Essays on Strategic Communication

by

Anton Kolotilin

B.Sc. Applied Physics and Mathematics, Moscow Institute of Physics and Technology 2005
M.Sc. Applied Physics and Mathematics, Moscow Institute of Physics and Technology 2007
M.A. Economics, New Economic School 2007

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Economics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2012

Signature of Author ....
Department of Economics
May 15, 2012

Certified by..
Robert S. Gibbons
Sloan Distinguished Professor of Management and Economics
Thesis Supervisor

Certified by.. ....
Muhamet Yildiz
Professor of Economics
Thesis Supervisor

Accepted by ....
Michael Greenstone
3M Professor of Environmental Economics
Chairman, Departmental Committee on Graduate Studies

Essays on Strategic Communication

by

Anton Kolotilin

Submitted to the Department of Economics
on May 15, 2012, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Economics

Abstract

The first chapter studies optimal information revelation with one-sided asymmetric information. A sender chooses ex ante how her information will be revealed ex post. A receiver obtains both public information and information revealed by the sender, and then takes one of two actions. The sender wishes to maximize the probability that the receiver takes the desired action. The sender optimally reveals only whether the receiver's utility is above a cutoff. The cutoff is such that the receiver is indifferent between the two actions when he learns that his utility is above the cutoff. The sender's welfare increases and the receiver's welfare does not change with the precision of the sender's information. The sender's welfare decreases and the receiver's welfare increases with the precision of public information.

The second chapter studies optimal information revelation with two-sided asymmetric information. A sender chooses ex ante how her information will be revealed ex post with the goal of persuading an informed receiver to take one of two actions. The sender faces a tradeoff between the frequency and the persuasiveness of messages: sending positive messages more often (in terms of the sender's private information) makes it less likely that the receiver will take the desired action (in terms of the receiver's private information). Under the optimal mechanism, the sender's and receiver's welfare is not monotone in the precision of the receiver's private information. I provide necessary and sufficient conditions when the full information revelation is optimal and when the no information revelation is optimal.

The third chapter (co-authored with Li Hao and Wei Li) studies a principal-agent problem where the only commitment for the uninformed principal is to restrict the set of decisions she makes following a report by the informed agent. Compared to no commitment, the principal improves the quality of communication from the agent. An ex ante optimal equilibrium for the principal corresponds to a finite partition of the state space, and each retained decision is suboptimal for the principal, biased toward the agent's preference. Generally an optimal equilibrium does not maximize the number of decisions the principal can credibly retain.

Thesis Supervisor: Robert S. Gibbons
Title: Sloan Distinguished Professor of Management and Economics

Thesis Supervisor: Muhamet Yildiz
Title: Professor of Economics

Contents

[1]**This chapter is co-authored with Li Hao and Wei Li.**

Acknowledgments

First of all, I thank my advisors, Robert Gibbons and Muhamet Yildiz, for their invaluable guidance and advice. I have been inspired by their dedication and wisdom.

I also thank Victor Chernozhukov, Glenn Ellison, Parag Pathak and Juuso Toikka for many interesting and insightful conversations. I have been fortunate to receive helpful comments from many other faculty members and visitors at MIT. I have greatly benefited from my co-authors, especially from Sergei Guriev, Li Hao, and Wei Li.

My classmates have made the past five years very enjoyable and interesting. I thank Gabriel Carroll, Denis Chetverikov, Michael Powell, Alexander Wolitzky, Juan Xandri, and Luis Zermeno for helpful conversations on my research projects.

Finally, I thank my parents, Dmitriy and Elena, and my brother, Dima, for their constant love and support.

Chapter 1

Optimal Information Revelation with Uninformed Receiver

1.1 Introduction

This chapter studies the optimal design of information revelation by a sender who has an interest in a non-contractible action taken by an uninformed receiver. The central questions of this chapter are: First, how much and which part of information is optimally revealed by the sender? Second, how do the sender's and receiver's expected utilities under the optimal revelation mechanism depend on information structure?

To answer these questions, I consider the following sender-receiver game. The receiver has a binary action choice: to act or not to act. The sender's utility depends only on the action taken by the receiver, and she prefers the receiver to act. The receiver's utility depends both on his action and on information. The receiver takes an action that maximizes his expected utility given his belief. He forms his belief based on public information and information revealed by the sender. The sender chooses ex ante how information will be revealed to the receiver ex post. Formally, she can choose any conditional distribution of messages given information. I call this distribution a *mechanism*. The sender chooses the mechanism that maximizes the ex ante probability that the receiver will act. No monetary transfers between the sender and receiver are allowed.

The drug approval process by the Food and Drug Administration (FDA) is a good example of my model. It takes on average between 8 to 12 years of studies and testing for a pharmaceutical company (PC) to get a new drug approved by the FDA. The FDA requires a PC to submit a research protocol for each test that the PC is planning to undertake. The research protocol includes both required tests (for example, Ames test on toxicity) and tests designed by the PC. The FDA closely monitors the record keeping and the adherence to the research protocol. If the test results are satisfactory, the PC submits an application that contains all the information obtained during the testing phase. Penalties for fraudulent or misleading claims are imposed. Finally, the FDA either approves the drug or rejects it.[1]

To see that my model is a good approximation of the drug approval process, let us reinterpret the PC as the sender, the FDA as the receiver, the FDA's approval decision as the receiver's action, the research protocol as the sender's choice of a mechanism, the results of the required tests as public information, and the results of the remaining tests as a message. Note that because of the length of the drug approval process, the PC has presumably little information at the beginning of the process. Furthermore, because of the large cost of the process (averaging more than $500 million) and large benefits of approval, the PC has strong incentives to optimally design tests to maximize the probability of the FDA's approval. In particular, the PC chooses dosage and characteristics of volunteer patients, such as gender, age, and health condition. Due to the FDA's regulation and close monitoring, the FDA essentially observes both the design and results (whether they are positive or negative) of all tests. Finally, based on the results of these tests, the FDA approves the drug if its benefits outweigh its costs and risks.

Note that I study a general problem of information revelation that fits many real-life examples well, such as scientific experiments by interested parties, grading policy chosen by a school, and accounting rules chosen by a firm. Of course, my model does not fit any of these examples perfectly. For example, in contrast to my model, the FDA has some commitment power in its approval decision, the PC cares not only about the probability of approval, the PC has constraints on its research protocol. However, to give economic meaning to my results, I will explain them using the drug approval example despite the inconsistencies between the model and the example.

Under the optimal design, the tests produce only two possible outcomes: a positive outcome that gives minimal evidence sufficient for approval or a negative outcome that gives maximal evidence against approving. More specifically, if the drug quality is above the cutoff, then the optimal tests produce the positive outcome that persuades the FDA to approve the drug, otherwise the tests produce the negative outcome that persuades the FDA to reject it. The cutoff quality is determined by the condition that the FDA is indifferent between approving and rejecting the drug

[1] The description of the drug approval process is taken from Lipsky and Sharp (2001).

when the FDA gets positive news from the PC.[2] This design maximizes the probability of drug approval given the feasibility constraint that the FDA cannot be fooled on average.

How does the PC's welfare (the probability of the drug approval) depends on information structure? First, if the PC is able to design more informative tests, then its welfare increases. This is because the PC can choose any mechanism, so it has free disposal of information. For example, if the PC is allowed to run tests on 500 subjects, it can actually run tests on only 100 of them. Second, if the FDA is optimistic about the drug quality, then the PC's welfare increases as well. This is because the PC finds it easier to persuade the FDA to approve the drug if the FDA believes that the drug is good in the first place. Interestingly, these two conditions are not only sufficient but also necessary for the PC's welfare to increase regardless of the FDA's opportunity cost of approving the drug. Finally, if public information becomes more precise, the PC's welfare decreases. That is, if there are more required tests that the PC must to carry out, then the probability that the FDA will approve the drug decreases because the PC has less freedom in manipulating the FDA.

How does the FDA's welfare (the expected quality of approved drugs) depend on information structure? Surprisingly, the FDA's welfare does not change if the PC is able to design more informative tests. This is because the optimal design of tests leaves no rent to the FDA. However, the FDA's welfare increases with the precision of public information. That is, the quality of approved drugs increases if the FDA requires the PC to run more tests.

Although the above monotone comparative statics results are intuitive, they do not generally hold in the large existing literature where the sender chooses what information to reveal when she already has her private information. In particular, they do not hold under cheap talk and verifiable communication. This happens because in this literature, the sender faces an incentive compatibility constraint on information revelation, which is missing in my model because the sender has commitment in information revelation.

What happens if the FDA has some private information that is supported by hard evidence? For example, the FDA has applications from other PCs that tested similar drugs. In this case, the PC optimally elicits all private information from the FDA and then implements the optimal design of tests as if this private information was public.

What happens if the PC has some private information before it submits the research protocol? For example, the PC develops a drug and carries out preclinical trials. In this case, the PC reveals recorded information and suppresses unrecorded information.

[2] This result is closely related to Kamenica and Gentzkow (2011), which I discuss in Section 1.3.1.

There is a large literature on strategic communication. In the cheap talk literature (Crawford and Sobel (1982)), the sender sends an unverifiable message to the receiver who then takes a non-contractible action, so it must be incentive compatible both for the sender to send her equilibrium message and for the receiver to take his equilibrium action. In the delegation literature (Holmstrom (1984)), the receiver can commit to how he uses the information revealed by the sender, so only the sender faces an incentive constraint. In contrast, in Rayo and Segal (2010), Kamenica and Gentzkow (2011), and this chapter, the sender can commit to how she reveals the information to the receiver, so only the receiver faces an incentive constraint.

Kamenica and Gentzkow (2011) consider a much more general model than mine, with an arbitrary set of actions, and arbitrary sender's and receiver's utility functions. They derive some interesting properties of the optimal mechanism. To completely characterize the optimal mechanism, I impose more structure on the problem that still fits many real-life examples well. Moreover, I derive general monotone comparative statics results that relate the sender's and receiver's expected utilities to information.

There is literature on equilibrium rather than optimal information revelation in environments similar to mine (e.g.,Lerner and Tirole (2006) and Benoit and Dubra (2011)). There is also literature on optimal information revelation in environments where monetary transfers are allowed (e.g., Bergemann and Pesendorfer (2007), and Eso and Szentes (2007)).

The rest of the chapter is organized as follows. Section 1.2 presents a model. Section 1.3 completely characterizes the optimal revelation mechanism and presents monotone comparative statics results. Section 1.4 extends the model to allow the receiver's verifiable private information and the sender's ex ante private information. Section 1.5 concludes. All proofs and technical details are relegated to the appendix.

1.2 Model

Consider a communication game between a female sender and a male receiver. The receiver takes a binary action $a = 0, 1$. Say that the receiver *acts* if he takes $a = 1$ and the receiver *does not act* if he takes $a = 0$. The sender's utility depends only on a, but the receiver's utility depends both on a and on (s, r), where components s and r denote the sender's and public types. Without loss of generality, the sender's utility is a, and the receiver's utility is u_0 if $a = 0$ and s if $a = 1$.[3] Before (s, r) is realized, the sender can commit to a mechanism that sends a message m to the receiver as a (stochastic) function of (s, r); specifically, the sender chooses the conditional distribution $\phi(m|s, r)$ of m given (s, r).

Assume that the set of messages M contains at least two elements m_0 and m_1, the set of sender's types S is $[\underline{s}, \overline{s}]$ where $\underline{s} < 0 < \overline{s}$, the set of public types R is an arbitrary set that satisfies regularity conditions that ensure that all conditional expectations exist. The information (s, r) has some joint distribution. For this distribution, assume that the marginal distribution $G(r)$ of r and the conditional distribution $F(s|r)$ of s given r admit strictly positive densities $g(r)$ and $f(s|r)$.

The timing of the communication game is as follows:

1. The sender publicly chooses a mechanism $\phi(m|s, r)$.

2. A tuple (m, s, r) is drawn according to ϕ, F, and G.

3. The receiver observes (m, r) and takes an action a.

The solution concept used is a Perfect Bayesian Equilibrium (PBE). At the third stage, the receiver forms a belief and acts if and only if the conditional expectation $\mathbb{E}_\phi[s|m, r]$ of s given (m, r) is at least u_0. (Note that a PBE requires that the receiver takes the sender's preferred action whenever he is indifferent between the two actions.) At the first stage, the sender chooses an *optimal mechanism* that maximizes her expected utility, the probability that the receiver acts.

Using the revelation principle, restrict attention to mechanisms that send only two messages: m_0 that persuades the receiver not to act and m_1 that persuades the receiver to act. Adopt the convention that $\phi(m_1|s, r)$ denotes the probability of the message m_1 given (s, r). Hereafter, all notions are in the weak sense. For example, increasing means not decreasing and higher means not lower.

1.3 Analysis

1.3.1 Optimal Mechanism

The optimal mechanism ϕ^* has a simple cutoff structure:

[3]Indeed, suppose that the receiver's utility is $u_0(s, r)$ if $a = 0$ and $u_1(s, r)$ if $a = 1$. Because the action is binary, only the difference $u_1(s, r) - u_0(s, r)$ matters for the receiver's choice of action, so $u_0(s, r)$ can be normalized to u_0 (or even to 0). Further, for any given r, which is observed both by the sender and the receiver, the sender's type can be transformed according to $u_1(., r)$.

Theorem 1.1 *The optimal mechanism is given by*

$$\phi^*(m_1|s,r) = \begin{cases} 1 \text{ if } s \geq \widehat{s}(r), \\ 0 \text{ if } s < \widehat{s}(r). \end{cases} \tag{1.1}$$

If $\int_{\underline{s}}^{\overline{s}} sf(s|r)\,ds \geq u_0$, *then* $\widehat{s}(r) = \underline{s}$; *otherwise* $\widehat{s}(r) < 0$ *is the unique solution to* $\int_{\widehat{s}(r)}^{\overline{s}} sf(s|r)\,ds = u_0$.

