THE LAMB SHIFT

by

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ABSTRACT

The Lamb shift is studied in this paper. First, the interaction between light and matter is discussed and it is pointed out how this interaction should shift the energy levels of an atom. The electromagnetic field is then quantized and the mechanism of the interaction of the electron with the zero-point radiation field is described. A perturbation procedure is developed and applied to the calculation of the Lamb shift in the same approximation that Bethe used in his first calculation: to the order α^2, non-relativistic, and in dipole approximation. The idea of renormalization is stressed as it arises during the calculation. The work of Kramers in building a structure-independent theory of the interaction of an electron with radiation is then described. The paper ends with a discussion of the idea of renormalization which has enabled the Lamb shift to be described successfully in the present theory.

TABLE OF CONTENTS

I. INTRODUCTION

For many years, the spectrum of hydrogen agreed, within experimental accuracy, with the prediction of Dirac's relativistic electron theory. In this theory the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states of hydrogen are degenerate. In 1947 Lamb and Retherford used an ingenious microwave technique for the comparison of these lines, publishing their first results in August of the same year. They found that the $2S_{\frac{1}{2}}$ state is higher than the $2P_{\frac{1}{2}}$ state by about 1000 megacycles per second. This shift is about 9% of the difference between the $2P_{\frac{3}{2}}$ and $2P_{\frac{1}{2}}$ states. With improved experimental techniques, the shift in the energy levels has been found to high precision. It is 1057.8 megacycles per second. The same effect has since been observed in the spectra of deuterium and singly-ionized helium.

Schwinger, Weisskopf, and Oppenheimer first suggested that the shift might be due to the interaction of the electron with the zero-point fluctuations of the radiation field. The possibility of such an effect must have been considered before this time; but, without experimental evidence to suggest it, no one had bothered to publish a

calculation. If anyone had made a rough calculation, he would have found the shift to be infinite. The calculation was first made by Bethe and published two weeks after the results of Lamb and Retherford. Bethe's calculation was non-relativistic, correct to the first order of the fine structure constant, and done in dipole approximation. At first the shift diverged linearly as the energy of interacting radiation. By making a mass renormalization, he was able to reduce this to a logarithmic divergence. Bethe disregarded light of energy greater than the rest energy of the electron, thus obtaining a finite shift. He guessed that a relativistic calculation would give a convergent shift without the necessity of arbitrarily cutting off at high energy, and later work has proved him right. According to his calculation, the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states should be 1040 megacycles per second apart. The calculation has since been done relativistically, to higher order, and with other small corrections. The result of all this is a value 1057.2 megacycles per second, in close agreement with experiment.

II. HAMILTONIAN

The energy (non-relativistic) of an electron moving in the Coulomb field of the hydrogen atom is

$$H_0 = \frac{p^2}{2m} + V,$$

where m is the reduced mass and

$$V = -\frac{e^2}{r}.$$

The eigenfunctions of this Hamiltonian are the well-known hydrogen atom wave functions,

$$\psi_s = \psi_{nlm} = R_{nl}(r)\, Y_{lm}(\theta, \phi)\, e^{-\frac{iE_s t}{\hbar}},$$

$$E_s = -\frac{me^4}{2\hbar^2 n^2},$$

where the sth state is characterized by the integers n, l, and m and has an energy E_s.

The energy in the electromagnetic field is

$$H_1 = \frac{1}{8\pi}\int (E^2 + B^2)\, d\tau.$$

This is the result of classical electromagnetic theory and is not a quantum mechanical operator. When the electromagnetic field has been quantized, H_1 is an operator having the eigenvalues E_p and the eigenfunctions F_p which completely describe the state of the field.

If the hydrogen atom and the radiation field exist together and interact, it is improper to consider them separately, as is done above. If there were no interaction,

the Hamiltonian would be a sum,

$$H = H_0 + H_1 ,$$

and would have the eigenfunctions,

$$\Psi_{\alpha\rho} = \psi_\alpha F_\rho ,$$

and the eigenvalues,

$$E_{\alpha\rho} = E_\alpha + E_\rho .$$

Spectra would not be affected by the field, for the energy of every electron state is shifted by a constant amount, regardless of the binding of the electron. In this case, the energy in the field is not a physical observable and its magnitude cannot be determined experimentally.

