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THE DYNAMIC RANGE OF

A PARAMETRIC AMPLIFIER

by

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B.S., Seoul University, Korea
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Submitted to the Department of Electrical Engineering on May 21, 1962 in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The application of the semiconductor capacitor diode to parametric amplifiers has received considerable interest in the past few years. This analysis considers the diode as a series combination of a constant loss resistance and a current controlled nonlinear elastance, there being good physical evidence for this choice.[7] As a direct consequence of the series circuit, the choice of restricting the currents to certain frequencies is made. A Fourier-series approach permits the large-signal behavior of the amplifier to be analyzed. The impedance representation of the varactor is used throughout. The relations among elastances are obtained. The gain is given. The dynamic range and phase shift are also obtained.

Thesis Supervisor: Robert P. Rafuse
Title: Assistant Professor of Electrical Engineering

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LIST OF SYMBOLS

Δ - elastance coefficient

γ - diode exponent

V_B - diode reverse breakdown voltage

ϕ - "contact" potential

$*$ - complex conjugate

S_k - Fourier component of elastance

M_k - normalized Fourier component of elastance

ω_c - cutoff frequency of the diode

S_{max} - maximum elastance of the diode

R_s - diode series resistance

ω_p - pump frequency

ω_i - idler frequency

ω_s - signal frequency

p - normalized pump frequency

i - normalized idler frequency

s - normalized signal frequency

δm_o - variation of normalized direct frequency component of elastance

α - $\delta m_o / i$

R_o - source resistance

R_i - idler resistance

P - power

T_d - diode temperature (of R_s)

T_n - 290^o K

T_a - antenna temperature

T_i - idler temperature (of R_i)

E_k - Fourier component of voltage

I_k - Fourier component of current

CHAPTER I

INTRODUCTION

1.1 History

Amplification can be achieved by vacuum tubes, transistors, semiconductor diodes, and many other devices. Each method of amplification has both advantages and disadvantages, with respect to dynamic range, frequency range, efficiency, device complexity, etc. The semiconductor diode parametric amplifier has been found to have large dynamic range, large frequency range, and low noise.[1, 2, 3, 4]

In 1939, parametric devices were treated by an electric-circuit concept in Hartley's[5] paper on nonlinear reactances. Until the discovery of the p-n junction diode, magnetic amplifiers were mainly the subjects of discussions concerning nonlinear reactance. During world war II, North[6] applied the concept of variable reactance to explain some of the characteristics of germanium diodes.

Since 1956,[1] the application of the semiconductor diode to low-noise microwave amplifiers has been investigated by many researchers. Many of their theoretical papers dealt with the lossless case. However, they should not ignore the losses completely in the analysis of the semiconductor capacitor diode.

A fundamental approach to the problem of parametric amplifier and many other applications of the semiconductor capacitor diode has been evolved by Rafuse.[7] His theory is based upon a series model for the varactor (variable reactor) - a constant resistance in series with a current controlled elastance, there being good physical evidence for this choice. As a direct consequence of the series circuit, the choice is made to restrict the currents to certain frequencies. He used a Fourier-series representation of the varactor elastance variation for the analysis of the amplifier. Penfield[2, 3] has derived best gain and the best excess noise figure for the small signal application of the varactor amplifier. From the results the optimum source resistance and the best pump frequency can be obtained. Rafuse[7] showed that the multiple-idler parametric amplifier does not give sufficient improvement over the single-idler one to warrant the extra circuit complexity required.

1.2 The varactor model and its implications

A model for the varactor is comprised of a constant series resistance and a capacitance varying with applied voltage as Fig. 1.1. This model follows from the physics of the semiconductor capacitor diode neglecting series lead inductance, case capacitance, and shunt conductance. Uhlir was probably the first to recognize and use such a model. And the measured behavior fits the model for the large-area junction diodes over a substantial range of retarding region and frequency.[7]

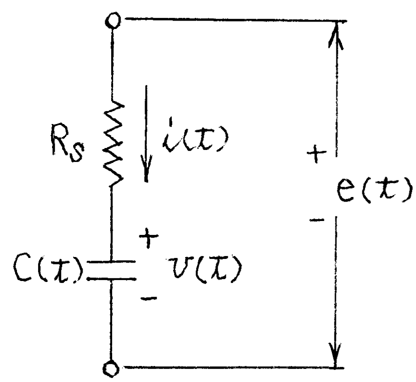


Fig. 1.1

Model of the varactor

With the varactor model, incremental capacitance c(t) is expressable as

$$c(v) = \frac{dq}{dv} = \frac{1}{\Delta(v + \phi)^{\gamma}} \qquad (1.1)$$

Δ = a constant dependent on diode geometry and doping

v = the applied <u>back</u> voltage

ϕ = the "contact" potential of the junction material at the depletion layer edges (a slowly varying function of v)

γ = an exponent dependent on doping geometry (1/3 for linear grading and 1/2 for abrupt)

There is a minimum value of capacitance set by the breakdown voltage of the diode, V_B. Therefore,

$$c_{min} = \frac{1}{\Delta(V_B + \phi)^\gamma} \tag{1.2}$$

With the elastance expression, from Eqs. 1.1 and 1.2

$$s(v) = S_{max} \left(\frac{v + \phi}{V_B + \phi}\right)^\gamma \tag{1.3}$$

Uhlir chose to specify a "cutoff" frequency

$$\omega_c = \frac{S_{max}}{R_s} \tag{1.4}$$

which will be independent of any operation for which the diode is to be utilized.

1.3 Mathematical formulation

From Eq. 1.1

$$\int dq = \int \frac{dv}{\Delta(v + \phi)^{\gamma}} \qquad (1.5)$$

which gives for the case $\gamma \neq 1$

$$q + q_{\phi} = \frac{1}{(1 - \gamma)\Delta} (v + \phi)^{1-\gamma} \qquad (1.6)$$

$$\left[q_{\phi} = \frac{1}{(1 - \gamma)\Delta} \phi^{1-\gamma} \right]$$

We may rearrange Eq. 1.6 to read

$$\left[\Delta (1 - \gamma)(q + q_{\phi}) \right]^{\frac{\gamma}{1 - \gamma}} = (q + q_{\phi})^{\gamma} \qquad (1.7)$$

Multiplying both sides of Eq. 1.7 by Δ, we recognize finally that

$$s(t) \equiv s(q) = \Delta \left[\Delta(1 - \gamma)(q + q_{\phi}) \right]^{\frac{\gamma}{1 - \gamma}} \qquad (1.8)$$

The maximum charge occurs at the breakdown voltage of the diode. Thus,

$$S_{max} = \Delta\left[\Delta(1 - \gamma)(Q_B + q_\phi)\right]^{\frac{\gamma}{1-\gamma}} \quad (1.9)$$

Equations 1.8 and 1.9 yield

$$s(q) = S_{max}\left[\frac{q + q_\phi}{Q_B + q_\phi}\right]^{\frac{\gamma}{1-\gamma}} \quad (1.10)$$

The elastance is now represented as a function of the charge rather than the voltage. This step is necessary because the varactor must be represented as a charge (or current) controlled elastance to be analyzed reasonably.[7]

The abrupt-junction diode has been chosen as the subject for this work. And Eq. 1.10, for $\gamma = 1/2$, becomes

$$s(q) = S_{max}\left[\frac{q + q_\phi}{Q_B + q_\phi}\right] \quad (1.11)$$

Currents are allowed to flow through the varactor for some chosen frequencies only according to the purpose of the varactor. The Fourier-series of the current can be written as

$$i(t) = \sum_{\substack{k=-N \\ k \neq 0}}^{N} I_k e^{j\omega_k t} \tag{1.12}$$

with

$$I_{-k} = I_k^*$$

where the asterisk represents a complex conjugate.

Substitution of

$$q(t) = q_o + \sum_{k=-N}^{N} \frac{I_k}{j\omega_k} e^{j\omega_k t} \tag{1.13}$$

where q_o is a constant, to Eq. 1.11 yields

$$s(t) = \frac{S_{max}}{Q_B + q_\phi} \left[q_\phi + q_o + \sum_{k=-N}^{N} \frac{I_k}{j\omega_k} e^{j\omega_k t} \right]$$

$$s(t) = S_o + \sum_{k=-N}^{N} S_k e^{j\omega_k t} \tag{1.14}$$

with $S_{-k} = S_k^*$

$s(t)$ has lower bound $s = S_{min}$ when the diode starts forward conduction. And S_{min} may approach zero if the varactor can be driven to the point $v = -\phi$ $(q = -q_\phi)$. This is a very good approximation for diodes characterizable by a γ.

The normalization of the elastance is defined as

$$m(t) \overset{\Delta}{=} \frac{s(t)}{S_{max}} = m_o + \sum_{k=-N}^{N} M_k e^{j\omega_k t} \tag{1.15}$$

where $m(t)$ is now constrained to the range

$$0 \leq m(t) \leq 1 , \qquad \text{if } S_{min} = 0 \tag{1.16}$$

We desire that the varactor be driven to both S_{max} and S_{min} to achieve a maximum efficiency. When the currents are sinusoidal, time average

$$< m(t) > = m_o \tag{1.17}$$

and $m_o = 1/2$ serves to specify the required d. c. bias voltage for achieving S_{max} and S_{min}, if $S_{min} = 0$.

From Eqs. 1.3, 1.11, and 1.15, for an abrupt-junction varactor,

$$\frac{v + \phi}{V_B + \phi} = \left[\frac{q + q_\phi}{Q_B + q_\phi} \right]^2 = m^2(t) \qquad (1.18)$$

If Eq. 1.18 is time averaged,

$$\frac{V_o + \phi}{V_B + \phi} = \left[m_o^2 + 2(m_1^2 + m_2^2 + \ldots) \right] \qquad (1.19)$$

where V_o is the external bias voltage and $m_k \equiv |M_k|$. Eq. 1.19 thus gives the required d.c. bias voltage for simultaneously attaining S_{max} and $S_{min} = 0$ (when $m_o = 1/2$).

We now have a Fourier-series representation for the elastance time variation, dependent upon the currents which are allowed to flow in the varactor. Now the current-voltage relationship for the varactor can be written as

$$e(t) = R_s\, i(t) + \int s(t)\, i(t)\, dt \qquad (1.20)$$

From Eqs. 1.12 and 1.14, Fourier-series of the voltage becomes

$$e(t) = \sum_{\substack{k=-2N \\ k\neq 0}}^{2N} E_k \, e^{j\omega_k t} \tag{1.21}$$

$$= R_s \sum_{\substack{k=-N \\ k\neq 0}}^{N} I_k \, e^{j\omega_k t} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \sum_{\ell=-N}^{N} \frac{I_k S_\ell}{j(\omega_k + \omega_\ell)} \, e^{j(\omega_k + \omega_\ell)t}$$

where

$$E_k = R_s I_k + \frac{1}{j\omega_k} \sum_{\substack{\ell=-N \\ k-\ell\neq 0}}^{N} I_{k-\ell} S_\ell$$

$$= I_k\left(R_s + \frac{S_o}{j\omega_k}\right) + \frac{1}{j\omega_k} \sum_{\substack{\ell=-N \\ k-\ell\neq 0 \\ \ell\neq 0}}^{N} I_{k-\ell} S_\ell \tag{1.22}$$

$$I_{-k} = I_k^* \,, \qquad S_{-k} = S_k^*$$

Leenov[8] was the first to recognize that the impedance basis was the most convenient for the series R-C diode. Penfield[2] was the first to use the impedance formulation directly from the beginning. Fortunately, he picked the elastance instead of the capacitance as the variable element. This choice lead to a very much simplified analysis. Rafuse[7] showed that a complete impedance formulation is the most convenient and most fruitful one for all devices utilizing time-varying (non-linear) semiconductor diode capacitors.

CHAPTER II

IMPEDANCES OF THE AMPLIFIER

2.1 The varactor conversion matrix

For three frequencies currents are allowed to flow the varactor. The three frequencies, a signal frequency ω_s, a pump frequency ω_p, and an idler frequency ω_i, are necessary and sufficient to have amplification with the relation $\omega_s = \omega_p - \omega_i$. Rafuse[7] showed that the multiple-idler parametric amplifier does not give sufficient improvement over the single-idler one to warrant the extra circuit complexity required.

The Fourier-series representations of the current and elastance variation have the forms, as in Eqs. 1.12 and 1.14, with single idler,

$$i(t) = I_s e^{j\omega_s t} + I_i e^{j\omega_i t} + I_p e^{j\omega_p t} + I_s^* e^{-j\omega_s t} + I_i^* e^{-j\omega_i t} + I_p^* e^{-j\omega_p t} \tag{2.1}$$

$$s(t) = S_o + S_s e^{j\omega_s t} + S_i e^{j\omega_i t} + S_p e^{j\omega_p t} + S_s^* e^{-j\omega_s t} + S_i^* e^{-j\omega_i t} + S_p^* e^{-j\omega_p t} \tag{2.2}$$

Fourier components of the voltage applied to the varactor are, from Eqs. 2.1, 2.2, and 1.22

$$E_s = \frac{S_o I_s}{j\omega_s} + \frac{I_p S_i^*}{j\omega_s} + \frac{S_p I_i^*}{j\omega_s} + R_s I_s$$

$$E_p = \frac{S_o I_p}{j\omega_p} + \frac{S_s I_i}{j\omega_p} + \frac{S_i I_s}{j\omega_p} + R_s I_p \qquad (2.3)$$

$$E_i = \frac{S_o I_i}{j\omega_i} + \frac{I_p S_s^*}{j\omega_i} + \frac{S_p I_s^*}{j\omega_i} + R_s I_i$$

Same relations can be expressed in matrix as

$$\begin{bmatrix} E_s \\ E_p \\ E_i \end{bmatrix} = \begin{bmatrix} (R_s + \frac{S_o}{j\omega_s}) & \frac{S_i^*}{j\omega_s} & 0 & \frac{S_p}{j\omega_s} & 0 \\ \frac{S_i}{j\omega_p} & (R_s + \frac{S_o}{j\omega_p}) & \frac{S_s}{j\omega_p} & 0 & 0 \\ 0 & \frac{S_s^*}{j\omega_i} & (R_s + \frac{S_o}{j\omega_i}) & 0 & \frac{S_p}{j\omega_i} \end{bmatrix} \begin{bmatrix} I_s \\ I_p \\ I_i \\ I_i^* \\ I_s^* \end{bmatrix}$$

This matrix relation is called "varactor conversion matrix" by Penfield.

2.2 Impedance

As the varactor model is chosen with a resistance and an elastance in series, impedances of the varactor in the three frequencies are bases to find the characteristics of the varactor.