Clearly, the optimal mechanism is conditioned on each piece of public information r. This implies that it does not matter whether the sender commits to a mechanism before or after the realization of r. I give the intuition for Theorem 1.1 conditional on some value r. If it is not possible to induce the receiver to always act, then the optimal mechanism induces the receiver to act if and only if his utility is above the cutoff. The cutoff is such that the receiver is indifferent to act whenever he acts. Intuitively, the optimal mechanism has two defining features: (i) it makes the receiver be indifferent to act whenever he acts; and (ii) it makes the receiver know whether his utility is above the cutoff. If the first feature were violated, then the receiver would strongly prefer to act whenever he acts. Thus, it would be possible to increase the probability that the receiver acts by sending m_1 for a slightly large set of types s. If the second feature were violated, then it would be possible to construct a mechanism that sends m_1 with the same probability but to higher types s. This mechanism would violate the first feature, so it would be possible to increase the probability that the receiver acts.

Theorem 1.1 extends when the distribution of (s,r) does not admit a density. The only difference is that the optimal mechanism may randomize over messages at the cutoff as the following example shows. Suppose that there is no public information and F is a discrete distribution that assigns probabilities $\frac{1}{3}$ and $\frac{2}{3}$ to 1 and -1. The optimal mechanism sends the message m_1 if $s = 1$, and the messages m_1 and m_0 with equal probabilities if $s = -1$. As a result, the receiver who gets m_1 is indifferent to act and the probability of m_1 is $\frac{2}{3}$.

Weaker versions of Theorem 1.1 appear in the literature. Lerner and Tirole (2006) show that the mechanism from Theorem 1.1 is optimal in a smaller class of feasible mechanisms in a more specific setting than mine. Kamenica and Gentzkow (2011) establish Theorem 1.1 for the above example. For a more general setting than mine, they derive interesting properties of the optimal mechanism, which can also be used to establish Theorem 1.1.

1.3.2 Comparative Statics without Public Information

For simplicity, assume in this section that there is no public information. Theorem 1.2 presents the monotone comparative statics results that relate the sender's and receiver's expected utilities under the optimal mechanism to the distribution of the sender's type. This theorem uses the standard definitions from the literature on stochastic orders. Let P_1 and P_2 be two distributions. *P_2 is higher than P_1 in the increasing convex order* if there exists a distribution P such that P_2 first-order stochastically dominates P and P is a mean-preserving spread of P_1.[4]

Theorem 1.2 [5]*Let F_1 and F_2 be two distributions of s that do not depend on r.*

1. *The sender's expected utility is higher under F_2 than under F_1 for all u_0 if and only if F_2 is higher than F_1 in the increasing convex order.*

2. *The receiver's expected utility is higher under F_2 than under F_1 for all u_0 if and only if $\mathbb{E}_{F_2}[s] \geq \mathbb{E}_{F_1}[s]$.*

Part 2 holds because the optimal mechanism is as uninformative as possible from the receiver's perspective as follows from Theorem 1.1. Indeed, under the optimal mechanism, if the receiver acts, then he either holds the prior belief or is indifferent to act. Thus, the receiver's expected utility under the optimal mechanism is $\max\{\mathbb{E}[s], u_0\}$, which is equal to his expected utility under a mechanism that sends the same message regardless of s.

Part 1 is more interesting. Suppose for the sake of argument that there is an underlying binary state ω that can take only two values $\omega_1 < 0$ and $\omega_2 > 0$. The receiver's utility is ω if he acts (and u_0 if he does not). The sender's type s is a noisy signal about ω and is normalized to the expectation of ω given s; specifically, the posterior $\Pr(\omega_2|s)$ is equal to $\frac{s-\omega_1}{\omega_2-\omega_1}$. If F is a mean-preserving spread of F_1, then distribution of posteriors under F is a mean-preserving spread of that under F_1 by the linearity of $\Pr(\omega_2|s)$ in s. This can be interpreted as F being more informative about the state (Blackwell (1953)). Since the sender can choose any mechanism in my model, all mechanisms under F_1 are also feasible under F, which immediately implies that the sender's expected utility is higher under F. If F_2 first-order stochastically dominates F, then F_2 is more favorable for acting and thus the sender can persuade the receiver to act more often. This intuition shows that F_2 being higher

[4]See Definition 1.1 and Lemma 1.1 in the appendix for more definitions and results on stochastic orders.

[5]*The assumption that distribution of s admits a density is not critical for this theorem. Further, as follows from the proof, it is straightforward to write a strong version of this theorem in which the sender's and receiver's expected utilities are strictly higher under F_2.*

than F_1 in the increasing convex order is sufficient for the conclusion that the sender's expected utility is higher under F_2. It turns out that this condition is also necessary if the conclusion is required to hold for all u_0.

Based on this intuition, the comparative statics results can be extended beyond this model as long as the sender can choose any mechanism at the ex ante stage. This assumption, however, is critical for the results.

Under a cheap talk version of my model, the sender would not be able to reveal any information because she always prefers the receiver to act. Thus, the sender's expected utility would not change as her information becomes more precise. More generally, Green and Stokey (2007) and Ivanov (2010b) show that the sender's expected utility may strictly decrease in the precision of her information. This happens because having less precise information may reduce the sender's incentive to misrepresent information.

Similarly, under a verifiable communication version of my model, the sender would completely reveal her information (Milgrom (1981)). Thus, by Theorem 1.1, it is optimal for the sender to know only whether the receiver's utility is above the cutoff, which is less informative than knowing the receiver's utility exactly. That is, the sender's utility may strictly decrease as her information becomes more precise.

1.3.3 Comparative Statics with Public Information

This section generalizes the previous section and obtains comparative statics results with respect to public information. To present the comparative statics results, we need to extend the stochastic orders introduced in Section 1.3.1 to the multidimensional case.[6] The multidimensionality arises because each piece of public information r is associated with a conditional distribution of the sender's type s. Therefore, to compare distributions of r, we essentially need to compare distributions of distributions of s.

Theorem 1.3 presents the monotone comparative statics results that relate the sender's and receiver's expected utilities to the distribution of public information. This theorem uses a new stochastic order. *P_2 is higher than P_1 in the increasing concave order* if there exists P such that P_2 first-order stochastically dominates P and P_1 is a mean-preserving spread of P.

[6] See Definition 1.2 and Lemma 1.2 in the appendix for definitions and results on multidimensional stochastic orders.

Theorem 1.3 *Let $\mathcal{P}$ be the set of distributions on $[\underline{s}, \overline{s}]$. Let r be identified with the conditional distribution F of s given r. Let G_1 and G_2 be two distributions of r such that $F(.|r)$ admits a strictly positive density for all r in the supports of G_1 and G_2.*

1. *Let a partial order on $\mathcal{P}$ be the increasing convex order; specifically, for all $r_1, r_2 \in \mathcal{P}$, r_2 is higher than r_1 if $F(.|r_2)$ is higher than $F(.|r_1)$ in the increasing convex order. If G_2 is higher than G_1 in the increasing concave order, then the sender's expected utility under the optimal mechanism is higher under G_2 than under G_1.*

2. *Let an order on $\mathcal{P}$ be the mean order; specifically, for all $r_1, r_2 \in \mathcal{P}$, r_2 is higher than r_1 if $\mathbb{E}_F[s|r_2] \geq \mathbb{E}_F[s|r_1]$. If G_2 is higher than G_1 in the increasing convex order, then the receiver's expected utility under the optimal mechanism is higher under G_2 than under G_1.*

We first discuss part 1 of Theorem 1.3. If G_2 is higher than G_1 in the increasing concave order, then there exists G such that G_2 first-order stochastically dominates G and G_1 is a mean-preserving spread of G. First, G_2 puts a higher probability than G on public types r that result in a higher probability that the receiver acts conditional on r by Theorem 1.2 part 1. Therefore, the unconditional probability that the receiver acts is also higher under G_2. Second, based on the intuition from the previous section, the receiver is less informed about s under G than under G_1, and, therefore, it is easier to persuade him to act. Intuitively, since the sender can choose any mechanism in my model, she can use a mechanism that sends two messages sequentially. The first stage of this mechanism will then make public information more precise and the second stage can be designed to reveal the optimal amount of information given more precise public information.

We now turn to part 2 of Theorem 1.3. Theorem 1.1 implies that the receiver has the same expected utility under the optimal mechanism and the mechanism ϕ_0 that sends the same message regardless of s. If G_2 is higher than G_1 in the increasing convex order, then there exists G such that G_2 first-order stochastically dominates G and G is a mean-preserving spread of G_1. First, G_2 puts a higher probability on public types r that are more favorable for acting by Theorem 1.2 part 2. Thus, overall, G_2 is more favorable for acting than G_1. Taking into account that the receiver's utility from not acting is fixed, and the receiver acts only if his expected utility from acting conditional on his information exceeds his expected utility from not acting, we get that the receiver's expected utility under ϕ_{no} is higher under G_2 than under G. Second, public information is more precise under G than under G_1. Therefore, under ϕ_{no}, the receiver takes a more appropriate action and, thus, the receiver's expected utility is higher under G than under G_1.

Although the receiver's expected utility increases with the precision of public information, the social welfare does not necessarily increase even if it includes the sender's expected utility with a very small weight. Indeed, suppose that initially there is no public information. As public information appears, the marginal increase in the receiver's expected utility is 0 by the envelope theorem (Radner and Stiglitz (1984)), but the marginal decrease in the sender's expected utility is strictly positive.

1.4 Extensions

1.4.1 Receiver's Verifiable Private Information

In this section, the receiver is allowed to have verifiable private information. As usual, verifiable information is the information that cannot be lied about but can be concealed. In this case, the sender extracts the receiver's information at no cost and then reveals her information optimally as if the receiver's type was public. Therefore, all results of Section 1.3 apply.

To illustrate this result, assume that the type r is privately known by the receiver rather than publicly known. In other respects, the environment is the same as in Section 1.2. In particular, players, actions, information structure, and preferences are the same. In addition, assume that the set of receiver's types R is given by $[\underline{r}, \overline{r}]$ and is ordered in such a way that $\widehat{s}(r)$ is strictly increasing in r where $\widehat{s}(r)$ is given by Theorem 1.1.

Similarly to Milgrom (1981), assume that the set of receiver's reports is $N(r) = [\underline{r}, r]$. That is, the receiver can report any type that is lower than his true type. Informally, the report n can be viewed as the receiver's claim that his true type r is at least n and the receiver's claims are required to be truthful in that r must belong to $[n, \overline{r}]$.

Now a mechanism ϕ sends a message m to the receiver as a (stochastic) function of (s, n). Finally, the timing of the game is as follows: 1. The sender publicly chooses a mechanism $\phi(m|s, n)$. 2. The receiver's type r is drawn according to G. 3. The receiver makes a report n. 4. A vector (m, s) is drawn according to ϕ and F. 5. The receiver gets a message m and takes an action a.

Theorem 1.4 characterizes the unique PBE of this game:

Theorem 1.4 [7] *In the unique PBE, the receiver reports his true type $n = r$ and the sender chooses the optimal mechanism ϕ^* given by Theorem 1.1.*

[7] *Formally, a PBE is not necessarily unique, but all PBE have the same equilibrium mapping from (s, r) to the receiver's action a. See the proof for details.*

This theorem shows that without loss of generality we can view the receiver's verifiable private information as public information. This result is in spirit of the standard unravelling result due to Milgrom (1981) who show that generally all verifiable private information is revealed in an equilibrium.

Note that the mechanism ϕ^* and truthful reporting of the receiver constitutes a PBE even if the sender has partial commitment in that she can choose a mechanism only after the receiver's report. However, this PBE is not unique in this new model. For example, there exists a PBE in which the receiver always reports $n = 0$.[8]

1.4.2 Sender's Ex Ante Private Information

In this section, the sender is allowed to have private information before she chooses a mechanism. As a result, the sender reveals all of her verifiable information and none of her unverifiable information. Thus, without loss of generality, the sender's verifiable information can be viewed as public information, and the sender's ex ante unverifiable information can be integrated out.

To illustrate these results, assume that the type r is privately known by the sender rather than publicly known. In other respects, the environment is the same as in Section 1.2. In addition, assume that $\widehat{s}(r) > \underline{s}$ for all $r \in R$ where $\widehat{s}(r)$ is given by Theorem 1.1 and the set of receiver's types R is given by $[\underline{r}, \overline{r}]$ and is ordered in such a way that $\widehat{s}(r)$ is strictly decreasing in r.

The timing of the game is as follows: 1. The sender's type r is drawn according to G. 2. The sender makes a report n. 3. The sender publicly chooses a mechanism $\phi(m|s,n)$. 4. A vector (m,s) is drawn according to ϕ and F. 5. The receiver gets a report n and a message m and takes an action a.

Theorem 1.5 characterizes the unique PBE for both verifiable and unverifiable information of the sender. If the sender's information is verifiable, then the set of her reports is $N(r) = [\underline{r}, r]$. If the sender's information is unverifiable, then the set of her reports is $N = [\underline{r}, \overline{r}]$ regardless of r.

Theorem 1.5 [9] *If the sender's ex ante private information is verifiable, then in the unique PBE, the sender reports $n = r$ and chooses the optimal mechanism ϕ^* given by Theorem 1.1.*

[8]Indeed, suppose that the sender believes that each out-of-equilibrium report $n \neq 0$ is made by the receiver with type $r = n$. Note that under such a belief, the sender chooses a mechanism $\phi^*(m|s,n)$ for any $n \neq 0$. Thus, the receiver's interim expected utility from reporting $n \neq 0$ is $\max\{u_0, \mathbb{E}[s|r]\}$, which is smaller than that from reporting $n = 0$.

[9]*Again, a PBE is not unique, but the equilibrium mapping from (s,r) to a is unique.*

If the sender's ex ante private information is unverifiable, then in the unique PBE, the sender reports some fixed n regardless of r and chooses the optimal mechanism ϕ^{**} given by Theorem 1.1 where $F(s|r)$ is replaced with $\int_R F(s|r)\, g(r)\, dr$ for all r.

The sender reveals all her verifiable private information is again due to the standard unravelling argument (Milgrom (1981)). The sender conceals all her unverifiable private information because regardless of her information she always wants to pretend that she has one piece of information rather than another. Note that if the sender could commit to a mechanism before realization of r, then by Theorem 1.1, the optimal mechanism would be ϕ^{**} where ϕ^{**} is defined in Theorem 1.5. That is, the full commitment optimum is achieved as the equilibrium outcome if the sender's information is unverifiable. This observation is consistent with Theorem 1.3, which shows that the sender's expected utility decreases with the precision of public information.