However, this is not the case, for there is an interaction between the electron and the field. The energy of an electron moving non-relativistically in the electromagnetic field is given by

$$H = \frac{1}{2}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + V,$$

where $\vec{A}$ is the vector potential of the field. This is the correct Hamiltonian because it leads to the Lorentz force law. The energy of the total system of field and particle is then

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + V + \frac{1}{8\pi}\int (E^2 + B^2)\, d\tau$$

$$H = \frac{p^2}{2m} + V + \frac{1}{8\pi}\int (E^2 + B^2)\, d\tau - \frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2mc^2}$$

$$H = H_0 + H_1 + H' ,$$

where H' is the energy due to the interaction of electron and field.

$$H' = -\frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2mc^2} .$$

It is this energy of interaction which gives rise to the Lamb shift. Because the shift is known, from experiment, to be very small, the use of perturbation theory to determine the effect of H' is justified.

However, the Lamb shift is not explainable in terms of classical electromagnetic theory. It is not an effect like the Zeeman or Stark effects which occurs when a magnetic or electric field is turned on and disappears. When the field is removed. In their experiments, Lamb and his collaborators have gone to great lengths to determine the effects of stray electric and magnetic fields and to subtract such effects from their results. They have been able to conclude that the shift is not caused by stray electric or magnetic fields. From the classical point of view, a radiation-free region is physically possible, although difficult to obtain. In such a region, the above Hamiltonian would reduce to H_o and no Lamb shift would be observed. In order to understand the Lamb shift, one must use quantum electrodynamics in place of the classical theory. Just as in quantum mechanics the position and momentum of a particle cannot be measured simultaneously with exactness, it is a result of quantum electrodynamics that the measurement of one component of the field introduces an uncertainty into

another component. It is impossible to know that all components of $\vec{E}$ and $\vec{B}$ are simultaneously zero in a certain region. It is not proper to consider a region of space as completely free of radiation, for the fields continually fluctuate about their zero value. It is these fluctuations which interact with the electron and give rise to the Lamb shift, when the effects of all externally applied fields have been accounted for.

III. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

The success of quantum mechanics has suggested that the quantum mechanical commutator relation might well be applied to the electromagnetic field. The problem at hand can not be solved when half of it is expressed quantum mechanically and the other half classically. Therefore, in this section, the electromagnetic field will be quantized. Only an outline of the procedure will be given, since the details can be found in texts.

A convenient gauge for this problem is the Coulomb gauge:

$$\nabla \cdot \vec{A} = 0 \qquad \phi = 0 \quad .$$

The vector potential then satisfies the equation:

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

and the electric and magnetic fields are given by:

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad .$$

The solution for $\vec{A}$ can be expanded in a sum of plane waves:

$$\vec{A} = \sum_k \sum_\lambda \alpha_{k\lambda} \vec{A}_{k\lambda} + \alpha^*_{k\lambda} \vec{A}^*_{k\lambda}$$

$$\vec{A}_{k\lambda} = \vec{\epsilon}_{k\lambda} e^{i\vec{k}\cdot\vec{r}}$$

$$\alpha_{k\lambda} = \kappa_{k\lambda} e^{-i\omega t} \qquad \nabla \cdot \vec{A}_{k\lambda} = 0 .$$

$\vec{k}$ points in the direction of motion of the plane wave and has a magnitude

$$k = \frac{\omega}{c} .$$

$\vec{\epsilon}_{k1}$ and $\vec{\epsilon}_{k2}$ are two unit vectors which form a right-handed system with $\vec{k}$ and serve to determine the polarization of each plane wave in the sum. λ takes on only two values, but the k-sum is over all directions and magnitudes of $\vec{k}$.

The $\vec{A}_{k\lambda}$'s are to obey periodic boundary conditions on the opposite walls of a large box of side L (running waves are desired). This is satisfied by putting

$$k_i = \frac{2\pi n_i}{L} \qquad i = x, y, z$$
$$n_i = 0, \pm 1, \pm 2, \cdots$$

and, as a result, the $\vec{A}_{k\lambda}$'s are orthogonal:

$$\int \vec{A}_{k\lambda}^* \cdot \vec{A}_{k'\lambda'} \, d\tau = L^3 \delta_{k\lambda, k'\lambda'} .$$

The amplitudes are now redefined so that

$$d_{k\lambda} = a_{k\lambda} \sqrt{\frac{2\pi c^2 \hbar}{L^3 \omega}} .$$

Then

$$\vec{A} = \sum_k \sum_\lambda \sqrt{\frac{2\pi c^2 \hbar}{L^3 \omega}} \, \vec{\epsilon}_{k\lambda} \left[a_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^* e^{-i\vec{k}\cdot\vec{r}} \right] .$$