To have direct relation between the elastance and the current in each frequency, from

$$i_k = I_k e^{j\omega_k t}$$

$$q_k = \frac{I_k}{j\omega_k} e^{j\omega_k t}$$

and Eqs. 1.2, 1.4, and 1.8

$$\Delta = \frac{S_{max}}{(V_B + \phi)^{1/2}} = \frac{\omega_c R_s}{(V_B + \phi)^{1/2}}$$

$$S_k = \Delta \Delta \frac{1}{2} \frac{I_k}{j\omega_k}$$

Hence, in the three frequencies

$$M_s = \frac{S_s}{S_{max}} = \frac{1}{2j}\left(\frac{\omega_c}{\omega_s}\right)\frac{R_s}{V_B + \phi} I_s$$

$$M_p = \frac{1}{2j}\left(\frac{\omega_c}{\omega_p}\right)\frac{R_s}{V_B + \phi} I_p \qquad (2.5)$$

$$M_i = \frac{1}{2j}\left(\frac{\omega_c}{\omega_i}\right)\frac{R_s}{V_B + \phi} I_i$$

From Eqs. 2.3 and 2.5, the impedance of the varactor in signal frequency is

$$\frac{E_s}{I_s} = R_s + \frac{S_o}{j\omega_s} + \frac{S_i^*}{j\omega_s}\frac{I_p}{I_s} + \frac{S_p}{j\omega_s}\frac{I_i^*}{I_s}$$

$$= R_s + \frac{S_o}{j\omega_s} + \frac{M_i^* \omega_c R_s}{j\omega_s}\frac{\omega_p}{\omega_s}\frac{M_p}{M_s} - \frac{M_p \omega_c R_s}{j\omega_s}\frac{\omega_i}{\omega_s}\frac{M_i^*}{M_s}$$

$$= R_s + \frac{S_o}{j\omega_s} + \frac{M_i^* M_p}{M_s}\frac{\omega_c}{j\omega_s} R_s \qquad (2.6)$$

Again, in idler frequency

$$\frac{E_i}{I_i} = R_s + \frac{S_o}{j\omega_i} + \frac{S_s^*}{j\omega_i}\frac{I_p}{I_i} + \frac{S_p}{j\omega_i}\frac{I_s^*}{I_i}$$

$$= R_s + \frac{S_o}{j\omega_i} + \frac{M_s^* M_p}{M_i}\frac{\omega_c}{j\omega_i} R_s \qquad (2.7)$$

With the constraint in the idler frequency circuit,

$$\frac{E_i}{I_i} = -R_i - j\omega_i L_i \qquad (2.8)$$

from Eq. 2.7

$$M_i = \frac{M_s^* M_p \frac{\omega_c}{j\omega_i} R_s}{-R_s - R_i - j\omega_i L_i - \frac{S_o}{j\omega_i}} \qquad (2.9)$$

Substituting Eq. 2.9 to Eq. 2.6

$$\frac{E_s}{I_s} = R_s + \frac{S_o}{j\omega_s} - \frac{m_p^2 \omega_c^2}{\omega_i \omega_s} \frac{R_s^2}{R_s + R_i - j\omega_i L_i - \frac{S_o}{j\omega_i}} \qquad (2.10)$$

with $m_p = |M_p|$ being already defined.

In pump frequency, the impedance is, with the same steps as in the other frequencies,

$$\frac{E_p}{I_p} = R_s + \frac{S_o}{j\omega_p} + \frac{M_s M_i}{M_p} \frac{\omega_c}{j\omega_p} R_s$$

$$= R_s + \frac{S_o}{j\omega_p} + \frac{m_s^2 \omega_c^2}{\omega_i \omega_p} \frac{R_s^2}{R_s + R_i + j\omega_i L_i + \frac{S_o}{j\omega_i}} \tag{2.11}$$

Up to here, the impedances are expressed without special conditions or assumptions which aim practical optimum operation.

The equivalent model of the varactor in the signal frequency can be directly recognized from Eq. 2.10.

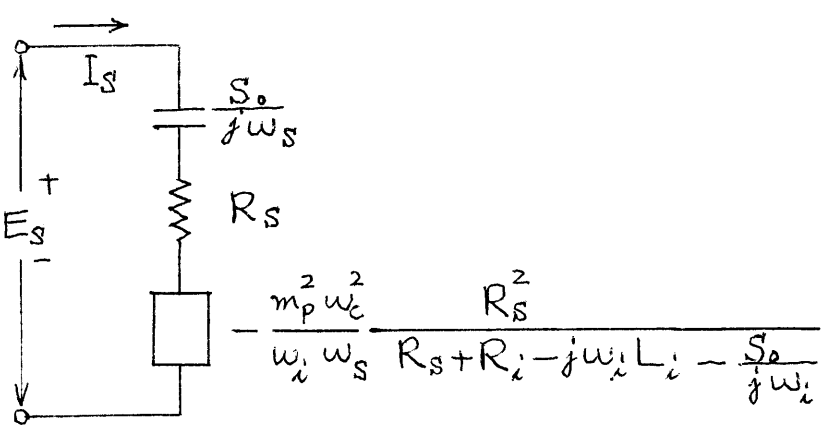


Fig. 2.1 Signal-frequency impedance model

2.3 Elastances and variables

The impedances obtained in the last section were functions of the three normalized elastances. To have further analysis of the amplifier, it is necessary to find constraints among the variables. Every normalized elastance is the function of the other normalized elastances. In this section the constraints will be derived through the variations of the normalized elastances from the initially tuned state when the signal is week, condition is optimum, and algebra is simple.

Suppose that we can imagine small signal variation from null. At the situation, tune the three frequency circuits as

$$j\omega_s L_s = -\frac{S_o}{j\omega_s}$$

$$j\omega_p L_p = -\frac{S_o}{j\omega_p} \qquad (2.12)$$

$$j\omega_i L_i = -\frac{S_o}{j\omega_i}$$

and assume $R_i = 0$. To the preceding conditions, make the initial bias and pump frequency normalized elastances

$$m_{o,\,in} \equiv 1/2$$

$$m_{p,\,in} \equiv 1/4 \tag{2.13}$$

to have maximum utilization of the varactor elastance range.

Then, with the condition $R_i = 0$ stated, from Eq. 2.11

$$\frac{E_p}{I_p} = R_s + \frac{S_o}{j\omega_p} + \frac{m_s^2\,\omega_c^2}{\omega_i\,\omega_p}\;\frac{R_s^2}{R_s + j\omega_i L_i + \dfrac{S_o}{j\omega_i}}$$

With the pump source impedance

$$Z_p = R_s + j\omega_p L_p$$

and the pump source voltage E_o as in Fig. 2.2, the pump current is

$$I_p = \frac{E_o}{2R_s + j\omega_p L_p + \dfrac{S_o}{j\omega_p} + \dfrac{m_s^2\,\omega_c^2}{\omega_i\,\omega_p}\;\dfrac{R_s^2}{R_s + j\omega_i L_i + \dfrac{S_o}{j\omega_i}}}$$

and from Eq. 2.5

$$M_p = \frac{1}{2j}\left(\frac{\omega_c}{\omega_p}\right)\frac{R_s}{V_B + \phi} \quad \frac{E_o}{2R_s + j\omega_p L_p + \frac{S_o}{j\omega_p} + \frac{m_s^2\,\omega_c^2}{\omega_i\,\omega_p}\;\frac{R_s^2}{R_s + j\omega_i L_i + \frac{S_o}{j\omega_i}}}$$

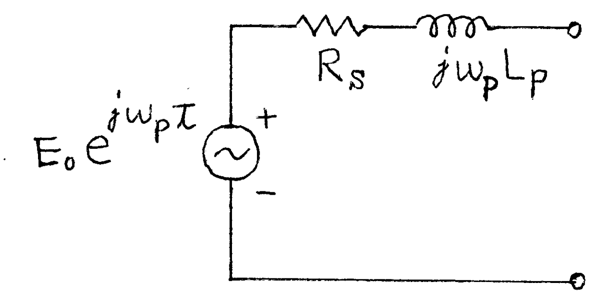


Fig. 2.2 Pump-source equivalent circuit

Since $m_{p,\,in} = 1/4$ when $m_s = 0$ from the last equation

$$1 = \left(\frac{\omega_c}{\omega_p}\right)\frac{|E_o|}{V_B + \phi} \qquad (2.14)$$

and

(2.15)

$$m_p = \frac{1/2}{\left| 2 + j\omega_p \frac{L_p}{R_s} + \frac{m_o \omega_c}{j\omega_p} + \frac{m_s^2 \omega_c^2}{\omega_i \omega_p} \frac{1}{1 + j\omega_i \frac{L_i}{R_s} + \frac{m_o \omega_c}{j\omega_i}} \right|}$$

Defining the variation of normalized bias elastance

$$\delta m_o \equiv m_o - m_{o,\,in}$$

and

$$\alpha \equiv \delta m_o \frac{\omega_c}{\omega_i} \qquad (2.16)$$

from Eq. 2.15,

$$m_p^2 = \frac{1/4}{\left[2 + \frac{m_s^2 \omega_c^2}{\omega_i \omega_p} \frac{1}{1 + \alpha^2}\right]^2 + \left[\frac{m_s^2 \omega_c^2}{\omega_i \omega_p} \frac{\alpha}{1 + \alpha^2} - \alpha \frac{\omega_i}{\omega_p}\right]^2}$$

(2.17)

From Eq. 2.9, applying the stated initial conditions,

$$m_i^2 = \frac{m_s^2\, m_p^2 \left(\frac{\omega_c}{\omega_i}\right)^2}{1 + \alpha^2} \tag{2.18}$$

Now, from Eq. 1.19

$$\frac{V_o + \phi}{V_B + \phi} = m_o^2 + 2(m_s^2 + m_i^2 + m_p^2) \tag{2.19}$$

Setting the bias voltage

$$\frac{V_o + \phi}{V_B + \phi} = m_{o,\,in}^2 + 2m_{p,\,in}^2$$

$$= \frac{3}{8} \tag{2.20}$$

from Eqs. 2.19 and 2.20

$$\frac{1}{8} = \delta m_o + \delta m_o^2 + 2(m_s^2 + m_i^2 + m_p^2) \tag{2.21}$$

Hence, Eqs. 2.21, 2.17 and 2.18 bring

$$\frac{1}{8} = \alpha \frac{\omega_i}{\omega_c} + \alpha^2 \left(\frac{\omega_i}{\omega_c}\right)^2 + 2m_s^2 + \frac{\frac{1}{2}\left(1 + \frac{m_s^2 \omega_c^2}{\omega_i^2} \frac{1}{1+\alpha^2}\right)}{\left[2 + \frac{m_s^2 \omega_c^2}{\omega_p \omega_i}\right]^2 + \left[\frac{m_s^2 \omega_c^2}{\omega_p \omega_i} \frac{\alpha}{1+\alpha^2} - \alpha \frac{\omega_i}{\omega_p}\right]^2} \quad (2.22)$$

This equation shows the variation of m_o as a function of m_s at any signal and pump frequencies. Therefore, any variable can be expressed as a function of the magnitude of signal frequency current at any frequency in operation.

2.4 Elastance equation and approximation

In the preceding section the equation between two normalized elastances was obtained. It is necessary to change the shape of the equation to examine the relations among normalized elastances and frequencies.

Define the three normalized frequencies

$$s = \frac{\omega_s}{\omega_c}, \qquad p = \frac{\omega_p}{\omega_c}, \qquad i = \frac{\omega_i}{\omega_c} \quad (2.23)$$

Then, the equation becomes

$$\frac{1}{8} = \alpha i + \alpha^2 i^2 + 2m_s^2 +$$

$$+ \frac{\frac{1}{2}\left[(1+\alpha^2)^2 p^2 i^2 + (1+\alpha^2) p^2 m_s^2\right]}{4(1+\alpha^2)^2 p^2 i^2 + 4(1+\alpha^2) p i m_s^2 + m_s^4 + \alpha^2 m_s^4 - 2(1+\alpha^2)\alpha^2 i^2 m_s^2 + (1+\alpha^2)^2 \alpha^2 i^4}$$

$$(2.24)$$

Multiplying the equation by the denominator of the last term,

$$0 = \left[(-1/4 + 4m_s^2)(4p^2 i^2 + 4p i m_s^2 + m_s^4) + (p^2 i^2 + p^2 m_s^2)\right]$$

$$+ \alpha\left[2i(4p^2 i^2 + 4p i m_s^2 + m_s^4)\right]$$

$$+ \alpha^2\left[2i^2(4p^2 i^2 + 4p i m_s^2 + m_s^4) + (-1/4 + 4m_s^2)(8p^2 i^2 + 4p i m_s^2 - 2i^2 m_s^2 + i^4 + m_s^4)\right.$$

$$\left. + (2p^2 i^2 + p^2 m_s^2)\right]$$

$$+ \alpha^3\left[2i(8p^2 i^2 + 4p i m_s^2 + m_s^4 - 2i^2 m_s^2 + i^4)\right]$$

$$+ \alpha^4\Big[2i^2(8p^2 i^2 + 4p\, i\, m_s^2 + m_s^4 - 2i^2 m_s^2 + i^4)$$

$$+ (-1/4 + 4m_s^2)(4p^2 i^2 - 2i^2 m_s^2 + 2i^4) + p^2 i^2\Big]$$

$$+ \alpha^5\Big[2i(4p^2 i^2 - 2i^2 m_s^2 + 2i^4)\Big]$$

$$+ \alpha^6\Big[(-1/4 + 4m_s^2)\, i^4 + 2i^2(4p^2 i^2 - 2i^2 m_s^2 + 2i^4)\Big]$$

$$+ \alpha^7\Big[\, 2i^5 \,\Big]$$

$$+ \alpha^8\Big[\, 2i^6 \,\Big]$$

(2. 25)

This equation is still too complicated.

An approximation to the equation is necessary which is accurate enough in certain ranges of the normalized elastance variations and frequencies.

Assume the possible regions,

$$m_s < \frac{1}{20}$$

$$\frac{1}{4} > [p, i] > \frac{1}{12} \qquad (2.26)$$

$$s < \frac{1}{10}\, i$$

This assumption is made on the bases of practical values which are determined from cutoff frequencies of abrupt junction diodes, optimum or near optimum pump frequency, usual signal frequency in application with high gain, and the bias and pump currents at Eq. 2.13. At a glance, realizing that α and m_s^2 vary in near order, we can disregard up to the terms with α^4 and m_s^8, then

$$
\begin{aligned}
0 = & \left[m_s^2(16p^2i^2 - pi + p^2) + m_s^4(16pi - 1/4) + 4m_s^6 \right] \\
& + \alpha\left[2i(4p^2i^2 + 4pim_s^2 + m_s^4) \right] \\
& + \alpha^2\left[8p^2i^4 - \frac{1}{4}i^4 + m_s^2(-pi + 8pi^3 + p^2 + 32p^2i^2 + \frac{1}{2}i^2 + 4i^4) \right] \\
& + \alpha^3\left[2i^3(8p^2 + i^2) \right]
\end{aligned}
\tag{2.27}
$$

To have further simplification, let us have rather rough idea about the value of α. Directly from Eq. 2.24

$$\frac{1}{8} \approx \alpha i + \alpha^2 i^2 + 2m_s^2 + \frac{1}{8}$$

$$0 \approx \alpha i + 2m_s^2 \tag{2.28}$$

From this fact, the terms with α^2 and α^3 are negligible to the term with α in Eq. 2.27. With this fact, recognizing

$$2pi \gg m_s^2$$

The equation can be

$$0 = \left[m_s^2 (16p^2 i^2 - pi + p^2) + m_s^4 (16pi - 1/4) \right]$$

$$+ \alpha \left[2i(4p^2 i^2 + 4pim_s^2) \right]$$

or

$$\alpha = -\left(\frac{m_s^2}{2i} \right) \frac{16p^2 i^2 - pi + p^2 + m_s^2 (16pi - 1/4)}{4p^2 i^2 + 4pim_s^2}$$

$$= -\left(\frac{m_s^2}{2i} \right) \left[4 + \frac{-pi + p^2 - \frac{1}{4} m_s^2}{4p^2 i^2 + 4pim_s^2} \right]$$

$$= -\left(\frac{m_s^2}{2i} \right) \left[4 + \frac{-si + 2si - \frac{1}{4} m_s^2 + s^2}{4pi(i^2 + si + m_s^2)} \right]$$

$$\approx -\left(\frac{m_s^2}{2i} \right) \left[4 + \frac{si - \frac{1}{4} m_s^2}{4pi^3} \right]$$

$$\approx -\left(\frac{m_s^2}{2i}\right)\left[4 - \frac{m_s^2}{16pi^3}\right] \tag{2.29}$$

$$\approx -\frac{2m_s^2}{i}$$

or

$$\delta m_o \approx -\frac{m_s^2}{2}\left(4 - \frac{m_s^2}{16pi^3}\right)$$

$$\approx -2m_s^2$$

The equation 2.29 is good enough approximation in the range given in Eq. 2.26. Eq. 2.28 is also valuable. Later, to analyze the phase shift of the amplifier Eq. 2.29 will be applied.