1.5 Conclusions

In this chapter, I have studied optimal information revelation mechanisms. I have imposed the following key assumptions. First, the sender can choose ex ante how her information will be revealed ex post in that she can choose any conditional distribution of messages given her information. Second, the receiver has a binary action choice. Third, the sender's utility depends on the receiver's action but does not depend on information. Fourth, the receiver is uninformed.

The optimal mechanism has a particularly simple structure. It reveals only whether the sender's utility is above the cutoff where the cutoff is such that the receiver does not get any rent from learning whether his utility is above or below it.

The sender's and receiver's welfare is monotone in information. The sender's welfare increases with the precision of her potential information and decreases with the precision of public information. The receiver's welfare does not change with the precision of the sender's potential information and increases with the precision of public information.

I have also analyzed the extensions where the sender and the receiver are allowed to have ex ante private information. However, in this chapter, I have not explored an environment in which the receiver has unverifiable private information. Generically, the receiver does have some private information at least by the time he takes an action. For example, the FDA carries out an independent review after receiving the application from the PC. Moreover, the PC is uncertain about preferences and beliefs of the FDA regarding the safety and efficacy of a new drug. Since the optimal mechanism leaves no rent to the receiver if the receiver is uninformed, as a trivial result, we get that the optimal mechanism is (weakly) more informative if the receiver is privately informed. The detailed analysis of this environment is the central goal of the next chapter.

1.6 Appendix: Proofs

Proof of Theorem 1.1. The optimal mechanism ϕ^* solves

$$\underset{\phi(m_1|s,r)\in[0,1]}{\text{maximize}} \int_{S\times R} f\left(s|r\right) g\left(r\right) \phi\left(m_1|s,r\right) drds$$

subject to

$$\int_S sf\left(s|r\right) \phi\left(m_1|s,r\right) \geq u_0.$$

The Lagrangian for this problem is given by:

$$\mathcal{L} = \int_{S\times R} \left(1+s\lambda\left(r\right)\right) f\left(s|r\right) g\left(r\right) \phi\left(m_1|s,r\right) drds,$$

where $\lambda\left(r\right) g\left(r\right)$ is a multiplier for the constraint. Since the choice variable $\phi\left(m_1|s,r\right)$ belongs to the unit interval, we get that $\phi\left(m_1|s,r\right) = 1$ if $s \geq -\frac{1}{\lambda(r)}$ and $\phi\left(m_1|s,r\right) = 0$ if $s < -\frac{1}{\lambda(r)}$ where $\lambda\left(r\right)$ is 0 if $\mathbb{E}_F\left[s|r\right] > u_0$ and is such that the constraint is binding if $\mathbb{E}_F\left[s|r\right] \leq u_0$. ■

Definition 1.1 *Let X_1 and X_2 be two random variables with distributions P_1 and P_2 on $[\underline{x}, \overline{x}]$. Say that*

1. *P_2 first-order stochastically dominates P_1 (denoted by $P_2 \geq_{st} P_1$) if $P_2\left(x\right) \leq P_1\left(x\right)$ for all x.*

2. *P_2 is a mean-preserving spread of P_1 (denoted by $P_2 \geq_{cx} P_1$) if there exist two random variables $\widehat{X}_2$ and $\widehat{X}_1$, defined on the same probability space, with distributions P_2 and P_1 such that $\mathbb{E}\left[\widehat{X}_2|\widehat{X}_1\right] = \widehat{X}_1$.*

3. *P_2 is higher than P_1 in the increasing convex order (denoted by $P_2 \geq_{icx} P_1$) if there exists a distribution P such that $P_2 \geq_{st} P \geq_{cx} P_1$.*

Lemma 1.1 *Let P_1 and P_2 be two distributions that admit densities on $[\underline{x}, \overline{x}]$.*

1. *$P_2 \geq_{st} P_1$ if and only if $\mathbb{E}\left[h\left(X_2\right)\right] \geq \mathbb{E}\left[h\left(X_1\right)\right]$ for all increasing functions h.*

2. $P_2 \geq_{cx} P_1$ *if and only if* $\mathbb{E}[h(X_2)] \geq \mathbb{E}[h(X_1)]$ *for all convex functions* h.

3. *The following statements are equivalent:*

   (a) $P_2 \geq_{icx} P_1$*; (b)* $\mathbb{E}[h(X_2)] \geq \mathbb{E}[h(X_1)]$ *for all increasing convex functions* h*; (c)* $\int_p^1 P_2^{-1}(\widetilde{p})\, d\widetilde{p} \geq \int_p^1 P_1^{-1}(\widetilde{p})\, d\widetilde{p}$ *for all* $p \in [0,1]$.

Proof of Lemma 1.1. See Shaked and Shanthikumar (2007) Section 1.A.1 for part 1, Section 3.A.1 for part 2, and Section 4.A.1 for part 3. ■

Proof of Theorem 1.2. I start by proving the first part. Let $\widehat{s}_i$ be given by Theorem 1.1 where F is replaced with F_i. If $F_2 \geq_{icx} F_1$ (Definition 1.1), then the sender can induce the receiver to act with higher probability under F_2 than under F_1 because

$$\int_{F_2^{-1}(F_1(\widehat{s}_1))}^{\overline{s}} sdF_2(s) = \int_{F_1(\widehat{s}_1)}^{1} F_2^{-1}(\widetilde{p})\, d\widetilde{p} \geq \int_{F_1(\widehat{s}_1)}^{1} F_1^{-1}(\widetilde{p})\, d\widetilde{p} = \int_{\widehat{s}_1}^{\overline{s}} sdF_1(s) \geq u_0,$$

where the equalities hold by the appropriate change of variables, the first inequality holds by Lemma 1.1 part 3(c), and the last inequality holds by Theorem 1.1. Conversely, if $F_2 \ngeq_{icx} F_1$, then there exists p such that $\int_p^1 P_2^{-1}(\widetilde{p})\, d\widetilde{p} < \int_p^1 P_1^{-1}(\widetilde{p})\, d\widetilde{p}$. Setting $u_0 = \int_{F_2^{-1}(p)}^1 sdF_2(s)$ and using an analogous argument, we get that the receiver acts with a strictly higher probability under F_1 than under F_2:

$$\int_{F_1^{-1}(p)}^{\overline{s}} sdF_1(s) = \int_p^1 F_1^{-1}(\widetilde{p})\, d\widetilde{p} < \int_p^1 F_2^{-1}(\widetilde{p})\, d\widetilde{p} = \int_{F_2^{-1}(p)}^{\overline{s}} sdF_2(s) = u_0.$$

Now I prove the second part. The receiver's expected utility under F_i is $\max\{\mathbb{E}_{F_i}[s], u_0\}$ by Theorem 1.1. Clearly, if $\mathbb{E}_{F_2}[s] \geq \mathbb{E}_{F_1}[s]$, then $\max\{\mathbb{E}_{F_2}[s], u_0\} \geq \max\{\mathbb{E}_{F_1}[s], u_0\}$ for all u_0. Conversely, if $\mathbb{E}_{F_2}[s] < \mathbb{E}_{F_1}[s]$, then $\max\{\mathbb{E}_{F_2}[s], u_0\} < \max\{\mathbb{E}_{F_1}[s], u_0\}$ for any $u_0 \in (\mathbb{E}_{F_2}[s], \mathbb{E}_{F_1}[s])$. ■

Definition 1.2 *Let* $\mathcal{P}$ *be the set of distributions on* $[\underline{x}, \overline{x}]$ *endowed with some partial order* $\geq_P$. *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be two random elements with distributions* Q_1 *and* Q_2 *on* $\mathcal{P}$. *Say that*

1. Q_2 *first-order stochastically dominates* Q_1 *(denoted by* $Q_2 \geq_{mst} Q_1$*) if* $\Pr_{Q_2}(\mathbf{X}_2 \in U) \geq \Pr_{Q_1}(\mathbf{X}_1 \in U)$ *for all measurable sets* $U \subset \mathcal{P}$ *such that* $P \geq_P P'$ *and* $P' \in U$ *imply* $P \in U$.

2. Q_2 *is a mean-preserving spread of* Q_1 *(denoted by* $Q_2 \geq_{mcx} Q_1$*) if there exist two random elements* $\widehat{\mathbf{X}}_2$ *and* $\widehat{\mathbf{X}}_1$*, defined on the same probability space, with distributions* Q_2 *and* Q_1 *such that* $\mathbb{E}\left[\widehat{\mathbf{X}}_2 | \widehat{\mathbf{X}}_1\right] = \widehat{\mathbf{X}}_1$.

3. Q_2 *is higher than* Q_1 *in the increasing convex order (denoted by* $Q_2 \geq_{micx} Q_1$*) if there exists a distribution* Q *such that* $Q_2 \geq_{mst} Q \geq_{mcx} Q_1$.

4. Q_2 *is higher than* Q_1 *in the increasing concave order (denoted by* $Q_2 \geq_{micv} Q_1$*) if there exists a distribution* Q *such that* $Q_2 \geq_{mst} Q$ *and* $Q_1 \geq_{mcx} Q$.

Lemma 1.2 *Let* Q_1 *and* Q_2 *be two distributions on* $\mathcal{P}$.

1. $Q_2 \geq_{mst} Q_1$ *if and only if* $\mathbb{E}[h(\mathbf{X}_2)] \geq \mathbb{E}[h(\mathbf{X}_1)]$ *for all increasing functions* h *in that* $h(P_2) \geq h(P_1)$ *for all* $P_1, P_2 \in \mathcal{P}$ *such that* $P_2 \geq_P P_1$.

2. $Q_2 \geq_{mcx} Q_1$ *if and only if* $\mathbb{E}[h(\mathbf{X}_2)] \geq \mathbb{E}[h(\mathbf{X}_1)]$ *for all convex functions* h *in that* $h(\alpha P_1 + (1-\alpha) P_2) \leq \alpha h(P_1) + (1-\alpha) h(P_2)$ *for all* $P_1, P_2 \in \mathcal{P}$ *and all* $\alpha \in (0,1)$.

3. $Q_2 \geq_{micx} Q_1$ *if and only if* $\mathbb{E}[h(\mathbf{X}_2)] \geq \mathbb{E}[h(\mathbf{X}_1)]$ *for all increasing convex functions* h.

4. $Q_2 \geq_{micv} Q_1$ *if and only if* $\mathbb{E}[h(\mathbf{X}_2)] \geq \mathbb{E}[h(\mathbf{X}_1)]$ *for all increasing concave functions* h.

Proof of Lemma 1.2. See Shaked and Shanthikumar (2007) Section 6.B.1 for part 1, and Section 7.A.1 for parts 2, 3, and 4. ■

Proof of Theorem 1.3. The probability that the receiver acts is $\int_R p^*(r)\, dG(r)$ where the conditional probability $p^*(r)$ that the receiver acts is given by $1 - F(\widehat{s}(r)|r)$ with $\widehat{s}(r)$ given by Theorem 1.1. The function p^* is increasing in r (in the increasing convex order) by Theorem 1.2 part 1. Moreover, p^* is concave in r. For concavity, it suffices to show that there exists a mechanism ϕ that induces the receiver to act with probability $\alpha p^*(r_1) + (1-\alpha) p^*(r_2)$ when the distribution of s is $\alpha F(s|r_1) + (1-\alpha) F(s|r_2)$. Without loss of generality, suppose that $\widehat{s}(r_1) \geq \widehat{s}(r_2)$. The required mechanism is as follows. If $s \geq \widehat{s}(r_1)$, the receiver gets the message m_1. If $s \in [\widehat{s}(r_2), \widehat{s}(r_1))$, the receiver gets the messages m_1 and m_0 with probabilities $p_1 \equiv \frac{(1-\alpha)(F(\widehat{s}(r_1)|r_2) - F(\widehat{s}(r_2)|r_2))}{\alpha(F(\widehat{s}(r_1)|r_1) - F(\widehat{s}(r_2)|r_1)) + (1-\alpha)(F(\widehat{s}(r_1)|r_2) - F(\widehat{s}(r_2)|r_2))}$ and $1 - p_1$, respectively. If $s < \widehat{s}(r_2)$, the receiver gets the message m_0. Since p^* is increasing and concave in r, part 1 of the theorem follows by Lemma 1.2 part 4.

Theorem 1.1 implies that the receiver's expected utility is

$$\int_R \max\left\{u_0, \int_{\underline{s}}^{\overline{s}} s dF(s|r)\right\} dG(r).$$

Both the constant u_0 and $\int_{\underline{s}}^{\overline{s}} sdF(s|r)$ are linear and increasing in r (in the mean order). Since the maximum of two increasing linear functions is increasing and convex, part 2 follows by Lemma 1.2 part 3. ■

Proof of Theorem 1.4. I start by showing that the described strategies constitute a PBE. If the receiver reports $n = r$, then his interim expected utility is $\max\{u_0, \mathbb{E}[s|r]\}$ as follows from Theorem 1.1. If the receiver reports $n < r$, then his interim expected utility is again $\max\{u_0, \mathbb{E}[s|r]\}$ because $\widehat{s}(r)$ is increasing in r. Thus, given the mechanism ϕ^*, it is a best response for the receiver to report his true type $n = r$. To see that it is optimal for the sender to choose ϕ^* at the first stage, note that ϕ^* is the optimal mechanism in the relaxed problem where r is publicly known, so ϕ^* gives a higher expected utility to the sender than all other feasible mechanisms.

To complete the proof, I show that in all PBEs, the sender chooses ϕ^* and the receiver reports $n = r$ if $\widehat{s}(r) > \underline{s}$. However, for r such that $\widehat{s}(r) = \underline{s}$, the receiver can make any feasible report in a PBE, but all these reports result in the message m_1 sent by the mechanism ϕ^*. Suppose to get a contradiction that there exists another PBE. In this PBE, the sender's expected utility is strictly less than in the above PBE because ϕ^* is the optimal mechanism in the relaxed problem. Consider a mechanism $\widetilde{\phi}$ that sends the message m_1 if and only if $s \geq \widehat{s}(r) + \delta$ where $\delta > 0$ is sufficiently small. Under this mechanism, the receiver strictly prefers to report his true type r and the sender's expected utility is arbitrarily close to that under ϕ^*. A contradiction. ■

Proof of Theorem 1.5. Suppose that given the sender's report n, r is distributed according to G_n. Given this report, the receiver believes that s is distributed according to $F_n(s) = \int_R F(s|r)\, dG_n(r)$. By sequential rationality, at the third stage, the sender chooses the optimal mechanism ϕ_n^* that sends m_1 if and only if $s \geq \widehat{s}_n$ where $\widehat{s}_n$ is given by Theorem 1.1 where $F(s|r)$ is replaced with $F_n(s)$.