Since, in a plane wave, $E^2 = \mathcal{B}^2$, the energy in the field is

$$H_1 = \frac{1}{8\pi} \sum_k \sum_\lambda \sum_{k'} \sum_{\lambda'} \int \frac{4\pi c^2 \hbar}{L^3 \omega} \left[\nabla \times \vec{\epsilon}_{k\lambda} \left(a_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^* e^{-i\vec{k}\cdot\vec{r}} \right) \right] \left[\nabla \times \vec{\epsilon}_{k'\lambda'} \left(a_{k'\lambda'} e^{i\vec{k}'\cdot\vec{r}} + a_{k'\lambda'}^* e^{-i\vec{k}'\cdot\vec{r}} \right) \right] d\tau .$$

Because of the orthogonality of the $\vec{A}_{k\lambda}$'s this reduces to

$$H_1 = \sum_k \sum_\lambda \frac{\hbar\omega}{2} \times 2 a_{k\lambda} a_{k\lambda}^* = \sum_k \sum_\lambda \frac{\hbar\omega}{2} \left(a_{k\lambda} a_{k\lambda}^* + a_{k\lambda}^* a_{k\lambda} \right) .$$

The symmetrized form is used because, after quantization, $a_{k\lambda}\, a_{k\lambda}^*$ and $a_{k\lambda}^*\, a_{k\lambda}$ are not the same.

Now real combinations of the $a_{k\lambda}$'s are introduced:

$$Q_{k\lambda} = \sqrt{\frac{\hbar}{kc}}\,(a_{k\lambda} + a_{k\lambda}^*) \qquad P_{k\lambda} = \dot{Q}_{k\lambda} = -i\omega\sqrt{\frac{\hbar}{kc}}\,(a_{k\lambda} - a_{k\lambda}^*).$$

Then

$$H_1 = \sum_k \sum_\lambda H_{1k\lambda} = \sum_k \sum_\lambda \tfrac{1}{2}\left(P_{k\lambda}^2 + \omega^2 Q_{k\lambda}^2\right)$$

$$\frac{\partial H_{1k\lambda}}{\partial Q_{k\lambda}} = -\dot{P}_{k\lambda} \qquad \frac{\partial H_{1k\lambda}}{\partial P_{k\lambda}} = \dot{Q}_{k\lambda}\,.$$

Thus the electromagnetic field can be represented by an infinite number of pairs of canonical coordinates, $P_{k\lambda}$ and $Q_{k\lambda}$, each set having a Hamiltonian which is the same as that of a simple harmonic oscillator. The field is quantized by quantizing each of these "oscillators." Formally, the quantization is performed by setting

$$[P_{k\lambda}, Q_{k\lambda}] = \frac{\hbar}{i}\,.$$

The method is the same as that used in the quantization of the simple harmonic oscillator, as is the result:

$$H_{1k\lambda} F_{k\lambda} = (n_{k\lambda} + \tfrac{1}{2})\hbar\omega F_{k\lambda} \qquad n_{k\lambda} = 0, 1, 2, \cdots\cdots.$$

Because the total Hamiltonian of the field is a sum, the total wave function is a product; the following results are obvious:

$$F_p = \prod_{k\lambda} F_{k\lambda}$$

$$H_1 F_p = \sum_k \sum_\lambda (n_{k\lambda} + \tfrac{1}{2})\hbar\omega F_p = E_p F_p$$

$$H_{1k\lambda} F_p = (n_{k\lambda} + \tfrac{1}{2})\hbar\omega F_p\,.$$

$n_{k\lambda}$ is called the number of photons of a certain $\vec{k}$ and λ in the field. The numbers of photons in each state are contained in the wave function of the field F_p. The operation of $H_{ik\lambda}$ on F_p reveals the number of photons of that particular $\vec{k}$ and λ that is present in the field. By algebraic manipulations, it can be shown that:

$$a_{k\lambda} F(\cdots n_{k\lambda} \cdots) = \sqrt{n_{k\lambda}}\ F(\cdots n_{k\lambda}-1 \cdots).$$

$$a^*_{k\lambda} F(\cdots n_{k\lambda} \cdots) = \sqrt{n_{k\lambda}+1}\ F(\cdots n_{k\lambda}+1 \cdots).$$

These relations will be useful later. $a_{k\lambda}$ is called the annihilation operator because, when it operates on the field, the number of photons of type $\vec{k}\lambda$ is reduced by one, as that one is absorbed by matter for example. $a^*_{k\lambda}$ is called the creation operator because it adds a photon of type $\vec{k}\lambda$ to the field, in the process of emission.