The author actually computed α, at $i = 1/4$ and $s < 10^{-3}$, through $0 < m_s < 1/16$, using Eq. 2.27, and found that Eq. 2.28 is good approximation in the case.

When $s > \frac{1}{10} i$, α can be obtained also as a function of m_s from Eq. 2.24. But, as it will be recognized at Eq. 3.8, at this case, the amplifier becomes unstable for a large gain.

CHAPTER III

THE DYNAMIC RANGE

3.1 Gain

We will assume that the amplifier is imbedded around signal loop in a lossless circulator, as shown in Fig. 3.1.

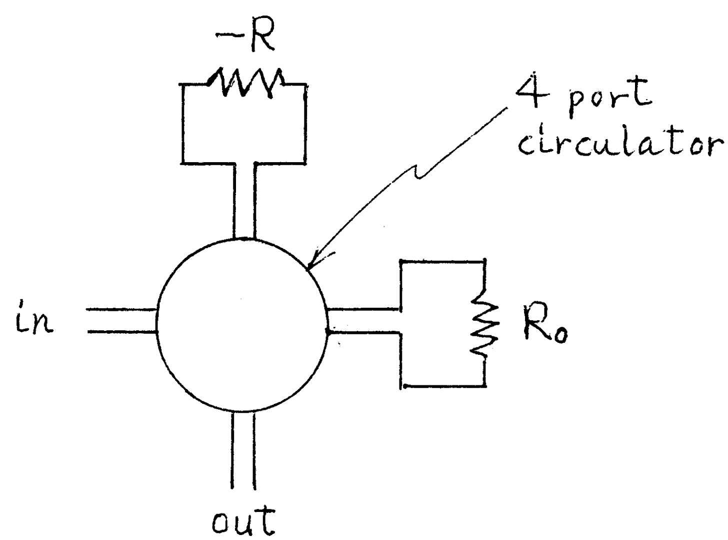


Fig. 3.1 Circulator imbedment of a parametric amplifier

The exchangable gain is

$$G = \left[\frac{R_o + R}{R_o - R}\right]^2 \qquad (3.1)$$

The resistance around signal loop is, from Eqs. 2.10 and 2.17,

$$R_o + R_s - \frac{\frac{1}{4}\frac{R_s}{si}\frac{1}{1+\alpha^2}}{\left[2 + \frac{m_s^2}{pi}\frac{1}{1+\alpha^2}\right]^2 + \left[\frac{m_s^2}{pi}\frac{\alpha}{1+\alpha^2} - \alpha\frac{i}{p}\right]^2} \tag{3.2}$$

Considering the ranges of the frequencies and parameters at Eqs. 2.26 and 2.29, the resistance has the following approximations

$$R_o + R_s - \frac{\frac{1}{4}\frac{R_s}{si}(1+\alpha^2)}{\left(4 + \frac{4m_s^2}{pi} + \frac{m_s^4}{p^2 i^2}\right) + \alpha^2\left(8 + \frac{i^2}{p^2} + \frac{4m_s^2}{pi} - \frac{2m_s^2}{p^2}\right)}$$

$$\approx R_o + R_s - \frac{\frac{1}{4}\frac{R_s}{si}(1+\alpha^2)}{4\left(1 + \frac{1}{4}\left[\frac{4m_s^2}{pi} + \frac{m_s^4}{p^2 i^2} + \alpha^2\left(8 + \frac{i^2}{p^2} + \frac{4m_s^2}{pi} - \frac{2m_s^2}{p^2}\right)\right]\right)}$$

$$\approx R_o + R_s - \frac{R_s}{16si}\left[1 - m_s^2 \frac{1}{pi} + m_s^4 \frac{3}{4p^2 i^2} - m_s^6 \frac{1}{2p^3 i^3}\right.$$

$$\left. - \alpha^2 (2 + \frac{i^2}{4p^2}) - \alpha^2 m_s^2 (\frac{2}{pi} - \frac{1}{2p^2})\right] \tag{3.3}$$

From Eqs. 3.1 and 3.3, with an approximation again,

$$\sqrt{G} \approx \frac{\frac{R_o}{R_s} - 1 + \frac{1}{16si}\left[1 - m_s^2 \frac{1}{pi}\right]}{\frac{R_o}{R_s} + 1 - \frac{1}{16si}\left[1 - m_s^2 \frac{1}{pi}\right]} \tag{3.4}$$

Put

$$\frac{R_o}{R_s} \equiv -1 + \frac{1}{16si} + r \tag{3.5}$$

then the equation is

$$\sqrt{G} = \frac{r - 2 + \frac{2}{16si} - m_s^2 \frac{1}{pi} \frac{1}{16si}}{r + \frac{1}{16si} m_s^2 \frac{1}{pi}}$$

$$= \frac{16\,s\,i\,r - 32\,s\,i + 2 - m_s^2 \frac{1}{pi}}{16\,s\,i\,r + m_s^2 \frac{1}{pi}} \qquad (3.6)$$

r is defined at Eq. 3. 5. To have large gain Eq. 3. 4 is good approximation, and this reason can be another criterion for the approximation. Even for small maximum gain (when $m_s^2 \approx 0$) , to

$$10 \log_{10} G_{max} > 6 \text{ db} \qquad (3.7)$$

$$\sqrt{G} \approx \frac{2 - 32\,s\,i}{16sir + m_s^2 \frac{1}{pi}}$$

$$\approx \frac{2\,i^2}{16\,s\,i^3\,r + m_s^2} \qquad (3.8)$$

To check whether Eq. 2. 8 is good enough approximation, take a case $i = 1/4$ and $s < 10^{-3}$ which is a significant one as will be shown in the following chapter. Then

$$\sqrt{G} = \frac{1}{2sr + 8m_s^2}$$

To check whether Eq. 3.8 is good enough approximation, take a case $i = \frac{1}{4}$ and $s < 10^{-3}$ which is a significant one as will be shown in the following chapter. Then

$$\sqrt{G} = \frac{1}{2sr + 8m_s^2}$$

This is good enough approximation to almost accurate one, which is obtained from Eqs. 3.1 and 3.3:

$$\sqrt{G} = \frac{1 - 8m_s^2}{2sr + 8m_s^2 - 56m_s^4}$$

Again, to check Eq. 3.8 at $m_s = 0$ with Eq. 3.6 about frequencies, take a case $i = \frac{1}{4}$. Then Eq. 3.6 becomes when $m_s^2 = 0$

$$\sqrt{G}_{max} = \frac{2sr - 4s + 1}{2sr} \qquad (3.9)$$

And, from Eq. 3.8

$$\sqrt{G}_{max} = \frac{1}{2sr} \qquad (3.10)$$

Figure 3.2 is about Eq. 3.9. The figure confirms that Eq. 3.10 is good approximation to Eq. 3.9.

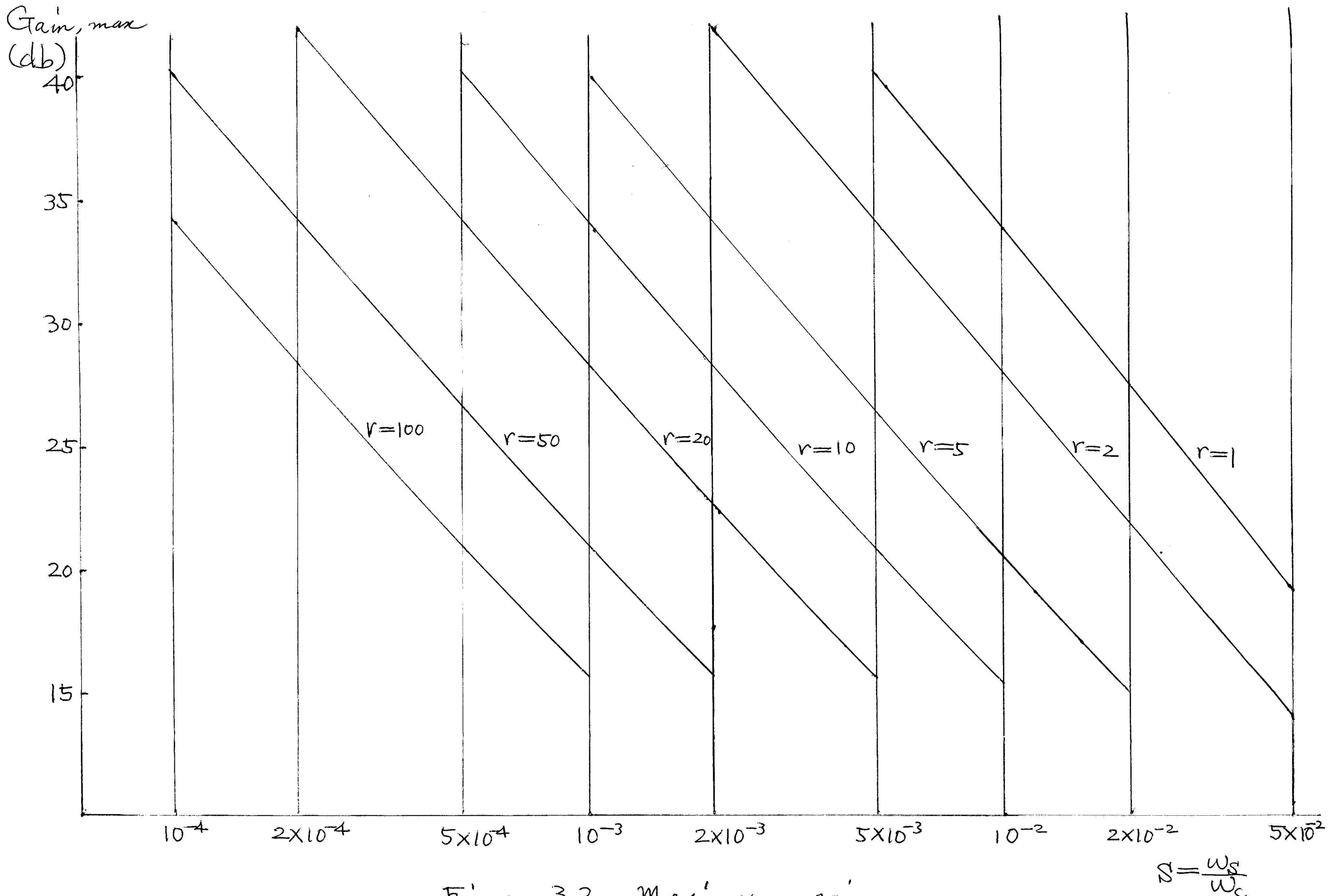


Figure 3.2 Maximum gain

To analyze dynamic range, it is necessary to have a simple equation of exchangeable gain like Eq. 3.8. And, the equation is derived as a reliable one and checked at an important case.

From Eqs. 3.5 and 3.8, smaller s brings larger gain. This meets a fundamental concept of the amplifier.

3.2 Noise Figure

When there is no signal to analyze noise power, it is sufficient to have same procedure as in small signal case worked by Rafuse[7]. Directly following his results, assuming $R_i = 0$ and impedances are initially tuned as in Eq. 2.12, the signal and idler impedances are

$$Z_{\omega_s} = R_s + \frac{S_o}{j\omega_s} - \frac{m_p^2 \omega_c^2}{\omega_s \omega_i} R_s$$

$$Z_{\omega_i} = R_s + \frac{S_o}{j\omega_i} - \frac{m_p^2 \omega_c^2}{\omega_s \omega_i} \frac{R_s^2}{R_s + R_o} \qquad (3.11)$$

and the squared-magnitude reverse current gain is

$$\left| \frac{I_s}{I_i} \right|^2 = \left(\frac{m_p \omega_c}{\omega_i} \right)^2 \frac{R_s^2}{(R_s + R_o)^2} \qquad (3.12)$$

We shall define noise sources in series with R_o and R_s as

$$\overline{e_o^2} = 4kT_o R_o \Delta f$$

$$\overline{e_r^2} = 4kT_d R_s \Delta f \tag{3.13}$$

where

T_o and T_d are source and diode temperatures in ^oK

k is Boltzmann's constant,

$$1.38 \times 10^{-23} \text{ joules}/^oK$$

Δf is an incremental positive frequency bandwidth in cps.

The portion of signal tank noise current due to idler noise is

$$\overline{I_{si}^2} = \overline{I_{ii}^2} \left|\frac{I_s}{I_i}\right| = \frac{T_d R_s}{R_{\omega_i}^2} \cdot 4k\Delta f \left(\frac{m_p \omega_c}{\omega_i}\right)^2 \frac{R_s^2}{(R_s + R_o)^2} \tag{3.14}$$

The other internal noise current flowing in the signal frequency is due to R_s at ω_s. It is,

$$\overline{I_{ss}^2} = \frac{T_d R_s}{(R_o + R_{\omega_s})^2} \cdot 4k\Delta f \tag{3.15}$$

And, the source noise current flowing in the signal circuit is

$$\overline{I_{so}^2} = \frac{T_o R_o}{(R_o + R_{\omega_s})^2} \cdot 4k\Delta f \tag{3.16}$$

Now, the exchangeable excess noise figure is [9]

$$(F_e - 1) = \frac{\overline{I_{ss}^2} + \overline{I_{si}^2}}{\overline{I_{so}^2}} \tag{3.17}$$

Inserting the results from Eqs. 3.14, 3.15, 3.16, and 3.11,

$$(F_e - 1) = \frac{T_d}{T_o} \frac{R_s}{R_o} \left[1 + \left(\frac{m_p \omega_c}{\omega_i} \right)^2 \right] \tag{3.18}$$

3.3 Dynamic Range

An estimate of the minimum detectable signal can be made by considering, T_a being antenna temperature,

$$P_{es,\ min} = (F_e - 1) \cdot kT_o \Delta f + kT_a \Delta f \tag{3.19}$$

This power brings same strength of signal power as noise (themal) power at the output. The term "Dynamic range" is used here as the input power range from the minimum input power to the power which drops the gain of the amplifier by 3 db from the maximum gain. This upper bound is defined as an estimate of the maximum possible signal power, $P_{es',\ max}$.

