\pm\pi)$, points where the Fermi velocity vanishes leading to well-known singularities in the density of states. As an example of the subtleties that may arise, consider the mutual information between two regions separated by a large distance compared to their size along the y axis. A little thought reveals that the leading contribution to the mutual information from the Fermi surface, in the form presented above, vanishes in this geometry. In fact, the mutual information is not totally zero; indeed, it cannot be since the fermion correlation function does not vanish in the limit of large y. However, the fermion correlation function does decay faster $(1/y^2)$ than the result for a nonsingular Fermi surface $(1/y^{3/2})$. In this sense, the vanishing of the leading "Fermi surface" part of the mutual information is physical, telling us that there are no extremely long ranged correlations such as would come from a nonsingular Fermi surface. When the Fermi surface geometry does lead to extremely long ranged correlations, such as along the nesting vector (π,π) in the half-filled example, then the mutual information formalism above perfectly captures the requisite decay. The simple lesson is that there are always subleading terms in all quantities which can become visible if the Fermi surface contribution happens to vanish. We must simply always remember that the formalism developed in this paper captures only the leading Fermi surface contribution to the physics.

The number fluctuations are in principle easy to observe; we simply need to prepare a clean Fermi liquid, perhaps in an optical lattice, and repeatedly count the number of fermions in a given subsystem to evaluate ΔN^2. If the subregion can be made large enough we have a quite precise prediction for ΔN^2. Another route to measuring the entropy may be to actually physically affect the separation of a given subregion from a larger system. We are in the process of studying the details of this process.

Given their experimental ubiquity, it is gratifying to have control of the quantum information content of Fermi liquids. In the larger framework of many-body entanglement, Fermi liquids have taught us how to produce one class of highly entangled quantum states. However, much remains to be understood about the role of entanglement in deeply quantum mechanical phases, especially gapless phases like those explored here.[24,44–47] Like Fermi liquids, these phases should violate the boundary law for entanglement entropy.[18,20] And the search continues for other highly entangled phases of quantum matter.

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