It will be observed that there is a zero-point energy in the field.

$$H_1 F_0 = E_0 F_0 = \sum_k \sum_\lambda \left(\frac{\hbar\omega}{2}\right) F_0 .$$

The fact that this zero-point energy is infinite is not pleasing. It isn't a serious difficulty in itself, for this energy is not a physical observable. F_0 is the zero-point wave function. It contains the information that $n_{k\lambda} = 0$ for all states. Since

$$a_{k\lambda} F_0 = 0$$

$$a^*_{k\lambda} F_0 = F(\cdots\cdots 1_{k\lambda} \cdots\cdots),$$

it is not possible for a photon to be absorbed from the zero-point field; a photon may be emitted. It is this emission of a photon by the electron (and its immediate

reabsorption) that is regarded as the mechanism by which the electron interacts with the zero-point field.

It must be emphasized that this mechanism of photon emission and absorption is a very convenient way of handling radiation problems. It is the exact analogue of the classical methods which employ energy conservation in problem solving. Just as, in classical physics, these same problems can be solved in principle by using a force law, it is possible in quantum electrodynamics to work directly with the field fluctuations. It is a result of this theory that the components of the fields obey commutation rules, not all of which are zero. This very fact that all field components are not simultaneously measurable, gives rise to the fluctuations in the field strengths. Welton has done a non-relativistic calculation of the Lamb shift on this basis. Considering the field fluctuations, he determined the mean-square displacement of the electron from its unperturbed orbit and showed how this "Brownian motion" of the electron reduced the effective potential of the nucleus. As would be expected, he arrived at the same Lamb shift expression that is obtained by the more common procedure.

IV. PERTURBATION PROCEDURE

Now that the electromagnetic field has been quantized, recall

$$H = H_0 + H_1 + H' = \frac{p^2}{2m} + V + H_1 - \frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2mc^2} .$$

H' will be regarded as a small perturbation and the eigenfunction Ψ of H will be expanded in the eigenfunctions of the Hamiltonian $H_0 + H_1$.

$$\Psi = \sum_s \sum_p a_{sp}\, \psi_s F_p .$$

Restricting the problem to the zero-point field and a particular state m of the atom:

$$H\Psi = E\Psi = (E_m + E_0 + \Delta E)\Psi$$

$$(H_0 + H_1)\psi_m F_0 = (E_m + E_0)\psi_m F_0 .$$

Ψ and E are expanded in terms of increasingly higher order, with a constant λ, whose degree labels the order of a term, as follows:

$$\Psi = \Psi^{(0)} + \lambda \Psi^{(1)} + \lambda^2 \Psi^{(2)} + \cdots\cdots$$

$$E = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots\cdots$$

and the various orders of Ψ are defined by

$$\Psi^{(1)} = \sum_s \sum_p a_{sp}^{(1)}\, \psi_s F_p$$

$$\Psi^{(2)} = \sum_s \sum_p a_{sp}^{(2)}\, \psi_s F_p$$

etc.

In the Hamiltonian itself, the terms $H_0 + H_1$ are of the zeroth order; and the perturbation is, of course, of the first order in itself.

$$H = H_0 + H_1 + \lambda H'.$$

The expansions are then inserted into the equation

$$H\Psi = E\Psi$$

and terms of the same degree in λ (thus of the same order) are equated, giving a series of relations:

$$(H_0+H_1)\Psi^{(0)} = E^{(0)}\Psi^{(0)}$$

$$(H_0+H_1)\Psi^{(1)} + H'\Psi^{(0)} = E^{(0)}\Psi^{(1)} + E^{(1)}\Psi^{(0)}$$

$$(H_0+H_1)\Psi^{(2)} + H'\Psi^{(1)} = E^{(0)}\Psi^{(2)} + E^{(1)}\Psi^{(1)} + E^{(2)}\Psi^{(0)}.$$

The first term says, as expected, that to the zeroth order there is no perturbation:

$$\Psi^{(0)} = \psi_m F_0 \qquad E^{(0)} = E_m + E_0 .$$

When the expansion for $\Psi^{(1)}$ and the now known values of $\Psi^{(0)}$ and $E^{(0)}$ are substituted into the second relation and certain manipulations are made, it is found that

$$E^{(1)} = \langle m\,0|H'|m\,0\rangle .$$

Thus to first order (the parameter λ is put equal to 1)

$$E = E^{(0)} + E^{(1)} = E_m + E_0 + \langle m0|H'|m0\rangle$$

$$\Delta E = \langle m\,0|H'|m\,0\rangle .$$

The third relation gives, upon substitution and manipulation,

$$E^{(2)} = \sum_{\alpha}\sum_{\beta} \frac{\langle \alpha\beta|H'|m0\rangle\langle m0|H'|\alpha\beta\rangle}{(E_m + E_0 - E_\alpha - E_\beta)}$$

where $\alpha, \beta \neq m, 0$.