I start by considering the case where the sender's information is verifiable. In this case, the sender r can make a report n only if $r \geq n$. Thus, the support of G_n does not intersect $[\underline{r}, n)$. Suppose to get a contradiction that there exists an equilibrium report n such that H_n is supported on $[\underline{r}_n, \overline{r}_n]$ with $\overline{r}_n > n$. This means that with a strictly positive probability, the sender $\overline{r}_n$ makes the report n and induces the receiver to act with probability $1 - F(\widehat{s}_n|\overline{r}_n)$. If this sender made the report $\overline{r}_n$ instead, then she would induce the receiver to act with a strictly higher probability because $\widehat{s}_{\overline{r}_n} < \widehat{s}_n$, which leads to a contradiction.[10] Thus, G_n assigns probability one to $r = n$,

[10]The inequality $\widehat{s}_{\overline{r}_n} < \widehat{s}_n$ holds because $\widehat{s}_{\overline{r}_n} \leq \widehat{s}(\overline{r}_n) < \widehat{s}_n$. Suppose to get a contradiction that $\widehat{s}_n < \widehat{s}(\overline{r}_n)$, then $\int_{\widehat{s}_n}^{\overline{s}} sdF_n(s) = \int_{\underline{r}_n}^{\overline{r}_n} \left(\int_{\widehat{s}_n}^{\overline{s}} sf(s|r)\, dr\right) dG_n(r) < 0$, contradicting the definition of $\widehat{s}_n$. The equality holds by Fubini's

which means that the sender reports $n = r$ for all r.

Now, I consider the case where the sender's information is unverifiable. Suppose to get a contradiction that there exist two equilibrium reports n_1 and n_2 such that $\widehat{s}_{n_1} < \widehat{s}_{n_2}$. Then the sender would always prefer to report n_1 regardless of r. A contradiction. ■

Theorem and the definition of $F_n(s)$. The inequality holds because $\widehat{s}(r) < 0$ is strictly decreasing in r and thus $\int_{\widehat{s}_n}^{\overline{s}} sf(s|r)\,ds < \int_{\widehat{s}(r)}^{\overline{s}} sf(s|r)\,ds = 0$ for all $r < \overline{r}_n$. Noting that the support of $G_{\overline{r}_n}$ does not intersect $[\underline{r}, \overline{r}_n)$ and using a similar argument gives $\widehat{s}_{\overline{r}_n} \leq \widehat{s}(\overline{r}_n)$.

Chapter 2

Optimal Information Revelation with Informed Receiver

2.1 Introduction

This chapter studies optimal information revelation from an informed sender to an informed receiver. Most of the literature on communication assumes that only the sender has private information.[1] But generically, each player has private information. The central goal of this chapter is to understand economic aspects of optimal information revelation when agents obtain information from various sources.

In my model, the receiver decides whether to act or not to act. The sender's utility depends only on the action taken by the receiver, and she prefers the receiver to act. The receiver's utility depends both on his action and on information. The receiver takes an action that maximizes his expected utility given his belief that is based on his private information and information revealed by the sender. Before obtaining her private information, the sender can commit to how her private information will be revealed to the receiver. Formally, the sender can choose any (stochastic) mapping from her information to messages, which I call a *revelation mechanism*. The sender chooses the revelation mechanism that maximizes the ex ante probability that the receiver will act.

For example, consider a school that chooses a grading policy for a student in order to persuade a potential employer to hire him. The school has a lot of freedom in choosing which part of available information about the student will appear on his transcript, which will be observed by the employer.

[1] Notable exceptions include Watson (1996) and Chen (2009).

Moreover, the school chooses its grading policy before it learns anything about the student. The employer obtains private information from conducting an employment interview. Similarly, the school uses the same grading policy for all students who apply to various jobs. In terms of my model, this also contributes to the receiver's private information.

There are many other examples that fit my model well, such as scientific experimentation by interested parties, and there are still other examples that fit my model somewhat imperfectly: An advertising agency emphasizes special attributes of a product to persuade consumers to buy it. A political party outlines its political platform to persuade citizens to vote for it. A sponsor of a proposal presents details of her proposal to persuade a committee to approve it. In these latter examples, the sender typically cannot directly commit to a revelation mechanism ex ante. However, as I discuss in the appendix, there are three other kinds of situations that are formally equivalent to assuming that the sender has commitment power.

Since the receiver has private information, he acts or does not act depending not only on a message received from the sender but also on his private information. Thus, from the sender's perspective, each message generates a probability distribution over receiver's actions. At the ex ante stage when the sender chooses how to reveal her information ex post, she faces a tradeoff between the frequency and the persuasiveness of messages that she sends ex post: sending positive messages more often (in terms of the sender's private information) makes it less likely that the receiver will act (in terms of the receiver's private information). The optimal revelation mechanism balances these two conflicting objectives. For example, the school may choose lower standards for getting good grades. In this case, more students will get good-looking transcripts. However, employers will rationally account for this and each student with a good-looking transcript will find it harder to get a job.

Interestingly, under the optimal revelation mechanism, the sender's and receiver's expected utilities are not monotone in the precision of the receiver's private information.[2] First, as the receiver becomes more informed, his expected utility may decrease despite the fact that he is the only player who takes an action that directly affects his utility. This happens because the optimal revelation mechanism changes with the precision of the receiver's private information, and the sender may prefer to reveal significantly less information as the receiver becomes more informed. Thus, it may be beneficial for employers to commit to less informative interviews because

[2] In contrast, in the previous chapter, I show that the sender's and receiver's expected utilities are monotone in information if the receiver does not have private information.

it motivates schools to design more informative grading policies.[3] Second, it may be easier for the sender to influence a better informed receiver. This happens because the sender may optimally choose to target only the receiver with positive private information. In this case, it becomes easier for the sender to persuade the receiver with more precise positive information.

Under a weak assumption that receiver's types can be ordered according to their willingness to act, the sender's problem of finding an optimal mechanism reduces to a linear program, which is similar to a transportation problem. Duality theory allows us to characterize when a candidate mechanism is optimal. For example, schools choose various grading policies and duality theory allows us to find primitive conditions on the environment that justify each particular choice of a grading policy. In particular, the full revelation is optimal if the sender prefers to separate any two of her types than to pool them. In contrast, the no revelation mechanism is optimal if the sender prefers to pool any three of her types than to pool two of them and reveal the third one. Under further assumptions, the amount of information that is optimally revealed is determined by the convexity properties of the distribution of the receiver's private information.

In the benchmark model, I assume that the receiver does not communicate with the sender. This assumption fits many real-life examples. In particular, the school gives the same transcripts to students, regardless of where students are applying for a job. Similarly, the employer conducts an interview after it receives a student's transcript. However, this assumption is not without loss of generality because the sender can potentially increase the probability that the receiver acts by conditioning the mechanism on the receiver's report. I give examples when allowing more general mechanisms with two-way communication helps the sender and when it does not.

Similar to this chapter, Ostrovsky and Schwarz (2010), Rayo and Segal (2010), and Kamenica and Gentzkow (2011) study environments where the sender can commit to a revelation mechanism. Kamenica and Gentzkow (2011) consider a much more general model than mine. However, they focus on the case where the receiver does not have private information. In contrast, the main focus of this chapter is on the case where the receiver does have private information, where both the results and analytical techniques are very different. Similar to this chapter, Rayo and Segal (2010) assume that the receiver has a binary action choice, but they allow the sender's utility to depend not only on the action but also on information. To make the analysis tractable, they impose a special assumption on the receiver's information structure that would make my model trivial in that the sender's expected utility would be the same under any revelation mechanism, as

[3] Arvey and Campion (1982) summarizes research on the reliability and validity of interviews.

follows from my Theorem 2.1 part 1. Similar to this chapter, Ostrovsky and Schwarz (2010) study information revelation in matching markets. The most important difference is that they study equilibrium rather than optimal information revelation.

The rest of the chapter is organized as follows. Section 2.2 develops a general model and discusses the commitment assumption. Section 2.3 presents two examples that illustrate the main tradeoff of the sender, non-monotone comparative statics, and informativeness of the optimal revelation mechanism. Section 2.4 partially characterizes the optimal revelation mechanism and derives primitive necessary and sufficient conditions for optimality of full revelation and no revelation mechanisms for the case where the sender's and receiver's information has a fairly general structure. Section 2.5 extends the model to allow two-way information revelation between the sender and receiver. Section 2.6 concludes. Appendix A completely characterizes the optimal revelation mechanism for the case where the sender's and receiver's information structure is binary. Appendix B contains all formal proofs. Appendix C discusses the sender's commitment assumption.

2.2 Model

Consider a communication game between a female sender and a male receiver. The receiver takes a binary action $a = 0, 1$. Say that the receiver *acts* if he takes $a = 1$ and the receiver *does not act* if he takes $a = 0$. The sender's utility depends only on a, but the receiver's utility depends both on a and on (r, s) where components r and s denote the receiver's and sender's types, respectively. That is, the sender's utility is a, and the receiver's utility is $au(r, s)$ where u is a continuously differentiable function. Before s is realized, the sender can commit to a mechanism that sends a message m to the receiver as a (stochastic) function of her type s; specifically, the sender chooses the conditional distribution $\phi(m|s)$ of m given s. With a slight abuse of notation, the joint distribution of (m, s) is denoted by $\phi(m, s)$.

Note that the model assumes that the receiver does not communicate with the sender. This assumption fits many real-life examples. In particular, the school gives the same transcripts to students, regardless of where students are applying for a job. Similarly, the employer conducts an interview after it receives a student's transcript. However, this assumption is not without loss of generality because, the receiver has private information in my model. In this case, the sender can potentially increase the probability that the receiver acts by conditioning the mechanism on the receiver's report. Section 2.5 extends the model to allow communication from the receiver to the sender.

Assume that the set of messages coincides with the real line, the set of receiver's types R is $[\underline{r}, \overline{r}]$, and the set of sender's types S is $[\underline{s}, \overline{s}]$. The information (s, r) has some joint distribution. Unless stated otherwise, assume that for this distribution, the marginal distribution $F(s)$ of s and the conditional distribution $G(r|s)$ of r given s admit strictly positive continuously differentiable densities $f(s)$ and $g(r|s)$.

The timing of the communication game is as follows:

1. The sender publicly chooses a mechanism $\phi(m|s)$.
2. A vector (m, r, s) is drawn according to ϕ, G, and F.
3. The receiver observes (m, r) and takes an action a.

The solution concept used is a Perfect Bayesian Equilibrium (PBE). At the third stage, the receiver forms a belief about s and acts if and only if the conditional expectation $\mathbb{E}_\phi[u(r, s)|m, r]$ of s given (m, r) is at least 0. At the first stage, the sender chooses an *optimal mechanism* that maximizes her expected utility, the probability that the receiver acts.

Hereafter, use the following definitions and conventions. All notions are in the weak sense, unless stated otherwise. For example, increasing means not decreasing and higher means not lower. Two mechanisms are *equivalent* if they result in the same probability that the receiver acts. One mechanism *dominates* another mechanism if the former results in a higher probability that the receiver acts than the latter. *The full revelation mechanism* (denoted by ϕ_{full}) is a mechanism that sends a different message for each s. *The no revelation mechanism* (denoted by ϕ_{no}) is a mechanism that sends the same message regardless of s. The *survival function* $\overline{H}$ of a random variable with distribution H is defined as $\overline{H} \equiv 1 - H$.

2.3 Examples

In this section, I discuss two illustrative examples. For these examples, I derive the optimal mechanism and the central tradeoff that the sender faces. Further, I show that the sender's and receiver's expected utilities are non-monotone in information. Finally, I discuss what determines how much of information is optimally revealed.

2.3.1 Binary Example

In this example, the sender's and receiver's types are binary. Further, the sender is perfectly informed, but the receiver is imperfectly informed. That is, the sender knows the receiver's utility exactly, but the receiver only gets an imperfect signal about his utility.

More formally, the receiver's utility from acting is equal to the sender's type s that takes two values: $s = 1$ with probability $\frac{1}{5}$ and $s = -1$ with probability $\frac{4}{5}$. The receiver's type (equivalently signal) r also takes two values $r = 1$ and $r = -1$ according to the following conditional probabilities:

$$\Pr(r = 1|s = 1) = \Pr(r = -1|s = -1) = p.$$

The parameter p captures the precision of the receiver's private signal. For example, this may correspond to the quality of an interview conducted by an employer. Without loss of generality, assume that $p \in \left[\frac{1}{2}, 1\right]$. For a given mechanism, the receiver $r = 1$ assigns a higher probability that s is 1, than the receiver $r = -1$. Moreover, the difference in their assessments of the probability that s is 1 increases in p. Thus, p can be alternatively viewed as the measure of *polarization* between the *optimistic receiver* ($r = 1$) and the *pessimistic receiver* ($r = -1$).

A message m under a mechanism ϕ generates a posterior probability $\Pr_\phi(s|m)$ of s given m for each value s. The probability $\Pr_\phi(s = 1|m, r)$ that s is 1, given the receiver's message m and signal r, can be calculated using Bayes' rule. The receiver acts if $\Pr_\phi(v = 1|m, r) \geq \frac{1}{2}$. It is straightforward to calculate that upon receiving m, the optimistic receiver acts if $\Pr_\phi(s = 1|m) \geq 1 - p$, and the pessimistic receiver acts if $\Pr_\phi(s = 1|m) \geq p$. Clearly, if m induces the pessimistic receiver to act, it also induces the optimistic receiver to act. Thus, by the revelation principle, we can restrict attention to mechanisms with three messages: (i) $m_\emptyset$ that induces the receiver not to act regardless of his signal ($\Pr_\phi(s = 1|m_\emptyset) \in [0, 1 - p)$), (ii) m_1 that induces only the optimistic receiver to act ($\Pr_\phi(s = 1|m_1) \in [1 - p, p)$), and (iii) $m_{1,-1}$ that induces the receiver to act regardless of his signal ($\Pr_\phi(s = 1|m_{1,-1}) \in [p, 1]$). Because the sender's expected utility is equal to the probability that the receiver acts, she would strictly prefer to send $m_{1,-1}$ over m_1 and m_1 over $m_\emptyset$ if there were no constraints on how often she can send various messages.