From Eq. 3.18, when $m_p = \frac{1}{4}$

$$(F_e - 1) = \frac{T_d R_s}{T_o R_o} \left(1 + \frac{1}{16 i^2}\right)$$

And, with Eq. 3.19

$$P_{es,min} = k\Delta f \left[T_d \frac{R_s}{R_o} (1 + \frac{1}{16 i^2}) + T_a \right] \tag{3.20}$$

From Eq. 2.5, the square magnitude of the signal frequency normalized elastance is

$$m_s^2 = \frac{1}{4} (\frac{\omega_c}{\omega_s})^2 (\frac{R_s}{V_B + \phi})^2 I_s^2 \tag{3.21}$$

Defining

$$P_{norm} = \frac{(V_B + \phi)^2}{R_s} \tag{3.22}$$

with the output power

$$P_{out} = 2 I_s^2 R_o$$

equation 3.21 becomes

$$m_s^2 = \frac{1}{8} (\frac{\omega_c}{\omega_s})^2 \frac{P_{out}}{P_{norm}} \frac{R_s}{R_o} \tag{3.23}$$

When the gain is 3 db lower than the maximum,

$$m_{s_{max}}^2 = \frac{1}{8s^2} \frac{\frac{1}{2}G_{max} P_{es,max}}{P_{norm}} \frac{R_s}{R_o} \tag{3.24}$$

On the other hand, this $m_{s_{max}}^2$ can be obtained from Eq. 3.8 as

$$\frac{1}{\sqrt{2}} \sqrt{G_{max}} = \frac{2i^2}{16si^3 r + m_{s_{max}}^2}$$

But

$$\sqrt{G}_{max} = \frac{1}{8sir} \tag{3.25}$$

from the same equation. Hence, from the two preceding equations

$$\frac{1}{\sqrt{2}} \sqrt{G}_{max} = \frac{2i^2}{2i^2 \frac{1}{\sqrt{G}_{max}} + m_{s_{max}}^2}$$

and

$$m_{s,\,max}^2 = 2i^2 \frac{1}{\sqrt{G_{max}}} \left[\sqrt{2} - 1 \right]$$

or

$$m_{s,\,max}^2 = 0.828 \frac{i^2}{\sqrt{G_{max}}} \qquad (3.26)$$

Now, from Eq. 3.24 and 3.26

$$\frac{1}{16s^2} \frac{G_{max} P_{es,\,max}}{P_{norm}} \frac{R_s}{R_o} = 0.828 \frac{i^2}{\sqrt{G_{max}}}$$

or

$$P_{es,\,max} = 13.248 \frac{i^2 s^2 P_{norm} R_o}{G_{max}^{3/2} R_s} \qquad (3.27)$$

Define,

$$T_n = 290^{\circ} K$$

then

$$T_n k \Delta f = 4 \times 10^{-21} \Delta f \qquad (3.28)$$

From Eqs. 3.20, 3.27, and 3.28, the dynamic range is

$$DR = \frac{P_{es,\ max}}{P_{es,\ min}}$$

$$= \frac{13.248 i^2 s^2 P_{norm}}{4 \times 10^{-21} \Delta f G_{max}^{3/2} \left[\frac{T_d}{T_n} \frac{R_s^2}{R_o^2} \left(1 - \frac{1}{16 i^2}\right) + \frac{R_s}{R_o} \frac{T_a}{T_n} \right]}$$

$$= \frac{3.312 \times 10^{21} P_{norm}}{\Delta f G_{max}^{3/2} \left[\frac{T_d}{T_n} \frac{R_s^2}{s^2 R_o^2} \left(1 + \frac{1}{16_i^2}\right) + \frac{R_s}{i^2 s^2 R_o} \frac{T_a}{T_n} \right]} \qquad (3.29)$$

From Eq. 3.5 and 3.25

$$\frac{R_o}{R_s} \approx \frac{1}{16 s i} + \frac{1}{8 s i \sqrt{G_{max}}}$$

or

$$\frac{R_s}{R_o} = \frac{8 s i}{\frac{1}{2} + \frac{1}{\sqrt{G_{max}}}} \qquad (3.30a)$$

To here, the approximations were accurate enough. Let us examine

$$\frac{R_s^2}{R_o^2} \approx \frac{64 s^2 i^2}{\frac{1}{4} + \frac{1}{\sqrt{G_{max}}}} \qquad (3.30b)$$

When $10 \log_{10} G_{max} > 10$, this equation is very good. To $10 \log_{10} G_{max} = 6$ this is still acceptable one considering the participation in the Eq. 3.29. Furthermore, in the Eq. 3.29, usually T_a, which has coefficient $\frac{R_s}{R_o}$ in first order, almost determines DR comparing with T_d partici- pation in the other term of the denominator.

Substituting Eq. 3.30 to Eq. 3.29

$$DR = \frac{3.312 \times 10^{21} P_{norm}}{\Delta f G_{max}^{3/2} \left[\frac{64}{\frac{1}{4} + \frac{1}{\sqrt{G_{max}}}} \left(1 + \frac{1}{16 i^2}\right) \frac{T_d}{T_n} + \frac{8}{Si\left(\frac{1}{2} + \frac{1}{\sqrt{G_{max}}}\right)} \frac{T_a}{T_n} \right]}$$

$$= \frac{0.207 \times 10^{21} P_{norm}}{\Delta f G_{max}^{3/2} \left[\frac{16 + \frac{1}{i^2}}{1 + \frac{4}{\sqrt{G_{max}}}} \frac{T_d}{T_n} + \frac{1}{Si\left(1 + \frac{2}{\sqrt{G_{max}}}\right)} \frac{T_a}{T_n} \right]} \qquad (3.31)$$

Following usual expression, taking common logarithm of Eq. 3.31

$$10 \log_{10} DR = 10 \log_{10} P_{norm} - 15 \log_{10} G_{max} - 10 \log_{10} \Delta f$$

$$-10 \log_{10} \left[\frac{16 + \frac{1}{i^2}}{1 + \frac{4}{\sqrt{G_{max}}}} \frac{T_d}{T_n} + \frac{1}{Si\left(1 + \frac{2}{\sqrt{G_{max}}}\right)} \frac{T_a}{T_n} \right]$$

$$+ 203 \qquad (3.32)$$

when $\frac{1}{4} > \left[p \text{ or } i \right] > \frac{1}{12}$

$$s < \frac{1}{10} i$$

$$10 \log_{10} G_{max} > 6$$

where

$$P_{norm} = \frac{(V_B + \phi)^2}{R_s}$$

G_{max} = maximum gain

(when signal is negligibly small)

Δf = incremental positive frequency bandwidth in cps

$$s, i = \frac{\omega_s}{\omega_c}, \frac{\omega_i}{\omega_c}$$

T_d, T_a = diode, antenna temperature in ^{o}K

$T_n = 290^{o}K$

When T_a is comparable to T_d, T_d can be neglected in Eq. 3.32 at $s \ll i$. Following the equation, large si brings large dynamic range for the same maximum gains.

3.4 Examples on Dynamic Range

To examine the dynamic range, examples will follow.

Example 3.1: When

$$i = \frac{1}{4}$$

$$T_d = T_a = 290^oK$$

$$P_{norm} = 40W, \ \Delta f = 10^6$$

$$10 \log_{10} DR = 153 - 15 \log_{10} G_{max}$$

$$- 10 \log_{10} \left[\frac{8}{1 + \frac{4}{\sqrt{G_{max}}}} + \frac{1}{s(1 + \frac{2}{\sqrt{G_{max}}})} \right]$$

This example is one of usual case.

Figure 3.3 is on this example. As a comparison, we note that very good vacuum-tube amplifiers have dynamic ranges in the order of 80 db.

Example 3.2:

When

$$i = \frac{1}{4}$$

$$T_d = 290^oK, \ T_a = 3^oK$$

$$P_{norm} = 40\bar{W}, \ \Delta f = 10^6$$

$$10 \log_{10} DR = 153 - 15 \log_{10} G_{max} - 10 \log_{10} \left[\frac{8}{1 + \frac{4}{\sqrt{G_{max}}}} + \frac{\frac{3}{290}}{S(1 + \frac{2}{\sqrt{G_{max}}})} \right]$$

On this example antenna temperature affects the dynamic range noticeably. Figure 3.4 is on this one.

<u>Example</u> <u>3.3</u>:

When a certain dynamic range is desired, gain will be a function of signal frequency at other constants determined.

When

$$10 \log_{10} DR = 110$$

$$P_{norm} = 40\bar{W}, \ \Delta f = 10^6$$

$$T_d = T_a = 290^o K$$

$$0 = 43 - 15 \log_{10} G_{max}$$

$$- 10 \log_{10} \left[\frac{8}{1 + \frac{4}{\sqrt{G_{max}}}} + \frac{1}{S(1 + \frac{2}{\sqrt{G_{max}}})} \right]$$

For large gain,

$$0 \approx 43 - 15 \log_{10} G_{max} + 10 \log_{10} S$$

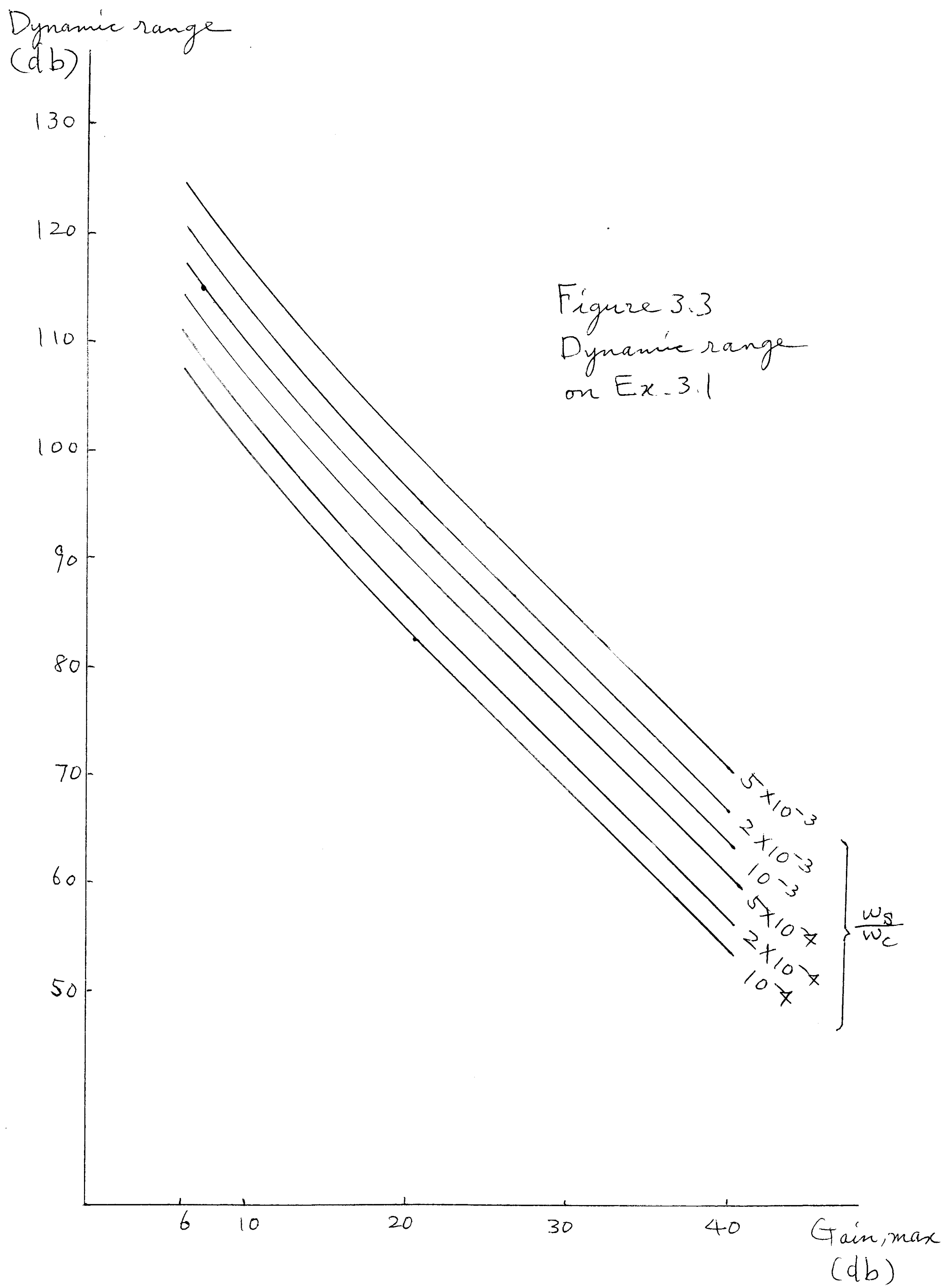


Figure 3.3
Dynamic range on Ex. 3.1

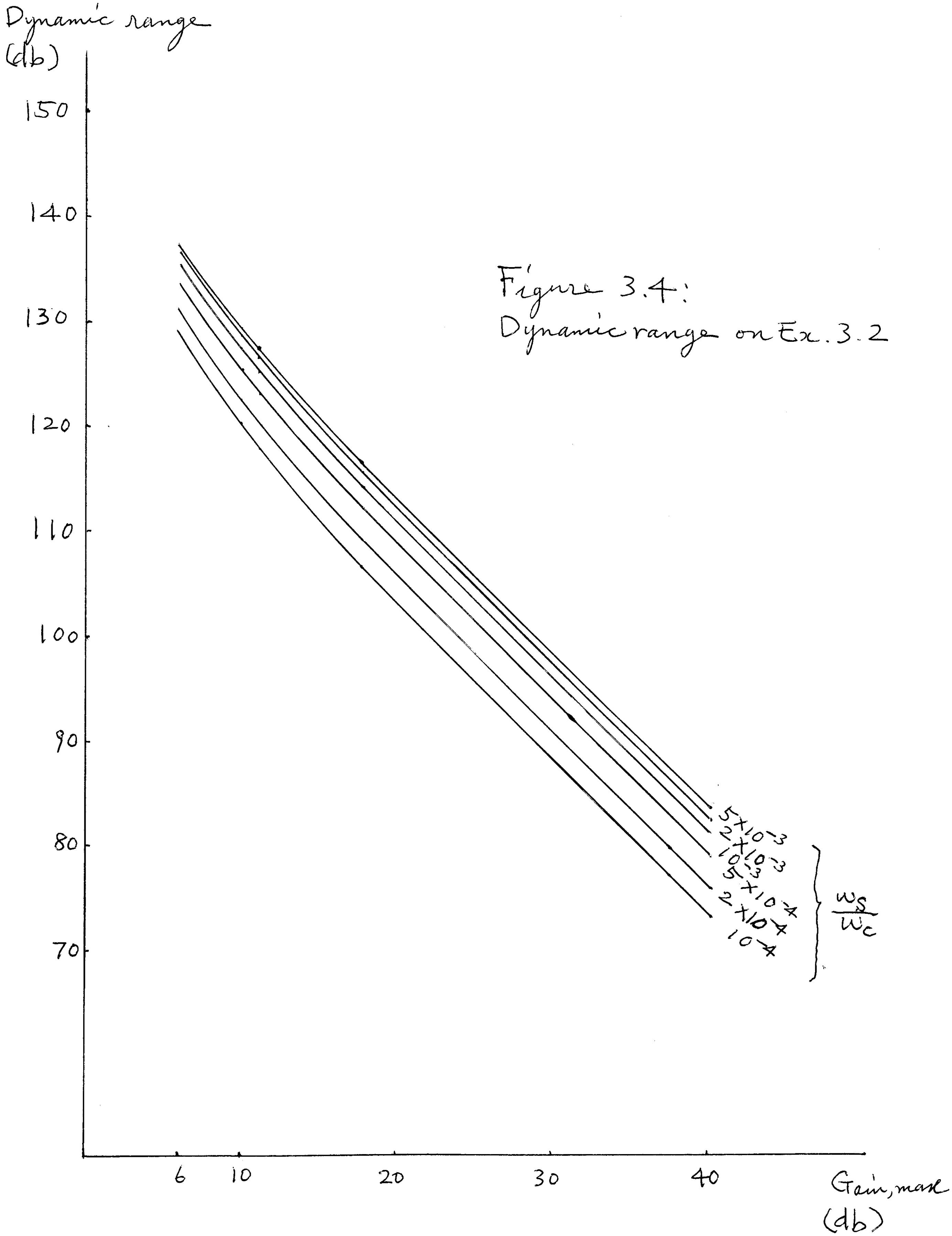


Figure 3.4: Dynamic range on Ex. 3.2

CHAPTER IV

PHASE SHIFT OF THE AMPLIFIER

4.1 Signal frequency reactance

From Eqs. 2.10 and 2.17, reactance around signal loop is

$$jX_s = jR_s \frac{1}{s} \alpha \left[- i + \frac{\frac{1}{4i} \frac{1}{1+\alpha^2}}{\left(2 + \frac{m_s^2}{pi} \frac{1}{1+\alpha^2}\right)^2 + \left(\frac{m_s^2}{pi} \frac{\alpha}{1+\alpha^2} - \alpha \frac{i}{p}\right)^2} \right] \tag{4.1}$$

The last term inside the bracket is, with a step by step algebra and approximation,

$$\frac{\frac{1}{4}(1+\alpha^2) p^2 i}{\left[(1+\alpha^2) p i 2 + m_s^2\right]^2 + \left[m_s^2 \alpha - (1+\alpha^2) \alpha i^2\right]^2}$$

$$\approx \frac{1}{16i} \left[1 - m_s^2 \frac{1}{pi} + m_s^4 \frac{3}{4p^2 i^2} - m_s^6 \frac{1}{2p^3 i^3} - \alpha^2 \left(2 + \frac{i^2}{4p^2}\right) - \alpha^2 m_s^2 \left(\frac{2}{pi} - \frac{1}{2p^2}\right) \right] \tag{4.2}$$

The criterion of approximation is the same discussed at sections 2.4 and 3.1.