Thus, to second order

$$E = E^{(0)} + E^{(1)} + E^{(2)}$$

$$E = E_m + E_0 + \langle m0|H'|m0\rangle + \sum_\alpha \sum_\beta \frac{\langle \alpha\beta|H'|m0\rangle\langle m0|H'|\alpha\beta\rangle}{(E_m + E_0 - E_\alpha - E_\beta)}$$

$$\Delta E = \langle m0|H'|m0\rangle + \sum_\alpha \sum_\beta \frac{\langle \alpha\beta|H'|m0\rangle\langle m0|H'|\alpha\beta\rangle}{(E_m + E_0 - E_\alpha - E_\beta)} \; .$$

V. CALCULATION OF ENERGY SHIFT

The interaction of the electron with the zero-point field was first described as the action of the zero-point fluctuations of $\vec{E}$ and $\vec{B}$. For the purpose of mathematical simplicity, this has been replaced by the mechanism of emission and reabsorption of a photon by the electron. Thus, in the sum over states of the field, the only states which occur are the ground state F_0 and the states F_1 which contain one photon. It will be clear that this is required mathematically in this calculation to the order e^2. This will not be true for calculations to higher orders; and, there, a more elaborate interaction mechanism must be used.

Since the second term of the perturbation is already of the order e^2, its contribution to the Lamb shift will be calculated in first order only. This term is

$$\frac{e^2 A^2}{2mc^2} = \frac{e^2}{2mc^2} \sum_k \sum_\lambda \sum_{k'} \sum_{\lambda'} \vec{A}_{k\lambda} \cdot \vec{A}_{k'\lambda'} .$$

In the first order, the orthogonality of the F's requires that k' and λ' be equal to k and λ. The above product contains terms in $a_{k\lambda}\, a_{k\lambda}$ and $a^*_{k\lambda}\, a^*_{k\lambda}$. These represent double emission and absorption and make no contribution because they cannot connect F_0 to itself. The term

$a_{k\lambda}^{*} a_{k\lambda}$ also contributes nothing because the action of the annihilation operator $a_{k\lambda}$ on F_0 gives 0. The energy shift reduces to

$$\Delta E = \frac{e^2}{2mc^2} \left\langle m\,0 \left| \sum_k \sum_\lambda \frac{2\pi\hbar c}{k L^3} a_{k\lambda} a_{k\lambda}^{*} \right| m\,0 \right\rangle$$

$$\Delta E = \frac{\pi \hbar e^2}{mc L^3} \sum_k \sum_\lambda \frac{1}{k} .$$

When the sum on k is replaced by an integral (as is done in a similar case later) this shift is seen to be infinite, diverging quadratically with k . A divergent shift like this is not very pleasing and points to the necessity for an improved theory. However, it does not stand in the way of the calculation. This shift, it will be noticed, is the same for all electrons, regardless of how they are bound and of how they are moving. Since it is always present in every situation and at every time, it is not a physical observable. For these reasons it need not be thought of in terms of the interaction but may be thought of as a self-energy which every electron possesses. This process by which an infinite quantity is "subtracted" from a physically significant situation and ascribed to the (unknown) structure of the electron is called renormalization. It plays a very important part not only in the understanding of the Lamb shift but of the whole of quantum electrodynamics.

The first term of the perturbation is of the order e , and so its effect should be calculated to second order.