The prior distribution of s, however, imposes a constraint on how often the sender can send various messages:

$$\sum_m \Pr{}_\phi(s = 1|m) \Pr{}_\phi(m) = \Pr(s = 1) = \frac{1}{5}, \tag{2.1}$$

where $\Pr_{\phi}(m)$ denotes the probability that m is sent under ϕ. Constraint (2.1) implies that to maximize the probability of the messages $m_{1,-1}$ and m_1, the sender should choose a mechanism that satisfies: $\Pr_{\phi}(s=1|m_{\emptyset})=0$, $\Pr_{\phi}(s=1|m_1)=1-p$, and $\Pr_{\phi}(s=1|m_{1,-1})=p$.[4] That is, $m_{\emptyset}$ gives the most possible evidence against acting; m_1 gives the minimal possible evidence to make the optimistic receiver act; and $m_{1,-1}$ gives the minimal possible evidence to make the pessimistic receiver act. These observations imply that the sender's expected utility simplifies to:[5]

$$2p(1-p)\Pr(m_1)+\Pr(m_{1,-1}), \tag{2.2}$$

and constraint (2.1) simplifies to:

$$(1-p)\Pr(m_1)+p\Pr(m_{1,-1})=\frac{1}{5}. \tag{2.3}$$

The sender's problem of finding the optimal mechanism can be viewed as a problem of maximizing the linear utility function (2.2) over probabilities $\Pr(m_{\emptyset})$, $\Pr(m_1)$, and $\Pr(m_{1,-1})$ subject to the budget constraint (2.3). That is, the price and the marginal utility of persuading the receiver not to act are both equal to 0; the price and the marginal utility of persuading only the optimistic receiver to act are equal to $1-p$ and $2p(1-p)$, respectively; and the price and the marginal utility of persuading the receiver to always act are equal to p and 1, respectively. Thus, the sender faces a tradeoff between the frequency and the persuasiveness of messages. Sending $m_{\emptyset}$ is free, but it persuades the receiver not to act. Sending m_1 is more expensive and it persuades the optimistic receiver to act. Finally, sending $m_{1,-1}$ is the most expensive, but it persuades the receiver to act regardless of his signal. This tradeoff is resolved by a choice of a mechanism that sends messages with the highest *marginal utility-price ratio*.

Figure 2-1 shows the sender's and receiver's expected utilities under the optimal mechanism. Naive intuition may suggest that (i) the sender's expected utility should decrease in p because it is harder to influence the better informed receiver and (ii) the receiver's expected utility should increase in p because the better informed receiver takes a more appropriate action. This naive intuition, however, does not take into account that the optimal mechanism changes with p, and it may reveal significantly less information if the receiver is better informed. Contrary to the naive

[4] Formally, the optimal mechanism for this example is derived in Proposition 2.6.

[5] Equation (2.2) is obtained using the fact that $m_{1,-1}$ induces the receiver to act with probability 1, m_1 induces the receiver to act with probability $p\Pr_{\phi}(s=1|m_1)+(1-p)\Pr_{\phi}(s=-1|m_1)$, and $m_{\emptyset}$ induces the receiver to act with probability 0.

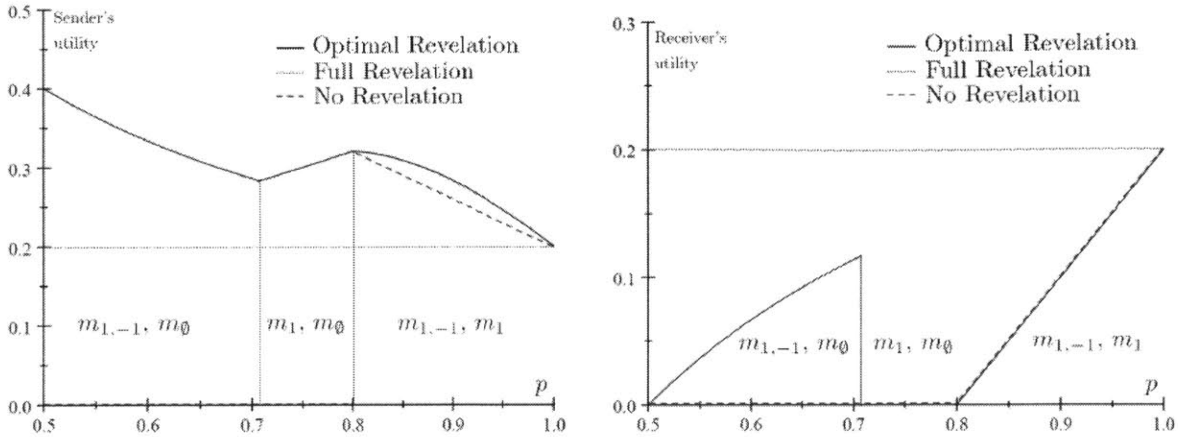

Figure 2-1: The sender's and receiver's expected utilities as a function of the precision of the receiver's private information.

intuition, this effect may result in the receiver being worse off and the sender being better off as the receiver becomes more informed. In fact, the receiver's expected utility jumps down to zero as p exceeds $\frac{1}{\sqrt{2}}$, and the sender's expected utility strictly increases in p for $p \in \left(\frac{1}{\sqrt{2}}, \frac{4}{5}\right)$.

I stress that these non-monotone comparative statics results with respect to the precision of information arise only when the receiver is privately informed. If the receiver does not have private information, then both the sender's and receiver's expected utilities are monotone in the precision of the sender's private and public information as I show in the previous chapter.

Figure 2-1 also sheds light on the extent to which information revelation can affect the receiver's action and on the informativeness of the optimal mechanism. As the left panel of Figure 1 shows, for a wide range of p, the probability that the receiver acts is considerably higher under the optimal mechanism than under the two benchmark mechanisms: the full revelation and no revelation mechanisms. As the right panel of Figure 1 shows, from the receiver's perspective, the optimal mechanism is maximally uninformative if $p = \frac{1}{2}$ or $p \in \left[\frac{1}{\sqrt{2}}, 1\right]$, and its informativeness gradually increases in p for $p \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. Indeed, if $p = \frac{1}{2}$ or $p \in \left[\frac{1}{\sqrt{2}}, 1\right]$, then the receiver's expected utility is the same under the optimal and no revelation mechanisms. If $p \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, then the receiver's expected utility under the optimal mechanism is strictly higher than under the no revelation mechanism and strictly lower than under the full revelation mechanism. I now explain the three forms that the optimal mechanism can take as p increases from $\frac{1}{2}$ to 1.

First, if the receiver's signal is imprecise in that p is close to $\frac{1}{2}$, then it is almost as cheap to persuade the pessimistic receiver to act so as to persuade the optimistic receiver to act, because the prices p and $1-p$ are close. Thus, the sender prefers to target the pessimistic receiver, so the optimal mechanism sends the messages $m_{1,-1}$ and $m_{\emptyset}$. As p increases, it becomes harder to persuade the pessimistic receiver to act and, thus, sending $m_{1,-1}$ becomes more expensive. As a result, the sender's expected utility $\frac{1}{5p}$, which is equal to the frequency of the message $m_{1,-1}$, decreases in p. The optimal mechanism gives no rent to the pessimistic receiver, but it gives a strictly positive rent to the optimistic receiver. The receiver's expected utility increases in p for two reasons. First, for a given mechanism, the better informed optimistic receiver gets a higher rent from acting when he receives $m_{1,-1}$. Second, the optimal mechanism makes $m_{1,-1}$ more favorable for acting in that $\Pr_{\phi}(s=1|m_{1,-1})=p$ increases in p. This effect further increases the rent of the optimistic receiver when he receives $m_{1,-1}$.

Second, as p exceeds $\frac{1}{\sqrt{2}}$ (but falls behind $\frac{4}{5}$), the polarization between the optimistic and pessimistic receivers becomes so high that it becomes much more expensive to persuade the pessimistic receiver to act than to persuade the optimistic receiver to act. Thus, the sender prefers to target only the optimistic receiver, so the optimal mechanism sends the messages m_1 and $m_{\emptyset}$. In other words, the sender switches from the more expensive and more persuasive message $m_{1,-1}$ to the less expensive and less persuasive message m_1. As p increases, the price $1-p$ of sending m_1 decreases because it becomes easier to persuade the optimistic receiver to act. As a result, the sender's expected utility $\frac{2}{5}p$ increases in p. The receiver's expected utility jumps down to 0 as p exceeds $\frac{1}{\sqrt{2}}$, and it stays at 0 because the optimal mechanism makes the receiver indifferent to act whenever he acts.

Third, as p exceeds $\frac{4}{5}$, the receiver's signal becomes so precise that the sender can persuade the optimistic receiver to act by revealing no information. Thus, the sender prefers to target the optimistic receiver with certainty and the pessimistic receiver with some probability, so the optimal mechanism sends m_1 and $m_{1,-1}$. As p increases further, the sender can persuade the pessimistic receiver to act more often, so the optimal mechanism sends $m_{1,-1}$ with a higher probability. But the probability of the receiver being optimistic decreases, so m_1 induces the receiver to act with a lower probability. In my example, the latter effect dominates the former, so the sender's expected utility decreases in p.[6] The receiver's expected utility increases in p because the better informed

[6] The latter effect would always dominate the former in a more general setting if the precision of the receiver's signal is sufficiently high. In this case, the sender can persuade the receiver to act, essentially, only if his utility is positive. But the sender can always achieve the same result by fully revealing the receiver's utility.

receiver takes a more appropriate action and the optimal mechanism gives no rent to the receiver.

The sender's tradeoff between the frequency and the persuasiveness of messages illustrated here carries on to a general version of the model. As I show in Section 2.4 and Appendix A, this tradeoff is resolved by the choice of messages with the highest marginal utility-price ratio as long as (i) the sender's signal has a binary structure, and (ii) one sender's signal is more favorable for acting than the other sender's signal, regardless of the receiver's signal. However, if (i) is violated, the tradeoff becomes more intricate because the budget constraint becomes multidimensional (Sections 2.3.2 and 2.4); and if (ii) is violated, the tradeoff becomes more intricate because the prices of some messages become negative (Appendix A).

2.3.2 Continuous Example

In this example, the receiver's utility is additive in sender's and receiver's type that are continuous and independent of each other. More formally, $u(r,s) = s - r$ where r and s are independently distributed according to distributions G and F that admit strictly positive continuously differentiable densities g and f everywhere. The supports are such the receiver $\underline{r}$ always acts ($\underline{r} < \underline{s}$) and the receiver $\overline{r}$ never acts ($\overline{r} > \overline{s}$).[7] For example, s may correspond to the ability of the student, and r to the employer's opportunity cost from hiring a student instead of an experienced worker.

I start this section by formulating the sender's problem as a problem of maximizing the expectation of $G(m)$ subject to the constraint that the distribution of m has to be less variable than the prior distribution F of s. Thus, the shape of the optimal mechanism is determined by the curvature of G. This observation allows us to relate the expected utilities to information structure.

Proposition 2.1 formulates the sender's problem. Note that a message m under a mechanism ϕ induces the receiver to act if and only if $r \leq \mathbb{E}_\phi[s|m]$. Thus, I identify each message m with the receiver's type that is just indifferent to act such that m induces the receiver to act if and only if $r \leq m$.

Proposition 2.1 *Let H denote the marginal distribution of m under the optimal mechanism. Then H*

$$\textit{maximizes} \int_{\underline{r}}^{\overline{r}} G(m)\, dH(m) \tag{2.4}$$

[7]This example is more general than it may seem. In particular, it includes the case where the receiver's expected utility depends on the belief about s only through the expectation of $h(s)$ for some function h, provided some regularity conditions are satisfied. To see this, note that in this case, the receiver is indifferent to act whenever $r = l\left(\int_S h(s)\, d\mu(s)\right)$ for some functions l and h and all distributions μ. An appropriate rescaling and reordering of S and R yields that $h(s) = s$ and $l(\mathbb{E}_\mu[s]) = \mathbb{E}_\mu[s]$, as required.

$$\text{subject to } F \text{ is a mean-preserving spread of } H. \tag{2.5}$$

The intuition for Proposition 2.1 is as follows. The objective function (2.4) is simply the probability that the receiver acts under the mechanism ϕ. If F is a mean-preserving spread of H, then F is more informative about the underlying (hypothetical) state than H (Blackwell (1953)). Since the sender has full commitment, she can garble her information to achieve any less informative distribution H than her prior distribution F. If she then fully reveals this garbled information to the receiver, then the distribution of m will be H. Conversely, since the sender cannot make her information more precise in any sense, then for any feasible mechanism, F is a mean-preserving spread of H.

Proposition 2.1 implies that the set of feasible mechanisms expands as F becomes more informative in the mean-preserving spread sense. Thus, the probability that the receiver acts under the optimal mechanism increases as F becomes more informative. In the two extreme cases, if F were to put probability one on some s, then the only feasible H would put probability one on $m = s$, but if F were to put strictly positive probabilities only on $\underline{s}$ and $\overline{s}$, then all marginal distributions H of m on S for which the mean of m is $\mathbb{E}_F[s]$ would be feasible.

Similarly to the previous example, the probability that the receiver acts (2.4) does not necessarily decrease as the receiver's private information becomes more precise in that G becomes more variable in the mean-preserving spread sense. To see this, consider F that puts probability one on s and note that $G(s)$ changes ambiguously. However, the probability that the receiver acts (2.4) unambiguously increases as the receiver's private information becomes more favorable for acting in that G increases pointwise (first-order stochastic dominance).

Proposition 2.2 shows that the optimal mechanism depends on convexity and concavity properties of G on the set S.