4.2 Phase shift of the amplifier

Through this work, impedance in three frequencies are assumed to be tuned when the signal does not flow the varactor. As a consequence of this, obviously phase shift is none when the signal is small enough.

From Eqs. 3.3 and and 4.2, the phase shift between input and output signal is, with the same kind of approximation as the one discussed at section 3.1,

$$\tan^{-1}\theta \approx \frac{-\frac{\alpha i}{s} + \frac{\alpha}{16 s i}\left[1 - m_s^2 \frac{1}{pi}\right]}{\frac{R_o}{R_s} + 1 - \frac{1}{16si}\left[1 - m_s^2 \frac{1}{pi}\right]} \tag{4.3}$$

This is practically accurate enough for $10 \log_{10} G = 6$.

From Eqs. 3.5 and 4.3

$$\tan^{-1}\theta = \frac{\alpha}{s} \frac{-i + \frac{1}{16i}\left[1 - \frac{m_s^2}{pi}\right]}{r + \frac{m_s^2}{pi} \frac{1}{16si}}$$

and from Eq. 2.29

$$\tan^{-1}\theta = -\frac{m_s^2}{2s}\,\frac{\left[4-\frac{m_s^2}{16pi^3}\right]\left[-1+\frac{1}{16i^2}\left(1-\frac{m_s^2}{pi}\right)\right]}{r+\frac{m_s^2}{pi}\frac{1}{16si}}$$

$$= -\frac{m_s^2}{2s}\,\frac{\left(\frac{1}{4i^2}-4\right)-\left(3+\frac{1}{16i^2}\right)\frac{m_s^2}{16pi^3}+\frac{m_s^4}{16^2p^2i^6}}{r+\frac{m_s^2}{pi}\frac{1}{16si}} \qquad (4.4)$$

From Eqs. 3.23 and 3.30a

$$m_s^2 = \frac{i}{s}\,\frac{P_{out}}{P_{norm}}\,\frac{1}{\frac{1}{2}+\frac{1}{\sqrt{G_{max}}}} \qquad (4.5)$$

Defining

$$\beta = \frac{m_s^2}{2i} = \frac{1}{s}\,\frac{P_{out}}{P_{norm}}\,\frac{1}{1+\frac{2}{\sqrt{G_{max}}}} \qquad (4.6)$$

with Eq. 3.25, Eq. 4.4 becomes

$$\tan^{-1}\theta = -2\beta \frac{(1 - 16i^2) - (3 + \frac{1}{16i^2})\frac{1}{2p}\beta + \frac{1}{16p^2 i^2}\beta^2}{\frac{1}{\sqrt{G_{max}}} + \frac{1}{p}\beta} \qquad (4.7)$$

The equation 4.7 is valid when

$$1/4 > i > 1/12$$

$$s < \frac{1}{10} i$$

$$10 \log_{10} G_{max} > 6$$

where

$$P_{norm} = \frac{(V_B + \phi)^2}{R_s}$$

G_{max} = maximum gain (when the signal is small)

When $i = 1/4$, keeping s and G_{max} as constants, the phase shift becomes minimum. Furthermore, at the situation, the phase shift increases as the signal increases having zero slope at zero signal. This fact is most interesting one.

When i is larger than 1/4 the phase shift becomes positive. The variable β is proportional to P_{ont} and inverse proportional to s . According to Eqs. 4.6 and 4.7, as G_{max} increases the phase shift increases in magnitude.

4.3 Maximum phase shift

While idler frequency determines the slope of increasing phase shift versus increasing signal, the maximum phase shift is defined as a phase shift when the gain drops 3db from the maximum gain. So, this value gives the order of the phase shift. From Eqs. 4.6 and 3.26

$$\beta_{max} \equiv \frac{m_{s\,max}^2}{2i}$$

$$= 0.414 \frac{i}{\sqrt{G_{max}}} \tag{4.8}$$

and

$$\tan^{-1}\theta_{max} = \tan^{-1}\theta \,\Big|_{\substack{at \\ \beta=\beta_{max}}} \tag{4.9}$$

with Eq. 4.7.

4.4 Examples on phase shift

Example 1

When $i = 1/4$ and $s < \frac{1}{10} i$, phase shift θ is

$$\tan^{-1}\theta = \frac{16\beta^2}{\frac{1}{\sqrt{G_{max}}} + 4\beta}$$

with

$$\beta = \frac{1}{s} \frac{P_{out}/P_{norm}}{1 + \frac{2}{\sqrt{G_{max}}}}$$

Figures 4.1a, b, c, d are on phase shift at maximum gains 40, 30, 20, 10 decibels each. They are brought together for comparison at 4.1d.

Example 2

When $i = 1/6$ and $s < \frac{1}{15} i$, the phase shift θ is

$$\tan^{-1}\theta = -2\beta \frac{\frac{5}{9} - \frac{63}{4}\beta + 81\beta^2}{\frac{1}{\sqrt{G_{max}}} + 6\beta}$$

Figures 4.2 a, b, c are on phase shift at maximum gains 30, 20, 10 decibels each. And they are drawn together at 4.2 c.

Example 3

When $i = 1/10$ and $s < \frac{1}{20} i$, the phase shift θ is

$$\tan^{-1} \theta = -2\beta \frac{0.84 - 46.2\beta + 624\beta^2}{\frac{1}{\sqrt{G_{max}}} + 10\beta}$$

Figures 4.3 a, b, c are on the phase shifts at maximum gains 30, 20, 10 decibels each. And they are together at 4.3 c.

The differences of slopes, convexities and magnitudes among all the figures are interesting.

Figures 4.4, 4.5 and 4.6 are groupings of preceding figures in same maximum gains and different idler frequencies. The restrictions on signal frequencies at Examples 2 and 3 came from algebraic conveniences. And, Eq. 4.7 satisfies to $s = \frac{1}{10} i$.

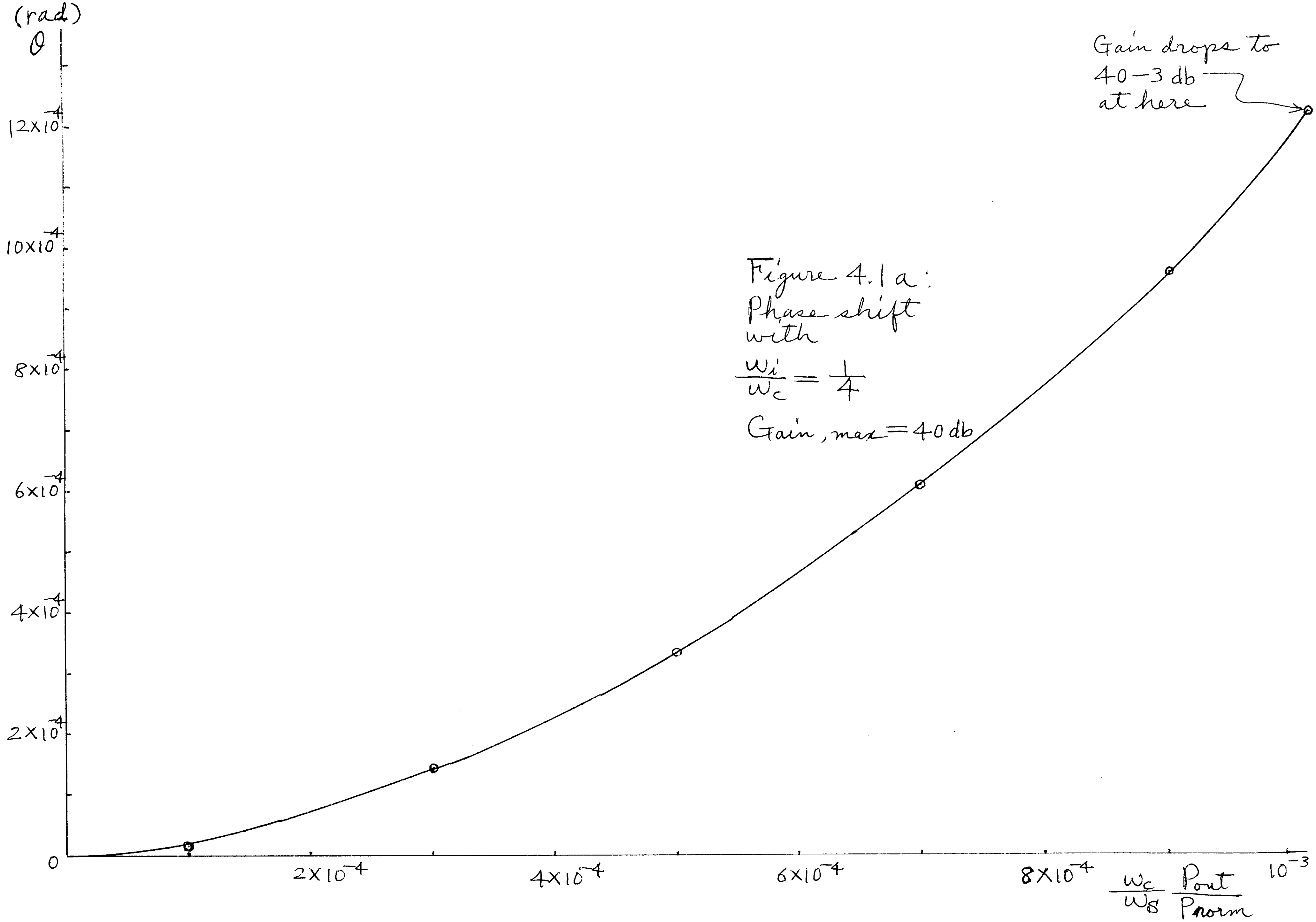
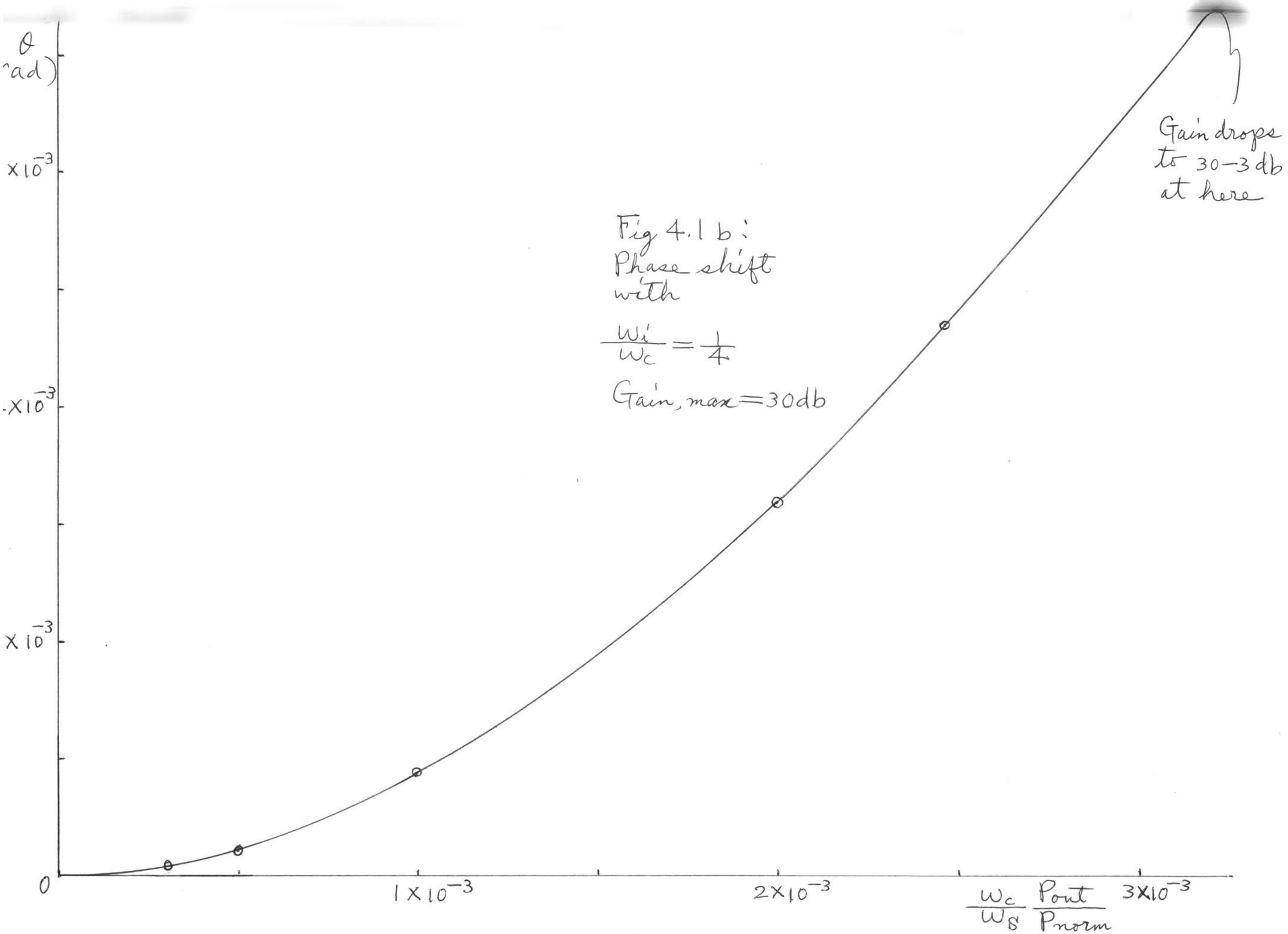
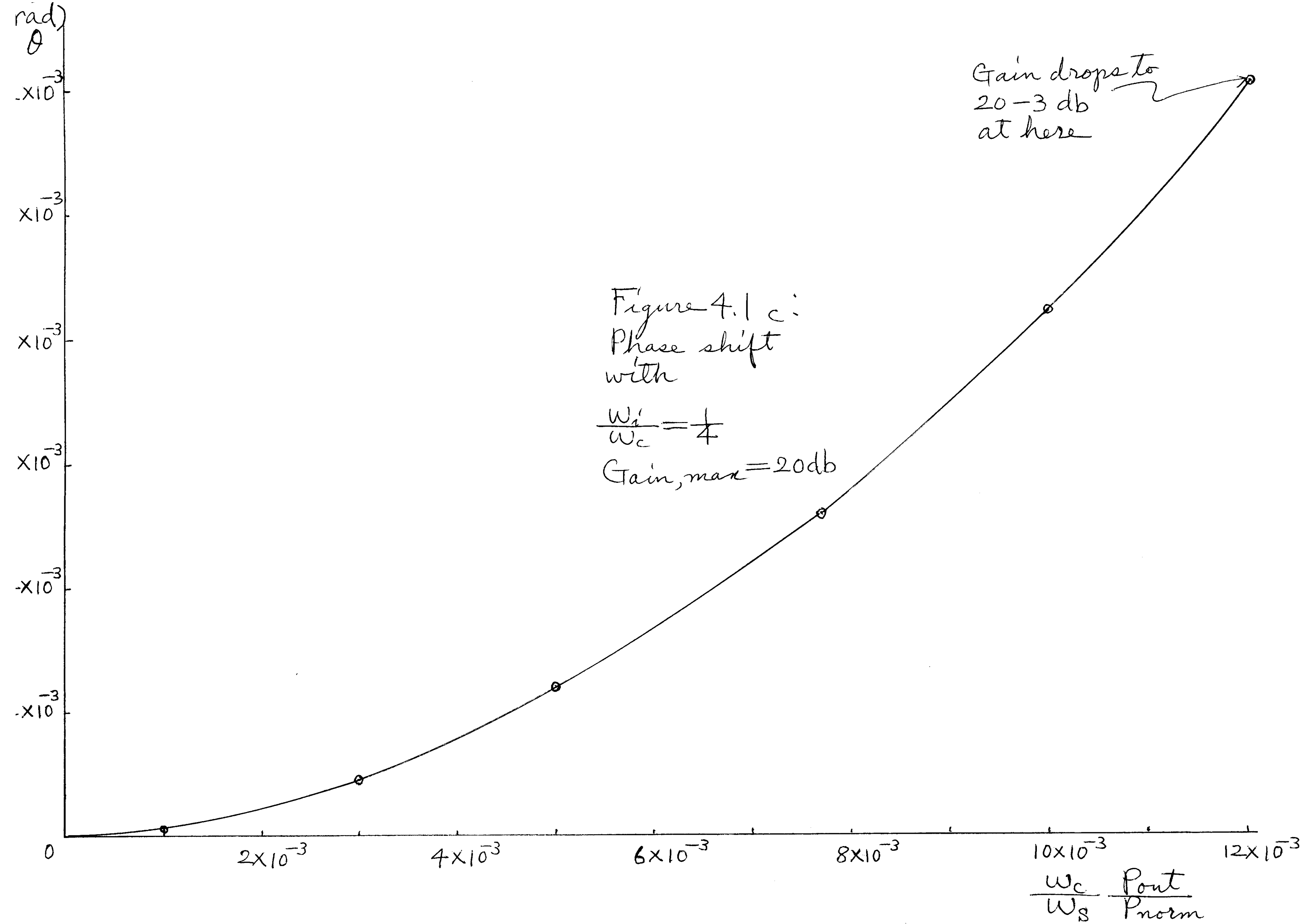