It is

$$-\frac{e}{mc}\vec{A}\cdot\vec{p} = -\frac{e}{mc}\sum_{k}\sum_{\lambda}\sqrt{\frac{2\pi\hbar c}{kL^3}}\left[a_{k\lambda}e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^{*}e^{-i\vec{k}\cdot\vec{r}}\right]\vec{\epsilon}_{k\lambda}\cdot\vec{p}\,.$$

It is clear that this gives no contribution to first order because

$$\Delta E \propto \langle 0 | a_{k\lambda}e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^{*}e^{-i\vec{k}\cdot\vec{r}} | 0\rangle$$

is zero by the orthogonality of the F's. The energy shift to second order is

$$\Delta E = \sum_{a}\sum_{p}\frac{\langle m0|-\frac{e}{mc}\vec{A}\cdot\vec{p}|ap\rangle\langle ap|-\frac{e}{mc}\vec{A}\cdot\vec{p}|m0\rangle}{E_m + E_0 - E_a - E_p}\,.$$

Once again, by orthogonality, the only state F_p that is possible is F_1 which is characterized by one photon of type $\vec{k}\lambda$. As before, the $\vec{A}$'s should be expanded over two sets of summation indices, $k\lambda$ and $k'\lambda'$; but

$\Delta E = 0$ unless

$$k' = k \qquad \lambda' = \lambda\,.$$

The energy difference in the denominator is then

$$E_p - E_0 = E_1 - E_0 = \sum_{k}\sum_{\lambda}\frac{\hbar\omega}{2} + \hbar kc - \sum_{k}\sum_{\lambda}\frac{\hbar\omega}{2} = \hbar kc\,.$$

Then the energy shift is

$$\Delta E = \frac{e^2}{m^2c^2}\frac{2\pi\hbar c}{L^3}\sum_{k}\sum_{\lambda}\frac{1}{k}\sum_{a}\frac{\langle m0|a_{k\lambda}e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{k\lambda}\cdot\vec{p}|a1\rangle\langle a1|a_{k\lambda}^{*}e^{-i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{k\lambda}\cdot\vec{p}|m0\rangle}{E_m - E_a - \hbar kc}.$$

First a's are allowed to operate and

$$\langle 0|a_{k\lambda}|1\rangle\langle 1|a_{k\lambda}^{*}|0\rangle = \langle 0|\pi|0\rangle\langle 1|\pi|1\rangle = 1\,.$$

Then the dipole approximation is made:

$$e^{i\vec{k}\cdot\vec{r}} = e^{-i\vec{k}\cdot\vec{r}} = 1$$

and the shift reduces to

$$\Delta E = \frac{e^2}{m^2c^2}\frac{2\pi\hbar c}{L^3}\sum_{k}\sum_{\lambda}\frac{1}{k}\sum_{a}\frac{|\langle m|\vec{\epsilon}_{k\lambda}\cdot\vec{p}|a\rangle|^2}{E_m - E_a - \hbar kc}\,.$$

$\vec{p}$ is replaced by $m\vec{v}$ and the sum over $\vec{k}$ is replaced by an integral:

$$\sum_{k} \longrightarrow \int \rho(k)\, k^2 dk\, d\Omega_k = \int \frac{L^3}{8\pi^3} k^2 dk\, d\Omega_k$$

$$\Delta E = \frac{e^2 \hbar}{4\pi^2 c} \sum_{\lambda} \int k\, dk\, d\Omega_k \sum_{a} \frac{|\langle m|\vec{\epsilon}_{k\lambda}\cdot\vec{v}|a\rangle|^2}{E_m - E_a - \hbar k c} .$$

The integration is carried out over all directions of $\vec{k}$, using $\vec{v}$ as a polar axis and adjusting the $\vec{\epsilon}$'s in each case so that one of them lies in the plane of $\vec{k}$ and $\vec{v}$ and the other is perpendicular to it. The result is

$$\Delta E = \frac{2e^2 \hbar}{3\pi c} \int k\, dk \sum_{a} \frac{|\langle m|v|a\rangle|^2}{E_m - E_a - \hbar k c} .$$

For large k, this expression diverges linearly. However, Bethe pointed out that this is not the true expression for the Lamb shift. The energy of the electron is shifted by this amount for two reasons: one, that it is bound to the hydrogen atom and two, that it has a certain average kinetic energy. A free electron with the same average kinetic energy will also undergo an energy shift because of its interaction with the zero-point field. This shift should properly be regarded as an electromagnetic mass effect, because it is an energy which the particle possesses because of its motion, regardless of how it may be bound. Thus it is not physically observable and does not contribute to the Lamb shift. The Lamb shift as it is observed in spectra is the difference between the shift for the bound electron and the shift for a free electron which has the same average kinetic energy. For a free electron, the shift is given by

the same formula; and, since the free electron wave functions are eigenfunctions of the momentum, only the diagonal element of v remains:

$$\langle m|v|a\rangle = 0 \quad , \quad a \neq m \qquad\qquad E_m = E_a \, , \, a=m$$

and, as a result,

$$\Delta E = \frac{2e^2\hbar}{3\pi c} \frac{1}{\hbar c} \int dk \sum_a |\langle m|v|a\rangle|^2 .$$