Proposition 2.2 *In this example:*

1. *All mechanisms are equivalent if and only if G is linear on S.*

2. *ϕ_{full} is optimal if and only if G is convex on S.*

3. *ϕ_{no} is optimal if and only if the concave closure* $\mathbf{G}$ *of G on S is equal to G at $r_{no} \equiv \mathbb{E}_F[s]$ in*

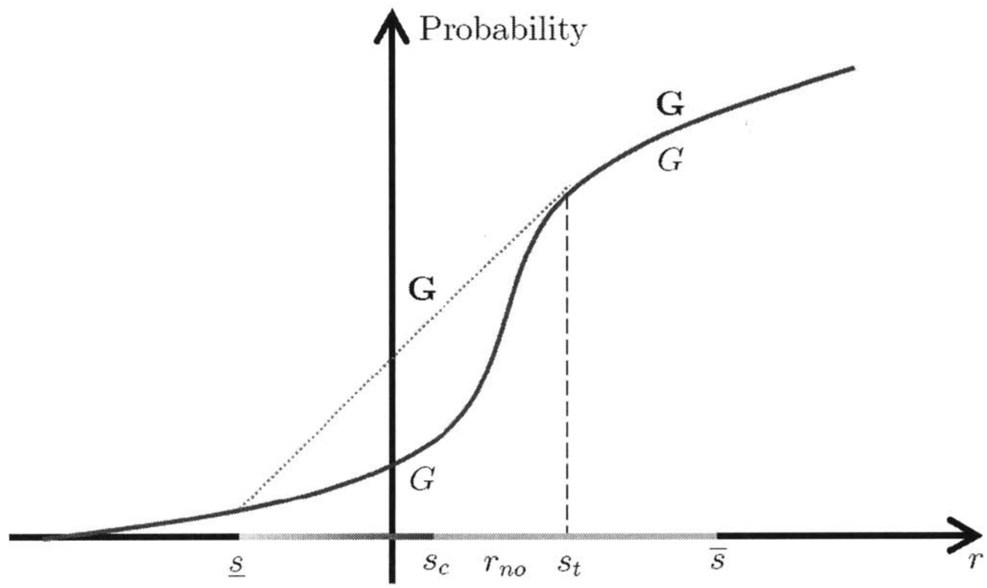

Figure 2-2: The distribution G and its concave closure $\mathbf{G}$ for $r \in S$.

that[8]

$$G(r_{no}) \geq \frac{r_2 - r_{no}}{r_2 - r_1} G(r_1) + \frac{r_{no} - r_1}{r_2 - r_1} G(r_2)$$

for all $r_1, r_2 \in S$ *such that* $r_{no} \in (r_1, r_2)$.

4. *If* G *is convex on* $[\underline{s}, s_i]$, *concave on* $[s_i, \overline{s}]$, *and* $\mathbf{G}(r_{no}) > G(r_{no})$, *then the optimal mechanism sends* $m = s$ *for* $s < s_c$ *and* $m = r_c \equiv \mathbb{E}[s | s \geq s_c]$ *for* $s \geq s_c$ *where* s_c *is uniquely pinned down by*

$$G'(r_c) = \frac{G(r_c) - G(s_c)}{r_c - s_c}.$$

The first three parts of Proposition 2.2 are immediate because the optimal mechanism is the solution to problem (2.4). First, if G is linear, then the sender is risk neutral, so all mechanisms are equivalent. Second, if G is convex, then the sender is risk loving, so the full revelation mechanism is optimal. Third, if G is concave, then the sender is risk averse, so the no revelation mechanism is optimal.

[8] *Intuitively, a concave closure of a function (defined on a convex set) is the smallest concave function that is everywhere greater than the original function.*

The last part of Proposition 2.2 derives the optimal mechanism under a natural assumption that the distribution G has an "S" shape as shown in Figure 2-2.[9] Let $\mathbf{G}$ denote the concave closure of G on S. Assume that $\mathbb{E}_F[s] < s_t$, otherwise ϕ_{no} is optimal by part 3. If F were to put strictly positive probabilities only on $\underline{s}$ and $\overline{s}$, then the optimal mechanism would send two messages $\underline{s}$ and s_t and the probability that the receiver acts would be $\mathbf{G}(r_{no})$. This mechanism, however, is not feasible for F that admits a density because s is equal to $\underline{s}$ with probability 0. Thus, the optimal mechanism reveals s for $s < s_c$ and sends the same message for all $s \geq s_c$ where the cutoff s_c is such that the sender is indifferent between revealing s_c or pooling it with $s \geq s_c$.

2.4 General Case

This section generalizes the examples of Section 2.3. The key assumption that is maintained throughout this section is that the receiver with a higher type is always more willing to act.[10] Section 2.4.1 develops necessary machinery and partially characterizes the optimal mechanism. Section 2.4.2 completely characterizes necessary and sufficient conditions for optimality of the two most important mechanisms, the full revelation and no revelation mechanisms.

2.4.1 Characterization of Optimal Mechanism

I start by discussing the key assumption. Then I turn to the characterization of the optimal mechanism. If the sender's type is binary, then similarly to the binary example, the optimal mechanism maximizes the linear utility function subject to the linear budget constraint. However, if the sender's type is not binary, then the budget constraint becomes multidimensional and it becomes hard to find the optimal mechanism. Nevertheless, the optimal mechanism still solves a linear program and thus duality theory applies. Using duality theory, I characterize general properties of the optimal mechanism and show how we can solve the inverse problem. Namely, the inverse problem is to find necessary and sufficient conditions on the primitives of the model that ensure that any candidate mechanism is optimal. For example, we can find primitive conditions on the environment that guarantee that a grading policy chosen by a school is optimal.

In this section, I impose the following *single-crossing assumption.* If the receiver with a given

[9] It is straightforward to characterize the optimal mechanism if the distribution G has more than one inflection points at which the curvature of G changes sign. Details are available on request.

[10] In the binary example, the optimistic receiver is more willing to act. In the continuous example, the receiver with a higher type $\widetilde{r} = -r$ is more willing to act.

type prefers to act upon receiving a certain message of an arbitrary mechanism, then the receiver with a higher type also prefers to act upon receiving the same message. Moreover, for each s there exists the unique and distinct type r such that the receiver r is indifferent to act. Formally, this assumption can be stated as follows. The function $v_H(r) \equiv \int_S \widetilde{u}(r,s)\,dH(s)$ crosses the horizontal axis once and from below for all distributions H that have the same support as F, where $\widetilde{u}(r,s) \equiv u(r,s)\,g(r|s)$. Moreover, $\widehat{r}(s)$ is strictly decreasing in s where $\widehat{r}(s)$ is the unique r that solves $u(r,s) = 0$.[11]

The single-crossing assumption allows us to restrict attention to mechanisms ϕ such that a message m induces the receiver to act if $r \geq m$. Both examples of Section 2.3 satisfy the single-crossing assumption, but the binary case with misalined preferences in Appendix A does not. To illustrate broad applicability of this assumption, Proposition 2.3 gives primitive sufficient conditions for a weak version of the single-crossing assumption.

Proposition 2.3 *If $u(r,s)$ is strictly increasing in both r and s, and the density $g(r|s)$ has the monotone likelihood ratio property in that $g(r_2|s_2)\,g(r_1|s_1) - g(r_2|s_1)\,g(r_1|s_2) \geq 0$ for all $s_2 \geq s_1$ and $r_2 \geq r_1$, then $v_H(r)$ crosses the horizontal axis at most once and from below for all distributions H.*

Before turning to the general problem where both the sender's and receiver's types are continuous, it is instructive to consider the case where the receiver's type is continuous but the sender's type is binary in that $G(r|S)$ admits a density $g(r|s)$ but F is supported on $\underline{s}$ and $\overline{s}$. For all $r \in \widehat{R} \equiv [\widehat{r}(\overline{s}), \widehat{r}(\underline{s})]$, denote $p(r)$ as the probability of $\overline{s}$ at which the receiver r is indifferent to act. Thus, in the optimal mechanism, the distribution H of messages

$$\text{maximizes} \int_{\widehat{R}} \Pr(r \geq m|m)\,dH(m)$$

$$\text{subject to} \int_{\widehat{R}} p(m)\,dH(m) = \Pr(\overline{s}).^{12}$$

[11] I believe that it is sufficient to assume that $v_H(r)$ crosses the horizontal axis at most once and in the same direction. Extending $u(r,s)$ to $\widetilde{R} \supset R$ for all s and making $g(r|s)$ infinitesimally small for all s and $r \notin R$ yields that $v_H(r)$ crosses the horizontal axis exactly once on $\widetilde{R}$. Reordering R yields that $v_H(r)$ crosses the horizontal axis from below. Considering H that puts probability one on s yields that $u(r,s)$ crosses the horizontal axis once for all s, so $\widehat{r}(s)$ is well defined. Finally, reordering S yields that $\widehat{r}(s)$ is decreasing.

[12] Explicitly, $p(r) = \frac{-\widetilde{u}(r,\underline{s})}{\widetilde{u}(r,\overline{s}) - \widetilde{u}(r,\underline{s})}$ and $\Pr(r \geq m|m) = p(r)\,\overline{G}(r|\overline{s}) + (1 - p(r))\,\overline{G}(r|\underline{s})$.

The objective function is the probability that the receiver acts and the constraint is the feasibility constraint that requires that posterior probabilities $\Pr(\overline{s}|m)$ average out to the prior probability $\Pr(\overline{s})$. Again, the objective function can be interpreted as a linear utility function and the constraint as a Bayesian budget constraint. As a result, the sender faces the same tradeoff between the frequency and the persuasiveness of messages as in the binary example of Section 2.3. Sending a lower message m is more expensive (the price $p(m)$ is higher), but it has a greater impact on the receiver (the marginal utility $\Pr(r \geq m|m)$ is higher). To resolve this tradeoff, the optimal mechanism sends at most two messages with the highest marginal utility-price ratio $\frac{\Pr(r\geq m|m)}{p(m)}$.[13]

In general (if the sender's type is continuous), the optimal mechanism is a distribution ϕ that

$$\text{maximizes} \int_{R\times S} \overline{G}(r|s)\, d\phi(r,s) \tag{2.6}$$

$$\text{subject to} \int_{R\times\widetilde{S}} d\phi(r,s) = \int_{\widetilde{S}} f(s)\, ds \text{ for any measurable set } \widetilde{S}\subset S, \tag{2.7}$$

$$\int_{\widetilde{R}\times S} \widetilde{u}(r,s)\, d\phi(r,s) = 0 \text{ for any measurable set } \widetilde{R}\subset R. \tag{2.8}$$

The objective function is the probability that the receiver acts under the mechanism ϕ. The first constraint (2.7) is the requirement that the marginal distribution of s for ϕ is F. Intuitively, (2.7) is a multidimensional Bayesian budget constraint. The second constraint (2.8) is the requirement that a message r makes the receiver r indifferent to act. Intuitively, (2.8) determines multidimensional prices of various messages.

The problem (2.6) is called the *primal problem.* This primal problem is a linear program, which is analogous to the mass transfer problem, an infinite dimensional extension of the well-known transportation problem. However, the transportation problem is notorious for being hard to analyze. The *dual problem* is to find bounded functions η and ν that

$$\text{minimize} \int_S \eta(s) f(s)\, ds \tag{2.9}$$

$$\text{subject to } \eta(s) + \widetilde{u}(r,s)\,\nu(r) \geq \overline{G}(r|s) \text{ for all } (r,s)\in R\times S. \tag{2.10}$$

Say that ϕ is *feasible* for (2.6) if it is a distribution that satisfies (2.7) and (2.8). Similarly, say that η and ν are *feasible* for (2.9) if they are bounded functions that satisfy (2.10). Feasible ϕ and

[13] The optimal mechanism is a solution to a linear program, so it is an extreme point of the constraint set. Thus, if s is binary, the optimal mechanism sends at most two messages.

(η, ν) that solve their respective problems (2.6) and (2.9) are called *optimal solutions.*

The properties of the primal and dual problems are intimately linked by duality theory. Only weak duality is used to establish the coming results:

Lemma 2.1 *If ϕ is feasible for (2.6), and (η, ν) is feasible for (2.9), then*

$$\int_S \eta(s) f(s)\, ds \geq \int_{R\times S} \overline{G}(r|s)\, d\phi(r,s)\,. \tag{2.11}$$

Moreover, if inequality (2.11) holds with equality, then ϕ and (η, ν) are optimal solutions, and

$$\int_{R\times S} \left(\eta(s) + \widetilde{u}(r,s)\,\nu(r) - \overline{G}(r|s)\right) d\phi(r,s) = 0. \tag{2.12}$$

Strong duality establishes the existence of an optimal mechanism and other interesting properties of optimal solutions to problems (2.6) and (2.9):

Lemma 2.2 *There exists an optimal mechanism ϕ, an optimal solution to the primal problem (2.6). There exists an optimal solution to the dual problem (2.9) such that functions η and ν are continuous. Inequality (2.11) holds with equality for these optimal ϕ and (η, ν).*

Now I show how, using duality theory, we can find necessary and sufficient conditions on the primitives u, F, and G that guarantee that a given mechanism is optimal. For a given candidate optimal mechanism ϕ, the complementarity condition (2.12) implies that (2.10) holds with equality ($\eta(s) = \overline{G}(r|s) - \widetilde{u}(r,s)\,\nu(r)$) at each (r,s) in the support of ϕ for a candidate optimal solution (η, ν) to the dual problem (2.9). Then we can find primitive conditions on $u(r,s)$, $F(s)$, and $G(r|s)$ such that the constraint (2.10) of the dual problem (2.9) is satisfied for all $(r,s) \in R \times S$ and some $\nu(r)$. Weak duality implies that these conditions are sufficient for ϕ to be optimal, and strong duality implies that these conditions are necessary for ϕ to be optimal. It turns out, however, that the necessary conditions can be easily established directly without the need for strong duality.

I conclude this section by introducing a few new definitions that are used in the next section. Let r_{no} be the unique r that solves $\int_S \widetilde{u}(r,s) f(s)\, ds = 0$. Note that the no revelation mechanism ϕ_{no} sends the same message r_{no} for all $s \in S$, whereas the full revelation mechanism ϕ_{full} sends the message $\widehat{r}(s)$ for each $s \in S$.