Figure 4.1a: Phase shift with $\frac{\omega_i}{\omega_c} = \frac{1}{4}$

Gain, max = 40 db



θ
(rad)
x10⁻³
x10⁻³
x10⁻³
0
1x10⁻³
2x10⁻³
3x10⁻³
Wc/Ws Pout/Pnorm
Fig 4.1 b: Phase shift with
Wi/Wc = 1/4
Gain, max = 30db
Gain drops to 30-3 db at here


rad)
θ
.X10^-3
X10^-3
X10^-3
X10^-3
-X10^-3
.X10^-3
0
2X10^-3
4X10^-3
6X10^-3
8X10^-3
10X10^-3
12X10^-3
Gain drops to
20−3 db
at here
Figure 4.1 c:
Phase shift
with
Wi'/Wc = 1/4
Gain, max = 20db
Wc/Ws Pout/Pnorm

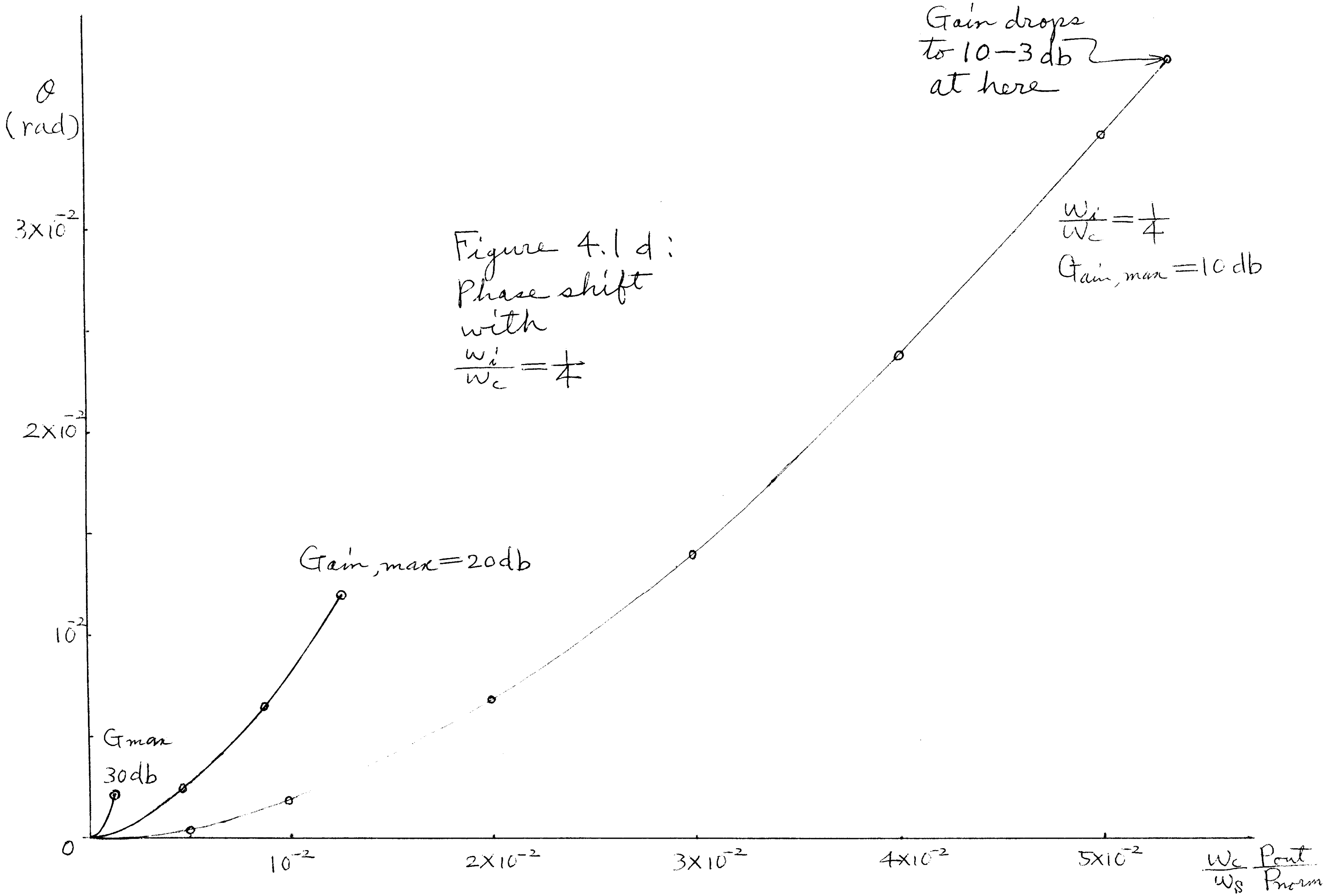
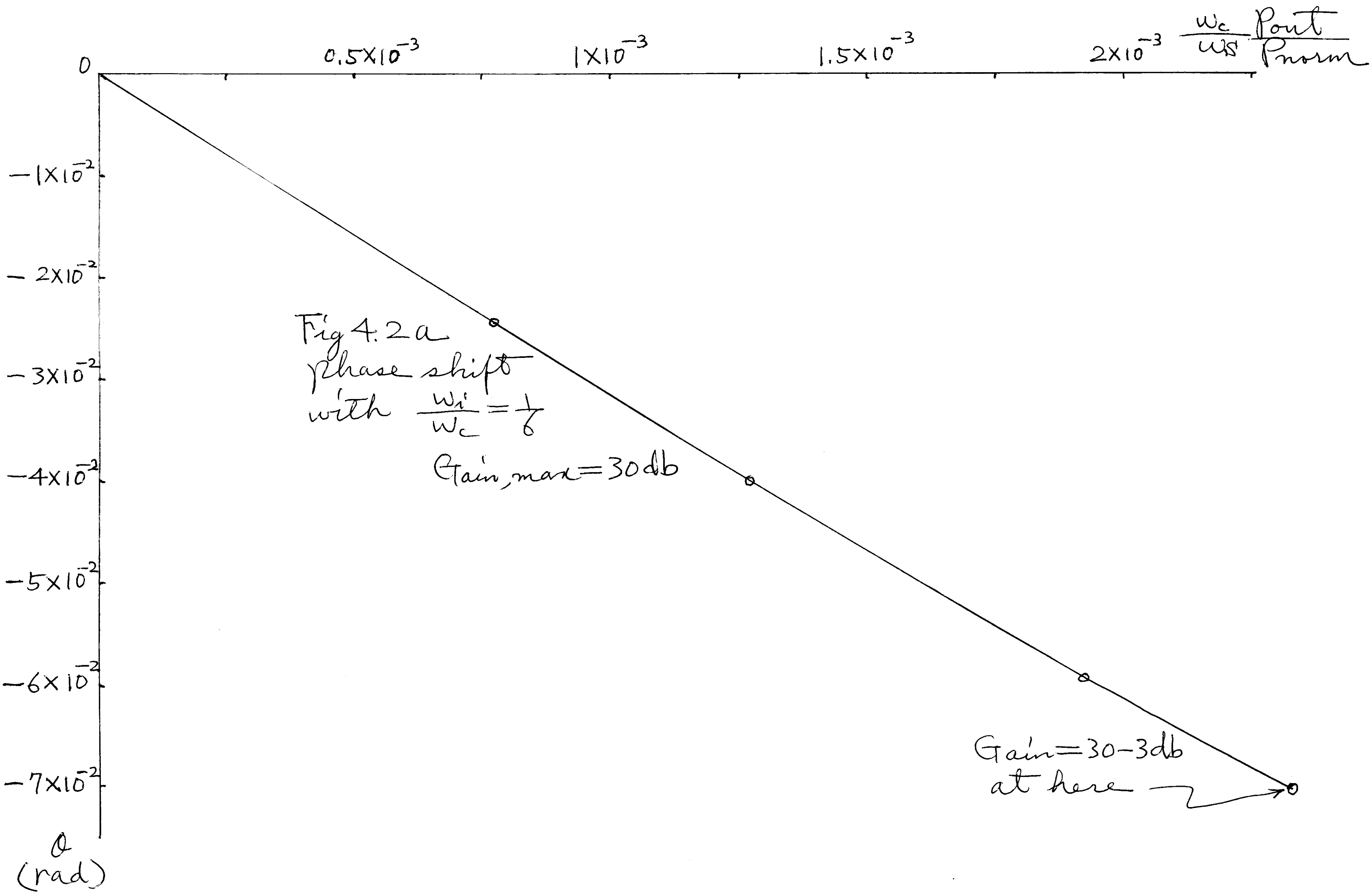


Figure 4.1 d: Phase shift with $\frac{\omega_i}{\omega_c} = \frac{1}{4}$



0
0.5X10^-3
1X10^-3
1.5X10^-3
2X10^-3
(wc/ws)(Pout/Pnorm)
-1X10^-2
-2X10^-2
-3X10^-2
-4X10^-2
-5X10^-2
-6X10^-2
-7X10^-2
θ (rad)
Fig 4.2a
phase shift
with wi/wc = 1/6
Gain, max = 30db
Gain = 30-3db
at here

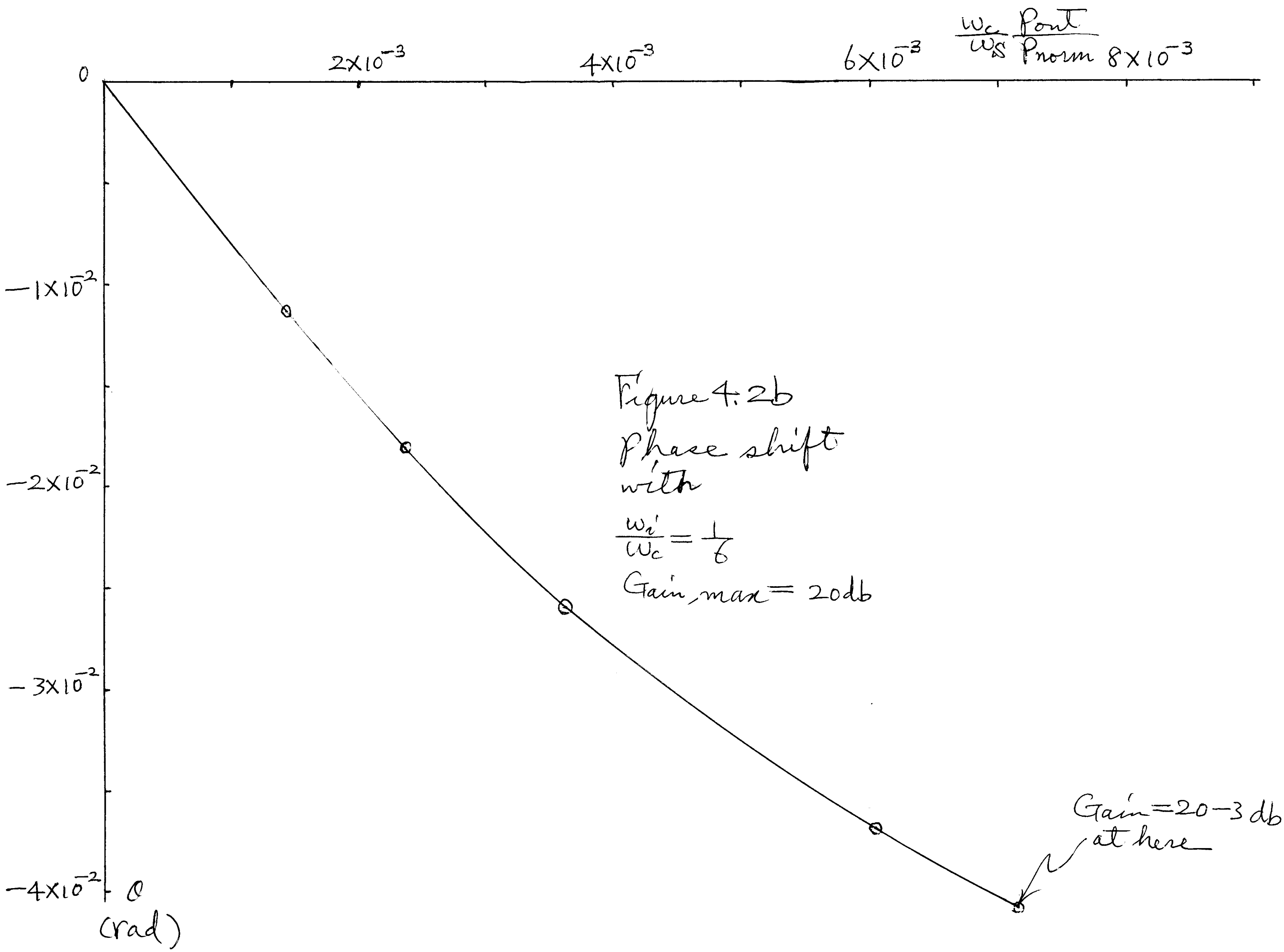
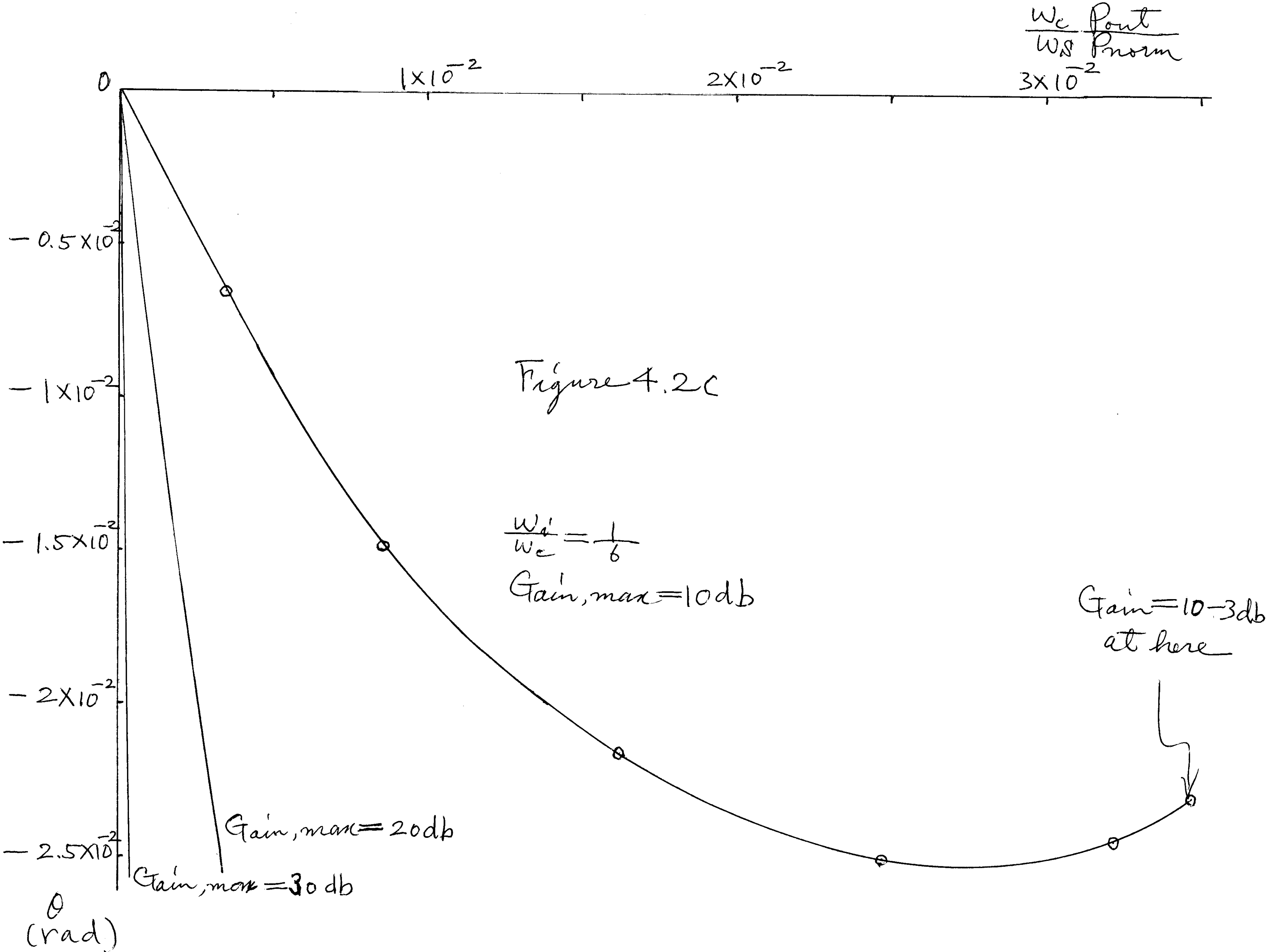