This is to be subtracted from the similar expression for the bound electron to give the correct expression for the Lamb shift, ΔE_L.

$$\Delta E_L = \frac{2e^2\hbar}{3\pi c} \sum_a |\langle m|v|a\rangle|^2 \int dk \left\{ \frac{k}{E_m - E_a - \hbar kc} - \frac{1}{\hbar c} \right\}$$

$$\Delta E_L = \frac{2e^2}{3\pi c^2} \int dk \sum_a |\langle m|v|a\rangle|^2 \frac{E_a - E_m}{E_m - E_a - \hbar kc} .$$

This expression diverges only logarithmically with increasing k. When this calculation is done relativistically, the shift converges. From relativistic considerations, Bethe decided to cut off the energy of the interacting photons at

$$\hbar K c = mc^2 .$$

This cut-off can be appreciated if the electron is regarded as having an extent of the order of its Compton wavelength, $\frac{\hbar}{mc}$. The cut-off then means that all light of wavelength smaller than the size of the particle should be disregarded. In this picture, such short wavelengths do not act upon the electron as a whole but affect only its internal structure, of which nothing is known. Thus the cut-off is justified.

If the integration over k is now performed,

$$\Delta E_L = -\frac{2e^2}{3\pi\hbar c^3} \sum_a |\langle m|v|a\rangle|^2 (E_a - E_m) \ln\left(\frac{E_a - E_m}{E_a - E_m + mc^2}\right)$$

and since $mc^2 >> E_a - E_m$

$$\Delta E_L = \frac{2e^2}{3\pi\hbar c^3} \sum_a |\langle m|v|a\rangle|^2 (E_a - E_m) \ln\left(\frac{mc^2}{E_a - E_m}\right)$$

which is Bethe's result.

This is, of course, the shift for only one state. If it were the same for all states, there would be no noticeable effect. However, the part which the state m plays in the formula makes it clear that different states are shifted in different ways. Calculations have shown this to be the case. S states are affected far more than others. It is with S states in mind (particularly the $2S_{\frac{1}{2}}$ state of hydrogen) that the preceding work has been done. To make the theory applicable to other states, the perturbation theory must be developed more carefully, keeping in mind the degeneracy of the unperturbed state.

VI. STRUCTURE-INDEPENDENT THEORY

Many of the problems which have arisen in the calculation of the Lamb shift and other radiative corrections were studied by Kramers and discussed by him at the Solvay Conference in 1948. Kramers was one of the first to suspect that the divergences which plagued quantum electrodynamics were due, in good part, to the point model of the electron. This model has, from the first, introduced an infinite self-energy. The self-energy is that energy in the field of a single electron:

$$U = \frac{1}{8\pi}\int_a^\infty E^2\, d\tau = \frac{e^2}{2}\int_a^\infty \frac{dr}{r^2}$$

and is clearly infinite if the radius is put equal to zero. Since quantum electrodynamics has been formulated with the correspondence principle always in mind, difficulties in the classical theory have been carried over into the quantized description. Kramers went to the root of the problem and attempted, in non-relativistic approximation, to make a better classical description of the interaction of matter and radiation.

Kramers was guided in his work by Lorentz' classical electron theory. Lorentz assumed the electron to be a rigid sphere of finite radius but was able to describe many

physical effects, using only two constants, the experimental mass and charge of the electron. The radius of the electron entered only into the electromagnetic mass, a constant in a non-relativistic approximation. It is the total mass, the sum of the inertial and electromagnetic masses, which is a physical observable. By introducing the total mass, Lorentz constructed a theory which depended only upon physically measurable constants and was independent of a knowledge of the structure of the electron.

Kramers' objective was to separate that part of classical electromagnetic theory which is (approximately) independent of the structure of the electron from that part which depends upon the unknown details of electronic structure. Although this is not, technically, a renormalization, it is effectively the same thing - and in a much more appropriate place. The need for renormalization in an advanced theory is often a result of a fault in the basic theory. If a structure-independent theory is used as a classical basis for quantum electrodynamics, those divergences which arise from a point model or some other structural assumption will not appear; and the corresponding renormalizations will not be necessary.