For any $s_1 < s_2$, and $r \in (\widehat{r}(s_2), \widehat{r}(s_1))$, the sender *prefers to reveal s_1 and s_2 than to pool* *them at r* if

$$\frac{\overline{G}\left(\widehat{r}\left(s_2\right)|s_2\right)-\overline{G}\left(r|s_2\right)}{\widetilde{u}\left(r,s_2\right)} \geq \frac{\overline{G}\left(\widehat{r}\left(s_1\right)|s_1\right)-\overline{G}\left(r|s_1\right)}{\widetilde{u}\left(r,s_1\right)}. \tag{2.13}$$

Similarly, the sender *prefers to pool* s_1 *and* s_2 *at* r *than to reveal them* if inequality (2.13) is reversed to "$\leq$". Finally, the sender *is indifferent to reveal* s_1 *and* s_2 *or to pool them at* r if (2.13) holds with equality. To provide the intuition for these definitions, suppose that the prior distribution of s puts probabilities $\frac{\widetilde{u}(r,s_2)}{\widetilde{u}(r,s_2)-\widetilde{u}(r,s_1)}$ on s_1 and $\frac{\widetilde{u}(r,s_1)}{\widetilde{u}(r,s_1)-\widetilde{u}(r,s_2)}$ on s_2. If inequality (2.13) holds, then the full revelation mechanism, which sends $\widehat{r}(s_1)$ and $\widehat{r}(s_2)$ for s_1 and s_2, respectively, dominates the no revelation mechanism, which sends the same message r for s_1 and s_2.

Say that s_1, s_2, s_3, r *are feasible* if $(s_3-s_2)(s_2-s_1)>0$, and there exists the prior distribution that puts probabilities p_1, p_2, p_3 on s_1, s_2, s_3 such that $\sum_{i=1}^{3} p_i\widetilde{u}\left(r_{no},s_i\right)=0$, and $\sum_{i=1}^{2} p_i\widetilde{u}\left(r,s_i\right)=0$.[14] For any feasible s_1, s_2, s_3, r, the sender *prefers to pool* s_1, s_2, s_3 *at* r_{no} *than to pool* s_1, s_2 *at* r *and to reveal* s_3 if for the above prior distribution, the no revelation mechanism, which sends r_{no} for s_1, s_2, s_3, dominates the mechanism that sends r for s_1, s_2 and $\widehat{r}(s_3)$ for s_3.[15] The obvious modifications of this definition are constructed similarly to the previous paragraph.

2.4.2 Optimality of Specific Mechanisms

By definition, a mechanism ϕ is optimal if and only if it dominates all feasible mechanisms. This gives trivial necessary and sufficient conditions for optimality of ϕ. However, to check these conditions, one needs to compare ϕ to all feasible mechanisms, which requires a lot of comparisons. It turns out that for the optimality of ϕ, it is necessary and sufficient to check that only certain deviations from ϕ do not increase the probability that the receiver acts.

Using duality theory, this section presents necessary and sufficient conditions for (i) all mechanisms to be equivalent, (ii) the full revelation mechanism ϕ_{full} to be optimal, and (iii) the no revelation mechanism ϕ_{no} to be optimal. There are at least two reasons that make mechanisms ϕ_{no} and ϕ_{full} prominent. First, if S did not have full commitment, then ϕ_{no} would be the unique equilibrium outcome if the sender's information were unverifiable in the sense of Crawford and

[14]The existence of such p_1, p_2, p_3 is equivalent to $(r_{no}-\widehat{r}(s_1))(r_{no}-\widehat{r}(s_3))<0$, and $(r-\widehat{r}(s_1))(r-\widehat{r}(s_2))<0$.

[15]Mathematically,

$$\begin{aligned}&\tfrac{1}{\widetilde{u}(r,s_2)}\left(\overline{G}\left(r_{no}|s_2\right)-\overline{G}\left(r|s_2\right)+\tfrac{\widetilde{u}(r_{no},s_2)}{\widetilde{u}(r_{no},s_3)}\left(\overline{G}\left(\widehat{r}\left(s_3\right)|s_3\right)-\overline{G}\left(r_{no}|s_3\right)\right)\right)\\ \geq\ &\tfrac{1}{\widetilde{u}(r,s_1)}\left(\overline{G}\left(r_{no}|s_1\right)-\overline{G}\left(r|s_1\right)+\tfrac{\widetilde{u}(r_{no},s_1)}{\widetilde{u}(r_{no},s_3)}\left(\overline{G}\left(\widehat{r}\left(s_3\right)|s_3\right)-\overline{G}\left(r_{no}|s_3\right)\right)\right).\end{aligned} \tag{2.14}$$

Sobel (1982), and ϕ_{full} would be the unique equilibrium outcome if the sender's information were verifiable in the sense of Milgrom (1981).[16] The second reason is that these two mechanisms are extremal as Proposition 2.4 shows.

Proposition 2.4 *Let the single-crossing assumption holds.*

1. *The receiver's expected utility under ϕ_{no} is strictly lower than under any other mechanism.*

2. *The receiver's expected utility under ϕ_{full} is strictly higher than under any other mechanism.*

A more informed receiver is better at maximizing his expected utility by taking a more appropriate action, so a weak version of Proposition 2.4 is immediate. The single-crossing assumption guarantees the strict version of Proposition 2.4. Note that the strict version does not hold when the receiver is uninformed. Indeed, in the previous chapter, I show that in the case of an uninformed receiver, the optimal mechanism is different from ϕ_{no}, yet the receiver's expected utility under the optimal mechanism is the same as under ϕ_{no}.

The next result, which is the main result of this section, characterizes when each of the above mechanisms is optimal.

Theorem 2.1 *Let the single-crossing assumption holds. Then:*

1. *All mechanisms are equivalent if and only if there exists a positive function $b(r)$ such that for all $r \in (\widehat{r}(\overline{s}), \widehat{r}(\underline{s}))$, $\widetilde{u}(r,s)$ is*

$$\widetilde{u}(r,s) = \frac{\overline{G}(\widehat{r}(s)|s) - \overline{G}(r|s)}{b(r)}. \tag{2.15}$$

2. *ϕ_{full} is optimal if and only if the sender prefers to reveal s_1 and s_2 than to pool them at r for all $s_1, s_2 \in S$ and $r \in (\widehat{r}(s_2), \widehat{r}(s_1))$, so that (2.13) holds.*

3. *ϕ_{no} is optimal if and only if the sender prefers to pool s_1, s_2, s_3 at r_{no} than to pool s_1, s_2 at r and to reveal s_3 for all feasible s_1, s_2, s_3, r, so that (2.14) holds.*[17]

[16] Under unverifiable communication, if the sender sent two different messages r_1 and r_2 in equilibrium, then the sender would strongly prefer to send $\min\{r_1, r_2\}$ regardless of s, which leads to a contradiction. Under verifiable communication, if the sender sent the same message r for two or more different s in equilibrium, then there would exist $\widetilde{s}$ such that the sender sent r for $\widetilde{s}$ but $u(r,\widetilde{s}) > 0$, which leads to a contradiction because the sender with $\widetilde{s}$ would strongly prefer to reveal $\widetilde{s}$.

[17] *As the proof of this part shows, the condition of this part can be replaced with two weaker conditions: (i) the sender prefers to pool s_1 and s_2 at r_{no} than to reveal them for all s_1 and s_2 such that $r \in (\widehat{r}(s_2), \widehat{r}(s_1))$; (ii) the condition of this part holds only for $s_3 \to s_{no}$ where s_{no} is the unique s that solves $u(r_{no}, s) = 0$.*

The "only if" parts of Theorem 2.1 are straightforward because for optimality of a candidate mechanism, we need to check all deviations from this mechanism, including those described in the theorem.

To provide the intuition for "if" parts of Theorem 2.1, I present Lemma 2.3, which is not used in the formal proof of the theorem. Note that a message m generates a lottery $\phi(.|m)$ over S that makes the receiver indifferent to act if $r = m$. This intuitive lemma shows that this lottery can be decomposed into simpler lotteries indexed by e in such a way that (i) the support of each lottery e contains at most two elements, and (ii) each lottery e makes the receiver r indifferent to act.

Lemma 2.3 *Let the single-crossing assumption holds. For each mechanism $\phi(r, s)$, there exists a mechanism $\varphi(m, s)$ with two dimensional messages $m = (r, e) \in R \times [0, 1]$ that satisfies the following properties:*

1. *For each $m = (r, e)$, $\int_S \widetilde{u}(r, s)\, d\varphi(s|m) = 0$.*
2. *For each m, the support of $\varphi(.|m)$ contains at most two elements of S.*

We now discuss each "if" part of Theorem 2.1 in turn. We start with part 1 of Theorem 2.1. If all mechanisms are equivalent, then the sender is indifferent to reveal s_1 and s_2 or to pool them at r for all possible s_1, s_2, and r, because mechanisms that differ only in that one reveals s_1 and s_2 and the other pools them at r, are equivalent. Therefore, all conditions (2.13) must hold with equality, which is equivalent to $\widetilde{u}(r, s)$ being given by (2.15), so the "only if" part follows. Conversely, if the sender is indifferent to reveal s_1 and s_2 or to pool them at r for all possible s_1, s_2, and r, then all mechanisms are equivalent because by Lemma 2.3 we can focus on mechanisms in which each message is sent only by some two types s_1 and s_2. Consider such a mechanism. Since the sender is indifferent to reveal s_1 and s_2 or to pool them, this mechanism is equivalent to the mechanism that differs only in that it reveals s_1 and s_2. Sequentially modifying the mechanism for each message, we get that any mechanism is equivalent to ϕ_{full}, so the "if" part follows.

As part 1 shows, all mechanisms are equivalent in the knife-edge case when $\widetilde{u}(r, s)$ has representation (2.15). If s and r are independent, then without loss of generality we can assume that r is uniformly distributed on $[-1, 0]$ and $\widehat{r}(s) = -s$.[18] In this situation, part 1 essentially states that all mechanisms are equivalent whenever $u(r, s) = \frac{r+s}{b(r)}$ for some positive function $b(r)$ where

[18] Redefining r as $-\overline{G}(r)$ and s as $\overline{G}(\widehat{r}(s))$ gives the required properties. It is clear that all functions are smooth and the single-crossing assumption holds for the model with redefined r and s because smooth increasing transformations are used.

$r \sim U[-1,0]$, and the support of s is contained in the interval $[0,1]$. Note that the continuous example of Section 2.3 assumes the same functional form of the receiver's utility, but it does not assume that r is uniformly distributed.[19]

Rayo and Segal (2010) analyze a similar model with a few distinctions. Similar to this chapter, they assume that actions are binary and the sender has full commitment. In contrast to this chapter, they allow the sender's utility to depend on both the action and the state. However, they assume that the utility of the receiver from acting is $u(r,s) = r+s$, where r is uniformly distributed on $[-1,0]$, and the support of s is contained in the interval $[0,1]$. This assumption allows them to bypass the effects analyzed in this chapter and get tractable results.

We now turn to part 2 of Theorem 2.1. The "only if" part holds because ϕ_{full} dominates the mechanism that differs only in that it pools s_1 and s_2 at r. The "if" part holds again because we can focus on mechanisms in which each message is sent only by some two types s_1 and s_2. Consider such a mechanism. Since the sender prefers to reveal s_1 and s_2 than to pool them, this mechanism is dominated by the mechanism that differs only in that it reveals s_1 and s_2. Sequentially modifying the mechanism for each message, we get that ϕ_{full} dominates all mechanisms, so the "if" part follows

Finally, we discuss part 3 of Theorem 2.1. The "only if" part holds because ϕ_{no} dominates the mechanism that differs only in that it pools s_1, s_2 at r and reveals s_3. I provide the intuition for a weaker version of the "if" part. Namely, if the sender prefers to pool s_1, s_2, s_3, s_4 at r_{no} than to pool s_1, s_2 at $r_{1,2}$ and to pool s_3, s_4 at $r_{3,4}$ for all feasible s_1, s_2, $r_{1,2}$, s_3, s_4, $r_{3,4}$, then ϕ_{no} is optimal. Again we can focus on mechanisms such that any message $r_{1,2} \leq r_{no}$ is sent only by some two types s_1 and s_2 and any message $r_{3,4} \geq r_{no}$ is sent only by some two types s_3 and s_4. Any such mechanism is dominated by the mechanism that differs only in that it sends the message r_{no} instead of $r_{1,2}$ and $r_{3,4}$. Sequentially applying this argument for pairs of messages, we get that ϕ_{no} dominates all mechanisms, so the weaker version of the "if" part follows.

2.5 Extensions: Two-Way Communication

This section extends the benchmark model of Section 2.2 to allow communication from the receiver to the sender. Assume that the sender has full commitment in that she chooses a mechanism before

[19] To map the example, we need to redefine r as $-r$ and drop $b(r)$ that does not affect the receiver's behavior for any message.

(s,r) is realized and therefore before the receiver makes a report to the sender. In this case, the revelation principle applies (Myerson (1982)).[20]

Thus, it is without loss of generality to consider the following timing: 1. The sender publicly chooses a mechanism, a conditional distribution $\gamma(m|s,n)$ of a message m given the sender's type s and the receiver's report n. 2. The receiver's type r is drawn according to G. 3. The receiver privately observes r and makes a report $n \in N$. 4. A vector (m,s) is drawn according to γ and F. 5. The receiver gets a message m and takes an action a.

Further, it is without loss of generality to focus on an incentive compatible direct mechanism. In a direct mechanisms, (i) the set of receiver's reports N coincides with the set R; and (ii) a mechanism γ sends only two messages m_1 and m_0.[21] In an incentive compatible direct mechanism, also, (iii) the receiver r prefers to report $n = r$; and (iv) the receiver prefers to act if he receives m_1 and not to act if he receives m_0, regardless of r and n.