Figure 4.2b
Phase shift with
$\frac{\omega_i'}{\omega_c} = \frac{1}{6}$
Gain, max = 20db



$\frac{W_c}{W_s} \frac{P_{out}}{P_{norm}}$
0
1×10^{-2}
2×10^{-2}
3×10^{-2}
-0.5×10^{-2}
-1×10^{-2}
-1.5×10^{-2}
-2×10^{-2}
-2.5×10^{-2}
θ (rad)
$\frac{W_i}{W_c} = \frac{1}{6}$
Gain, max = 10db
Gain = 10-3db at here
Gain, max = 20db
Gain, max = 30 db

Figure 4.2c

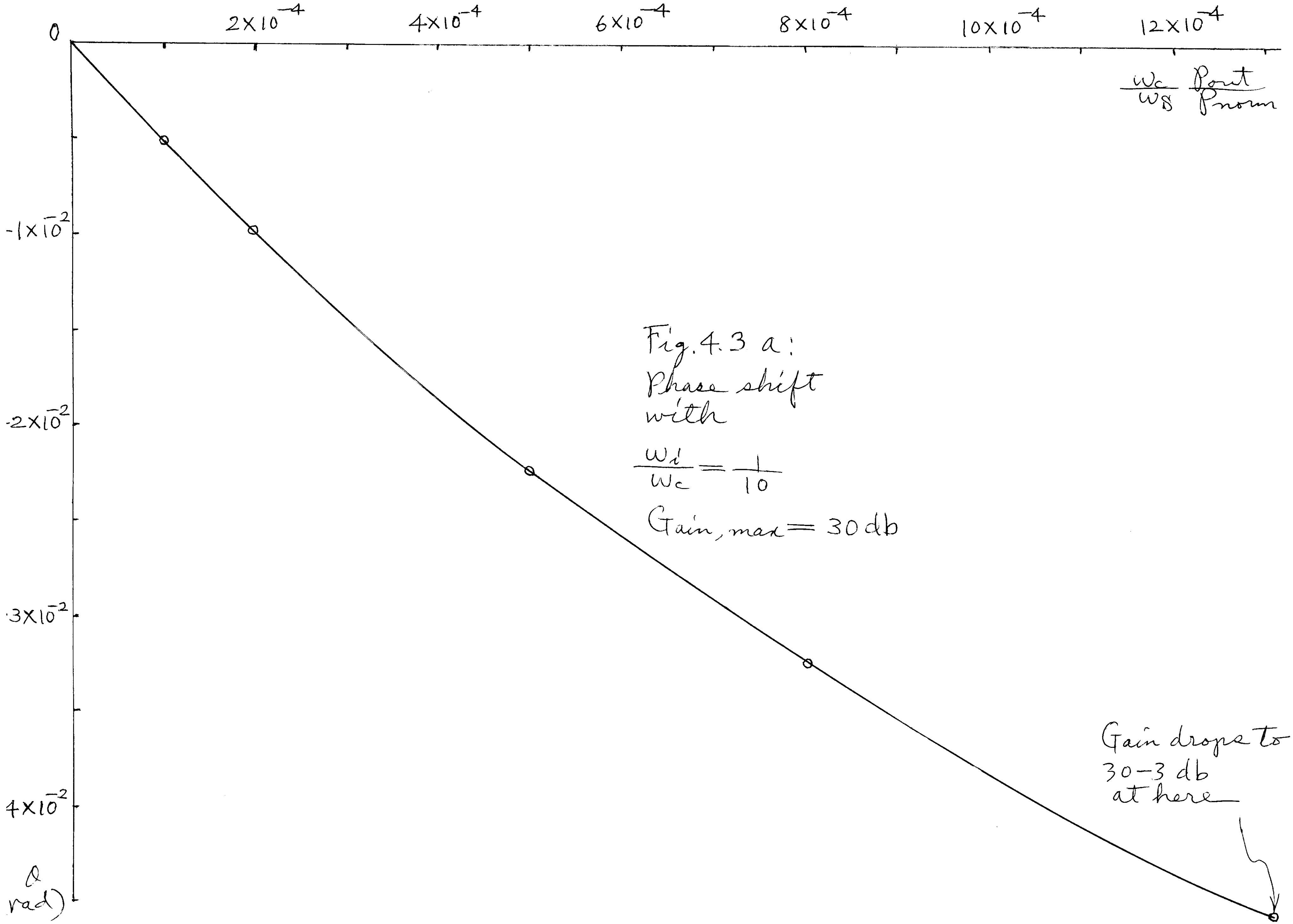


Fig. 4.3 a: Phase shift with $\frac{\omega_l}{\omega_c} = \frac{1}{10}$

Gain, max = 30 db

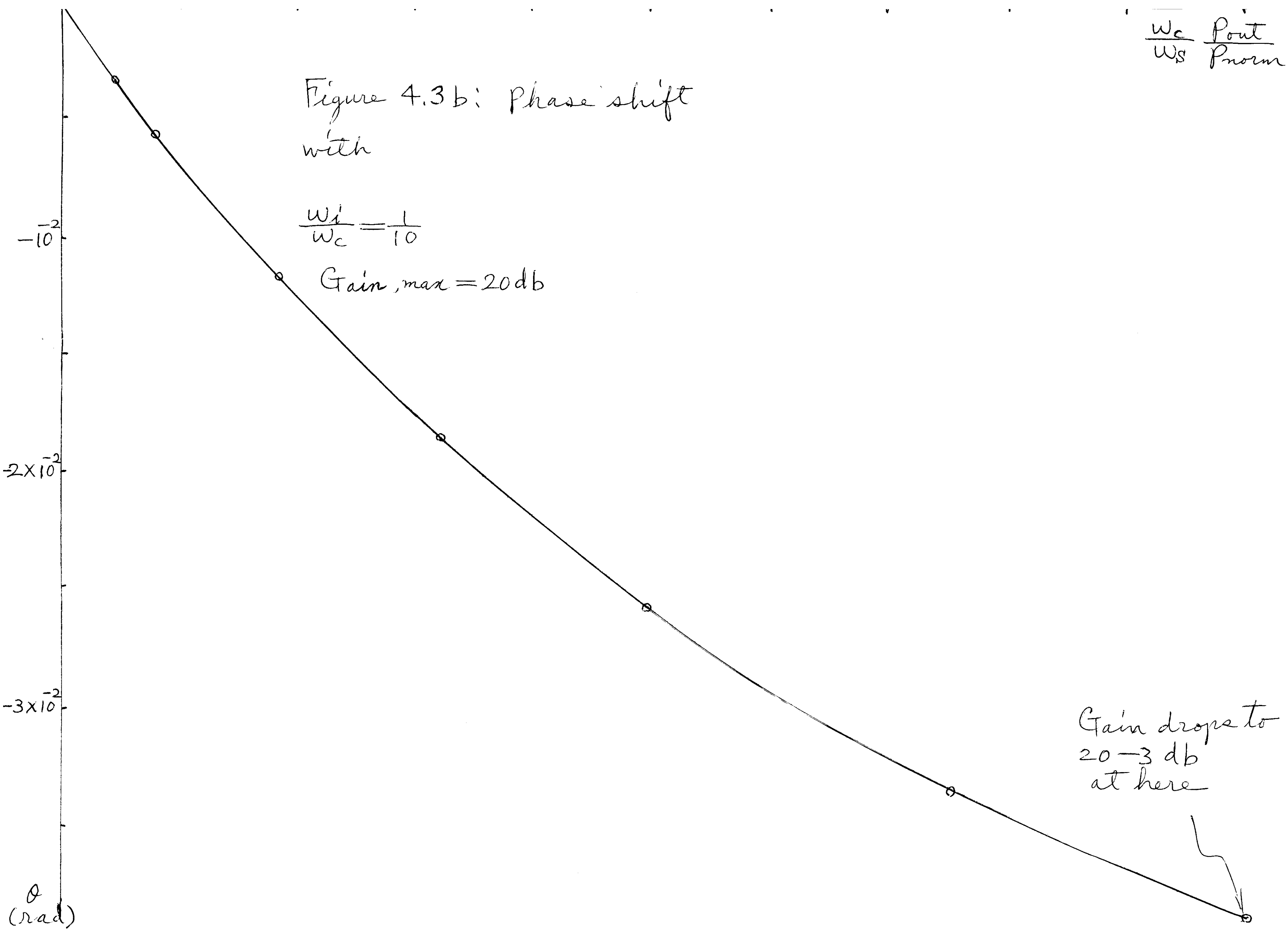
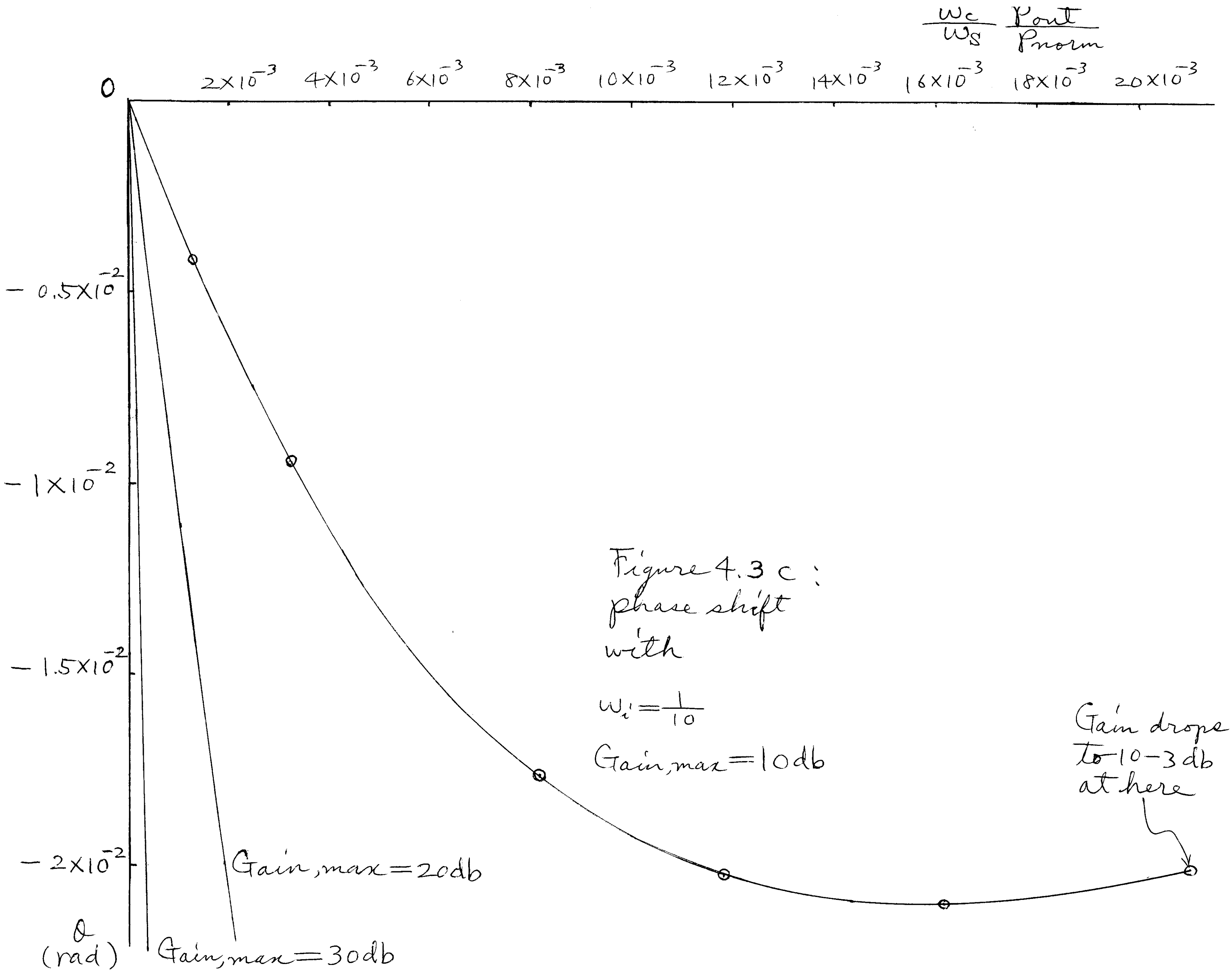


Figure 4.3b: Phase shift with

$$\frac{w_i}{w_c} = \frac{1}{10}$$

Gain, max = 20 db



$\frac{\omega_c}{\omega_s}$ $\frac{P_{out}}{P_{norm}}$
0
2X10^{-3}
4X10^{-3}
6X10^{-3}
8X10^{-3}
10X10^{-3}
12X10^{-3}
14X10^{-3}
16X10^{-3}
18X10^{-3}
20X10^{-3}
− 0.5X10^{-2}
− 1X10^{-2}
− 1.5X10^{-2}
− 2X10^{-2}
θ (rad)
Gain, max = 30db
Gain, max = 20db
Figure 4.3 c: phase shift with
$\omega_i = \frac{1}{10}$
Gain, max = 10db
Gain drops to 10−3 db at here