In his address, Kramers considered the motion of a single electron in the electromagnetic field. The electron was considered to be a rigid sphere of finite extent; thus

the treatment was, of necessity, non-relativistic. He divided the vector potential $\vec{A}$ into two parts: $\vec{A}_0$, the vector potential of the "proper" field, and $\vec{A}_1$, the potential of the external field. $\vec{A}_0$ is approximately the potential of a uniformly moving electron in radiation-free space. It is structure-dependent in the near vicinity of the electron. $\vec{A}_1$ may be regarded as the external field which interacts with the electron in a physical process. Although $\vec{A}_1$ depends upon $\vec{A}_0$:

$$\Box \vec{A}_1 = \ddot{\vec{A}}_0 ,$$

it was shown to be approximately structure-independent. The equation of motion of the electron was then brought into the form ($\vec{R}$ is the coordinate of the center of mass of the electron):

$$m\ddot{\vec{R}} = -e\vec{A}_1 + e\dot{\vec{R}} \times (\nabla \times \vec{A}_1) - \frac{\partial V}{\partial R}$$

where $\vec{A}_1$ and V are evaluated at the center of mass. If the field $\vec{A}_1$ is properly prescribed, this equation represents the motion of an electron in an external field in a manner which is approximately independent of the electron's structure. Kramers constructed the Hamiltonian which corresponds to this equation of motion. It is somewhat more complicated than the usual non-relativistic Hamiltonian which is used in preceding sections. Kramers showed that, when the Lamb shift is calculated in the same elementary manner used in the preceding section, but with his Hamiltonian, Bethe's final expression is obtained

without the necessity of renormalization at this point. This is a great triumph for the structure-independent theory.

In a very interesting digression, Kramers gave a semi-classical derivation of a Lamb shift for a harmonically bound electron. The external field $\vec{A}$, of a given frequency, was expressed as a spherical standing wave which was phase-shifted by an angle η which depended upon the frequency of the light and the binding of the electron. By enclosing the entire system in a large reflecting sphere, the k's were limited to a discrete, but infinite set. The set of k's is clearly dependent upon the phase shift η and therefore upon the binding of the electron. Kramers departed from a classical description only to consider the zero-point energy in the field. Although it is infinite for a harmonically bound electron or a free electron, Kramers showed that the difference in the zero-point energies in these two cases is finite and bears a close resemblance to Bethe's formula. The harmonic oscillator was a simple example. It appears that it is not mathematically feasible to make a semi-classical description of the hydrogen Lamb shift in the same way.

Van Kampen has continued the work of Kramers. He has derived Kramers' structure-independent Hamiltonian by a series of canonical transformations and has applied it to various radiation problems.

VII. DISCUSSION

Although Kramers' work has led to a better understanding of some of the problems of quantum electrodynamics, his method does not seem capable of providing a complete solution. For one thing, his theory is not quantum mechanical; yet all radiative calculations are. For this reason, all of the work done since 1948 has been directed toward perfecting the process of renormalization rather than toward making a structure-independent relativistic classical theory.

The difficulties which arise when the interaction of radiation and matter is considered have been traced to the use of a point electron. The divergent quantities have been reduced to two: an infinite self-mass and an infinite self-charge of the electron. The infinite self-mass is just that self-energy, calculated in a previous section, which is caused by the electron's interaction with the zero-point field. The infinite self-charge is a more sophisticated matter, which arises from Dirac's "hole" theory and has no non-relativistic analogue. However, its divergent nature is also due to the point model of the electron. In the process of renormalization, this mass and charge are considered as part of the experimental mass and

charge. Once the necessary renormalizations have been made, finite calculations of physically observable effects are possible. Resolution of the problem of the self-mass and self-charge will require some knowledge about the electron's structure. If it is really possible to separate the problem of the behavior of the electron from that of its structure, the theory is in good shape. It would seem that such a separation would not be possible in an exact treatment of the problem. However, as in much of physics, the interaction problem has proved far too difficult to be solved in any but an approximate way.

The situation is not quite as good as this, for quantum electrodynamics contains other divergent quantities which cannot be traced to a lack of understanding of electronic structure. A simple example is the infinite energy of the zero-point field, which is obviously independent of the presence of matter. Any idea that this might be due to the large size of the universe is seen to be false when it is recalled that quantization in a box or sphere gives rise to an infinite zero-point energy, regardless of the volume enclosed. One may say that there is no difficulty because the zero-point energy is not physically observable. However, a theory is not closed simply because it can predict all observable effects accurately. A theory cannot be wholly satisfactory if it contains implausible, though unobservable, elements.

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