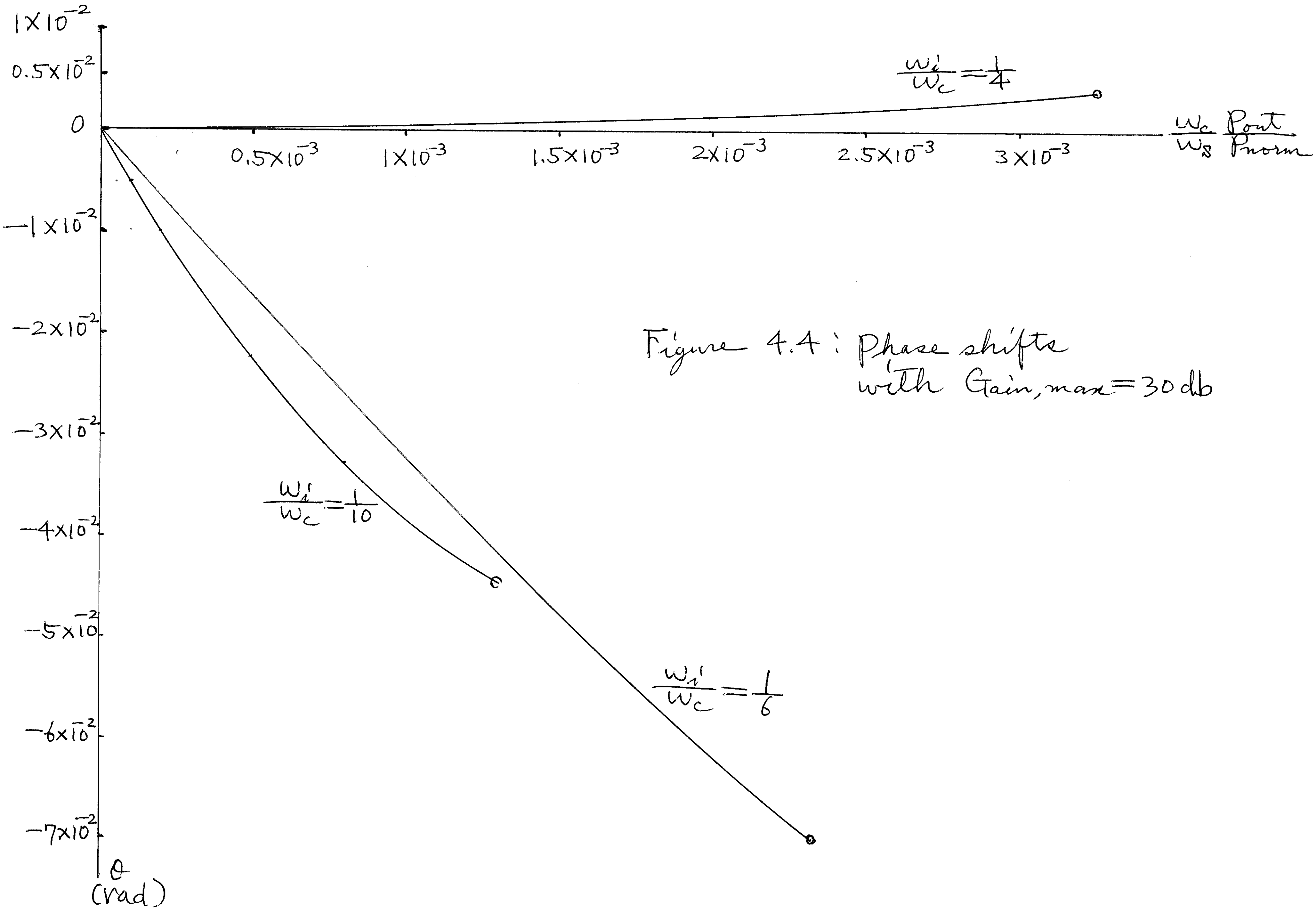


Figure 4.4: Phase shifts with Gain, max = 30 db

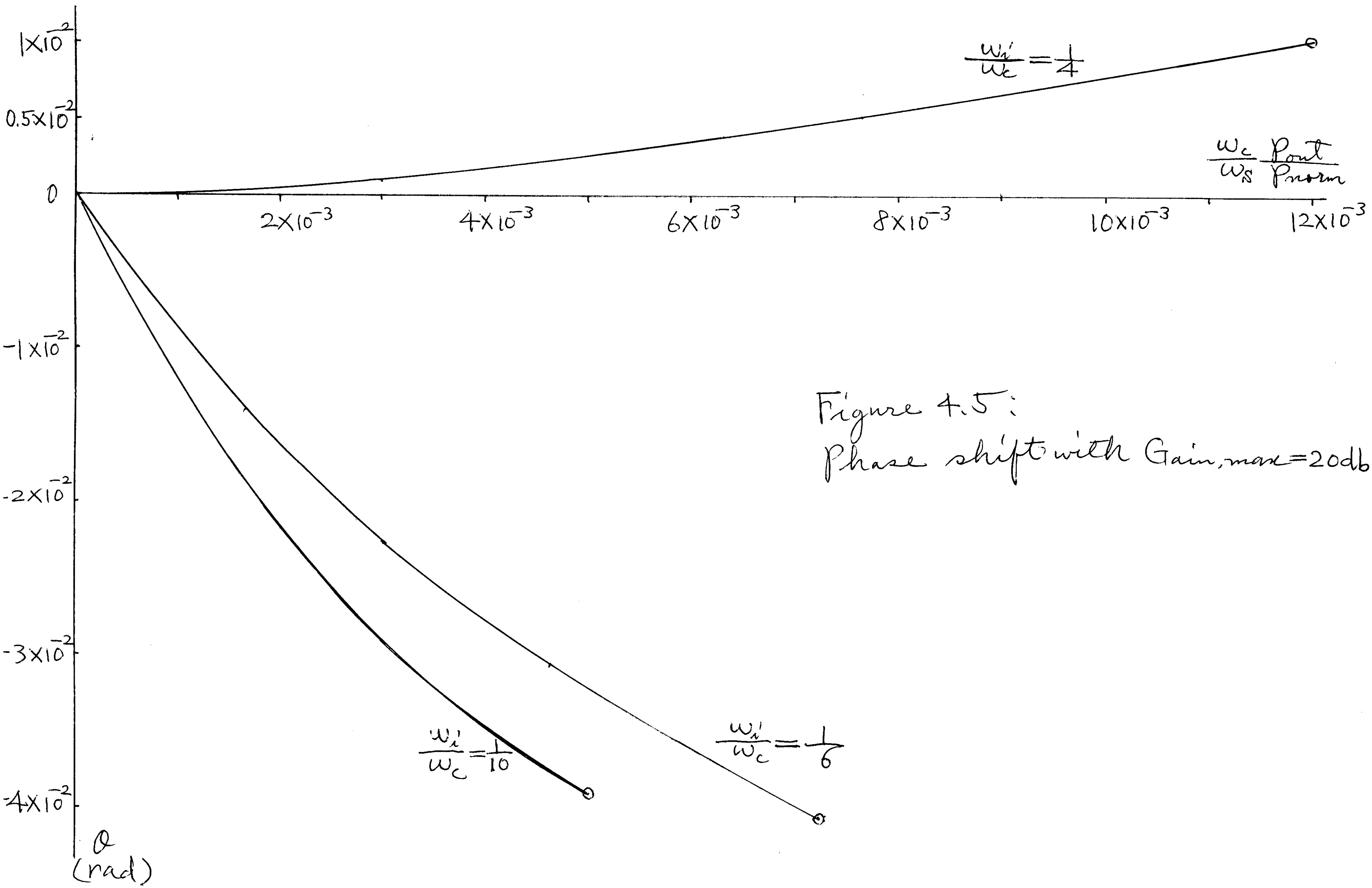
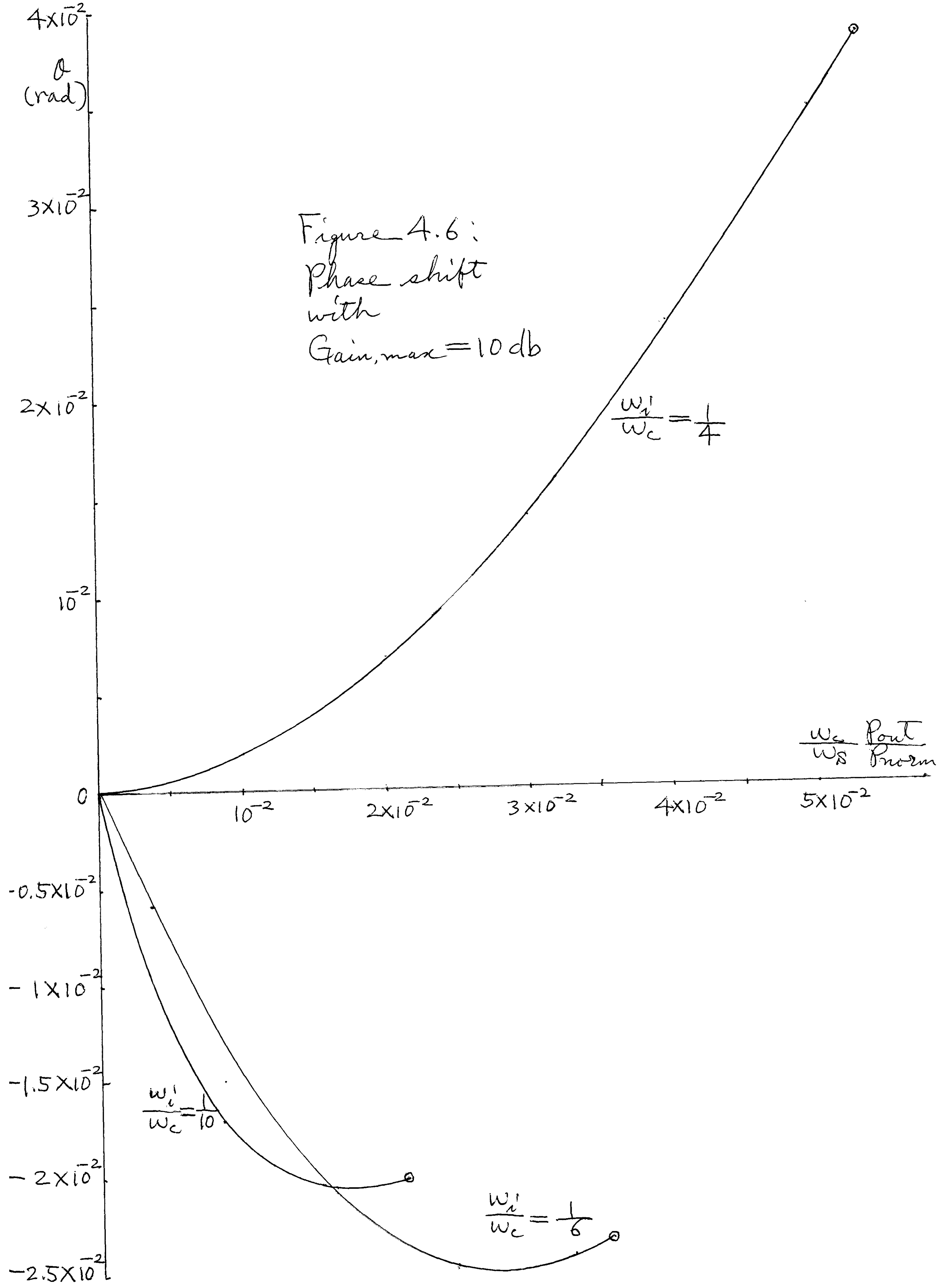


Figure 4.5:
Phase shift with Gain, max = 20db



4x10⁻²
θ (rad)
3x10⁻²
2x10⁻²
10⁻²
0
-0.5x10⁻²
-1x10⁻²
-1.5x10⁻²
-2x10⁻²
-2.5x10⁻²
10⁻²
2x10⁻²
3x10⁻²
4x10⁻²
5x10⁻²
$\frac{w_c}{w_s}\frac{P_{out}}{P_{norm}}$
Figure 4.6: Phase shift with Gain, max = 10 db
$\frac{w_i'}{w_c} = \frac{1}{4}$
$\frac{w_i'}{w_c} = \frac{1}{10}$
$\frac{w_i'}{w_c} = \frac{1}{6}$

CHAPTER V

CONCLUSIONS

5.1 Conclusions

As a varactor to form a parametric amplifier, abrupt junction diodes are selected. According to the physical structure of the diodes, a series model of the varactor with a constant resistance and a voltage controlled capacitance is determined.

As a consequence of the series model of the varactor, a choice is made to flow the current in specially selected frequencies only. A current controlled elastance has replaced the capacitance in the series varactor model. Fourier series analysis is used in signal, pump, and idler frequency circuits.

Normalized Fourier components of elastance variations were mathematical media to derive the dynamic range and phase shift from the impedances of three frequencies. Any one of four normalized elastance variations can determine the other three variations. But the accurate relations among them are too complicated and good approximations of them are valuable.

An approximation of direct frequency component of normalized elastance variation from the value when signal is weak,

$$\delta m_o = -\frac{m_s^2}{2} \left(4 - \frac{m_s^2}{16pi^3}\right) \tag{2.29}$$

is selected to calculate the phase shift of the amplifier and used throughout to approximate other equations. The approximation is accurate enough in the regions of parameters and frequencies,

$$m_s < \frac{1}{20}$$

$$\frac{1}{4} > [i, p] > \frac{1}{12} \tag{2.26}$$

$$s < \frac{1}{10} i$$

and this regions are wide enough for the amplifier in practical application.

The gain of the amplifier is obtained as

$$\sqrt{\text{Gain}} = \frac{2i^2}{16si^3 r + m_s^2} \tag{3.8}$$

where

$$r = \frac{R_o}{R_s} + 1 - \frac{1}{16si} \tag{3.5}$$

This is a good approximation, when maximum gain is larger than 6 db.

The dynamic range was given in the form, in db,

$$10 \log_{10} DR = 10 \log_{10} P_{norm} - 15 \log_{10} G_{max} - 10 \log_{10} \Delta f$$

$$- 10 \log_{10} \left[\frac{16 + \frac{1}{i^2}}{1 + \frac{4}{\sqrt{G_{max}}}} \frac{T_d}{T_n} + \frac{1}{si(1 + \frac{2}{\sqrt{G_{max}}})} \frac{T_a}{T_n} \right]$$

$$+ 203 \qquad (3.32)$$

For given P_{norm}, Δf, i, s, T_d and T_a, the dynamic range is inverse proportional to G_{max} (maximum gain when the signal is weak) when maximum gain is larger than around 20 db, and extremely increases for smaller gain than 15 db. The range is the input power range from the same power as noise power to the power which brings -3 db gain from the maximum gain of the amplifier.

The phase shift of the amplifier is obtained as, θ being the phase shift,

$$\tan^{-1}\theta = -2\beta \frac{(1 - 16i^2) - (3 + \frac{1}{16\,i^2})\frac{1}{2p}\beta + \frac{1}{16\,p^2\,i^2}\beta^2}{\frac{1}{\sqrt{G_{max}}} + \frac{1}{p}\beta} \tag{4.7}$$

with

$$\beta = \frac{1}{s}\frac{P_{out}}{P_{norm}}\frac{1}{1 + \frac{2}{\sqrt{G_{max}}}} \tag{4.6}$$

The idler frequency $i = \frac{\omega_i}{\omega_c}$ is important factor to determine the phase shift. And we should not forget that β is proportional to P_{out} divided by s for given i and G_{max}. Generally larger gain makes larger phase shift.

The dynamic range and phase shift are good approximation when the maximum gain is larger than 6 db, and they give considerable error at the 6 db, but they bring valuable results even at the 6 db maximum gain.

5.2 Future work

The following are among some of the problems which need future work.

The experimental study is needed to compare with the results obtained here. If the two kinds of results do not meet in some cases or regions of frequencies or powers it will be possible to find the factor and add them to the theory obtained here.

As a whole problem of the parametric amplifier, an analysis of noise, when large signal is present, will be interesting.

It is necessary to study the parametric amplifier in not sinusoidal signals.

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