Under certain assumptions, which, in particular, allow the binary example, it is without loss of generality to consider benchmark mechanisms where the receiver is not allowed to make reports:

Proposition 2.5 *In the binary example of Section 2.3, the set of mappings from (s,r) to the receiver's action a that can be supported by an incentive compatible mechanism is the same under $\phi(m|s)$ and under $\gamma(m|s,n)$.*

The following example shows that Proposition 2.5 does not hold generally:

Example 2.1 *Let $s = (s_1, s_2) \in \{0,1\} \times \{0,1\}$ and $r \in \{r_1, r_2, r_3\}$. Moreover, let all combinations of (s,r) be equally likely. Finally, let $u(s,r_1) = s_1 - 1$, $u(s,r_2) = s_2 - 1$, and $u(s,r_3) = \frac{3}{4} - s_1 s_2$.*

Consider the following mechanism of the two-way communication game:

$$\gamma^*(m_1|s,n) = \begin{cases} 1 \text{ if } n = r_1 \text{ and } s_1 = 1, \text{ or if } n = r_2 \text{ and } s_2 = 1, \text{ or if } n = r_3, \\ 0 \text{ otherwise.} \end{cases}$$

Intuitively, this mechanism reveals the component s_1 if the receiver reports r_1 and reveals the component s_2 if the receiver reports r_2. Clearly, under this mechanism, it is incentive compatible

[20] To map my model into Myerson (1982), assume that the principal is the sender who designs a mechanism for two agents. The first agent has type s, has no action to take, and always gets 0 utility. The second agent has type r, privately chooses $a = 0, 1$, and his utility is $au(t^R, t^S)$.

[21] In this case, $\gamma(m_1|s,n)$ denotes the probability of the message m_1.

for the receiver to truthfully report r and act whenever he receives m_1. Thus, the probability that the receiver acts under γ^* is:

$$\Pr{}_{\gamma^*}(a=1) = \Pr(r=r_1)\Pr(s_1=1) + \Pr(r=r_2)\Pr(s_2=1) + \Pr(r=r_3) = \frac{2}{3}.$$

However, the sender cannot induce the receiver to act with probability $\frac{2}{3}$ in the benchmark model. To see this, note first that the receiver r_1 acts only if he is certain that $s_1 = 1$, and the receiver r_2 acts only if he is certain that $s_2 = 1$. Thus, under any mechanism ϕ, the probability that the receiver acts cannot exceed $\frac{2}{3}$. The only possibility of how the sender could achieve this probability would be to reveal both s_1 and s_2, but in that case the receiver with r_3 would not act when $(s_1, s_2) = (1, 1)$.

2.6 Conclusions

In this chapter, I have studied optimal information revelation mechanisms with two-sided asymmetric information. The receiver bases his action not only on the information revealed by the sender but also on his private information. Thus, from the sender's perspective, each message results in a stochastic action by the receiver. The analysis reveals an important tradeoff between the frequency and the persuasiveness of messages. The optimal mechanism finds a balance between these two conflicting objectives. This balance is easiest to explain when the sender's information has a binary structure. In this case, the prior distribution of the sender's information imposes a budget constraint on the frequencies of various messages, whereas the distribution of the receiver's information determines the sender's expected utility, which is linear in the frequencies of various messages. The optimal mechanism sends messages with the highest marginal utility-price ratio.

I also derive interesting non-monotone comparative statics results with respect to the receiver's private information for the binary example in which the sender is perfectly informed but the receiver is partially informed. If the receiver's private information is either very precise or very imprecise, then the sender's expected utility decreases and the receiver's expected utility increases in the precision of the receiver's information. However, if the precision of the receiver's information is intermediate, then these results can be overturned. Surprisingly, the receiver may become worse off as he becomes more informed even though he is the only player who takes an action that directly affects his utility. Thus, if there is an earlier stage when the receiver can publicly choose how informed he will be, he may not want to be as informed as possible.

2.7 Appendix A: Binary Case

When the receiver has private information, in general, the problem of finding the optimal mechanism becomes complicated, as Section 2.4 suggests. This section fully characterizes the optimal mechanism when s and r are binary. More formally, assume that F puts strictly positive probabilities only on s_1 and s_2 and that $G(.|s_1)$ and $G(.|s_2)$ put strictly positive probabilities only on r_1 and r_2.

The binary case splits into two subcases. In the first subcase, one sender's signal is more favorable for acting than the other sender's signal, regardless of r. In this subcase, the sender faces a tradeoff between the frequency and the persuasiveness of messages. As a result, neither the full revelation nor the no revelation mechanisms are generally optimal. In the second subcase, different sender's signals arc favorable for acting depending on r. In this subcase, the sender faces a more subtle tradeoff, which results in a variety of possible optimal mechanisms. In particular, both the full revelation and no revelation mechanisms can generally be optimal.

Using the revelation principle, for any mechanism, we can find an equivalent mechanism that sends at most four messages: (i) $m_\emptyset$ that induces the receiver not to act for all r, (ii) m_1 that induces the receiver to act only if $r = r_1$, (iii) m_2 that induces the receiver to act only if $r = r_2$, and (iv) $m_{1,2}$ that induces the receiver to act for all r.

For notational simplicity, this section uses different notation than the rest of the chapter. In particular, denote $p_j \equiv \Pr(s_j)$, $p_{i|j} \equiv \Pr(r_i|s_j)$, $u_{ij} \equiv u(r_i, s_j)$, $\widetilde{u}_{ij} \equiv u_{ij}p_{i|j}$, $k_i = \frac{\widetilde{u}_{i1}}{\widetilde{u}_{i1}-\widetilde{u}_{i2}}$, and $\phi^j_K \equiv \Pr_\phi(m = m_K, s = s_j)$ for $i, j = 1,2$ and $K = \{\emptyset\}, \{1\}, \{2\}, \{1,2\}$. Indexes i and j are reserved for r and s, respectively. Note that k_i is the cutoff posterior belief $\Pr(s_2)$ at which the receiver r_i is indifferent to act because

$$\mathbb{E}[u|r_i] = \frac{\widetilde{u}_{i1}(1 - \Pr(s_2)) + \widetilde{u}_{i2}\Pr(s_2)}{\Pr(r_i)} = 0.$$

Aligned Preferences

If one sender's signal is more favorable for acting than the other sender's signal, regardless of r, then the analysis is analogous to the binary example of Section 2.3. In particular, the sender faces a tradeoff between the frequency and the persuasiveness of messages, which is resolved by the choice of a mechanism that sends messages with the highest marginal utility-price ratio.

To make the analysis non-redundant, assume that $u_{i1} < 0 < u_{i2}$ for $i = 1, 2$, $k_2 < k_1$, and

$p_2 < k_1$. Strict inequalities rule out non-generic cases. Inequalities $u_{i2} > u_{i1}$ and $k_2 < k_1$ can be obtained by relabelling elements of S and R, respectively. If u_{i1} and u_{i2} had the same sign for some i, then the receiver r_i would take the same action regardless of the mechanism and the analysis would be as if the receiver is uninformed as in the previous chapter. Finally, if $p_2 \geq k_1$, the no revelation mechanism would induce the receiver to act for all r, and, thus, it would be optimal.

Under this assumption, the optimal mechanism can take the three forms that were identified in Section 2.3, as follows from:

Proposition 2.6 *If $u_{i1} < 0 < u_{i2}$, $k_2 < k_1$, and $p_2 < k_1$, then the optimal mechanism sends two messages.*

1. *If $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} \geq \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$, it sends $m_{1,2}$ and $m_\emptyset$: $m_{1,2}$ with certainty if $s = s_2$ and with probability $-\frac{\widetilde{u}_{12}}{\widetilde{u}_{11}} \frac{p_2}{p_1}$ if $s = s_1$.*
2. *If $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} < \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$ and $p_2 < k_2$, it sends m_2 and $m_\emptyset$: m_2 with certainty if $s = s_2$ and with probability $-\frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} \frac{p_2}{p_1}$ if $s = s_1$.*
3. *If $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} < \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$ and $p_2 \geq k_2$, it sends m_2 and $m_{1,2}$: m_2 with probability $\frac{\widetilde{u}_{21}}{p_2} \frac{\widetilde{u}_{11} p_1 + \widetilde{u}_{12} p_2}{\widetilde{u}_{12} \widetilde{u}_{21} - \widetilde{u}_{11} \widetilde{u}_{22}}$ if $s = s_2$ and with probability $\frac{\widetilde{u}_{22}}{p_1} \frac{\widetilde{u}_{11} p_1 + \widetilde{u}_{12} p_2}{\widetilde{u}_{11} \widetilde{u}_{22} - \widetilde{u}_{12} \widetilde{u}_{21}}$ if $s = s_1$.*

In all cases, $m_\emptyset$ reveals s_1 in that $\Pr_\phi(s_2|m_\emptyset) = 0$; m_2 makes the receiver r_2 indifferent to act in that $\Pr_\phi(s_2|m_2) = k_2$; and $m_{1,2}$ makes the receiver r_1 indifferent to act in that $\Pr_\phi(s_2|m_{1,2}) = k_1$. The receiver's expected utility under the optimal mechanism is strictly greater than that under the no revelation mechanism only in case 1.[22]

The intuition for Proposition 2.6 is analogous to that in the binary example of Section 2.3. The receiver r_i acts upon receiving a message m under a mechanism ϕ if $\Pr_\phi(s_2|m) \geq k_i$. If the message m persuades the receiver r_1 to act, it also persuades the receiver r_2 to act because $k_2 < k_1$ by assumption. Thus, we can restrict attention to mechanisms with the three messages $m_\emptyset$, m_2, and $m_{1,2}$. To maximize the probability that the receiver acts, each message of the optimal mechanism either makes the receiver exactly indifferent to act for some r ($\Pr_\phi(s_2|m_{1,2}) = k_1$ and $\Pr_\phi(s_2|m_2) = k_2$) or makes the receiver certain that $s = s_1$ so that it is optimal not to act ($\Pr_\phi(s_2|m_\emptyset) = 0$).

[22] *If s and r are independent, then $p_{i|2} = p_{i|1} = \Pr(r_i)$, so $\widetilde{u}_{ij}$ can be replaced with u_{ij} for all $i, j = 1, 2$ in all expressions.*

Thus, the sender's problem is to maximize the probability that the receiver acts:

$$\left(k_2 p_{2|2} + (1 - k_2)\, p_{2|1}\right) q_2 + q_{1,2}$$

over probabilities $q_\emptyset$, q_2, and $q_{1,2}$ of the messages $m_\emptyset$, m_2, and $m_{1,2}$ subject to the constraint imposed by the prior distribution of s:

$$k_2 q_2 + k_1 q_{1,2} = p_2.$$

Similar to Section 2.3, we can interpret k_2 and k_1 as unit prices of sending m_2 and $m_{1,2}$, and the probabilities $\left(k_2 p_{2|2} + (1 - k_2)\, p_{2|1}\right)$ and 1 as the marginal utilities of sending m_2 and $m_{1,2}$. If $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} \geq \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$, then the marginal utility-price ratio is highest for $m_{1,2}$, and the sender prefers to send $m_{1,2}$ than m_2, so the optimal mechanism sends $m_{1,2}$ and $m_\emptyset$. If $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} < \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$, then the ratio is highest for m_2, and the sender prefers to send m_2 than $m_{1,2}$. The optimal mechanism depends on whether the no revelation mechanism induces the receiver r_2 to act or not. If so $(p_2 \geq k_2)$, then it sends the messages m_2 and $m_{1,2}$, otherwise it sends the messages m_2 and $m_\emptyset$.

Misaligned Preferences

The main goal of this section is to illustrate the variety of possible optimal mechanisms in the case where different sender's signals are favorable for acting depending on the receiver's signal. Note that this case violates the single-crossing assumption of Section 2.4. For example, a school knows whether a student is good at natural sciences or liberal arts, but the school does not know which of these two qualities are valued by the employer.

All forms that the optimal mechanism can take are characterized by Proposition 2.7. Similar to the previous section, to make the analysis non-redundant, I impose certain assumptions.

Proposition 2.7 *If $u_{12} < 0 < u_{11}$, $u_{21} < 0 < u_{22}$, and $p_2 > k_1$, the optimal mechanism sends at most two messages.*

1. *If $k_2 \leq k_1$, it sends m_2 and $m_{1,2}$. The message m_2 reveals s_2 in that $\Pr_\phi(s_2|m_2) = 1$ and the message $m_{1,2}$ makes the receiver r_1 indifferent to act in that $\Pr_\phi(s_2|m_{1,2}) = k_1$.*

2. *If $k_2 > k_1$, then depending on parameters, it sends either only m_2 or both m_2 and m_1. If it sends both m_2 and m_1, there are four cases in which each message m_i either reveals s_i in that $\Pr_\phi(s_i|m_i) = 1$, or makes the receiver r_i indifferent to act in that $\Pr_\phi(s_2|m_i) = k_i$.*

I only sketch the intuition for this proposition because it is quite tedious and involves the consideration of many cases. Note that in this case, a message m that assigns a higher probability to s_2 is more persuasive for the receiver r_2, and less persuasive for the receiver r_1. The messages m_1 and m_2 are always feasible because revealing s_1 induces the receiver r_1 to act, and revealing s_2 induces the receiver r_2 to act. However, if $k_2 \leq k_1$ (part 1 of Proposition 2.7), then the message $m_{1,2}$ is feasible, but the message $m_\emptyset$ is not. In this case, the sender wants to send $m_{1,2}$ as often as possible. As a result, the optimal mechanism sends two types of messages: those that give minimal possible evidence to make the receiver act regardless of his signal, and those that reveal s. In contrast, if $k_1 < k_2$ (part 2 of Proposition 2.7), then the message $m_\emptyset$ is feasible, but the message $m_{1,2}$ is not. In this case, the optimal mechanism can take five different forms, which, in particular, include the no revelation and full revelation mechanisms.

2.8 Appendix B: Proofs

Proof of Proposition 2.1. For any mechanism ϕ, $r = \mathbb{E}_\phi[s|r]$, which implies that F is a mean-preserving spread of H. Conversely, if F is a mean-preserving spread of H, then s has the same distribution as $r + z$ for some z such that $\mathbb{E}[z|r] = 0$. Define $\phi(\widetilde{r}, \widetilde{s}) = \Pr(r \leq \widetilde{r}, r + z \leq \widetilde{s})$ for all $(\widetilde{r}, \widetilde{s}) \in R \times S$. The marginal distribution of ϕ is F and $\mathbb{E}_\phi[s - r|r] = \mathbb{E}[z|r] = 0$, ϕ is a feasible mechanism. Finally, $\int_{-\infty}^{\infty} G(m)\, dH(m)$ is simply the probability that the receiver acts. ∎

Proof of Proposition 2.2. Clearly, this example satisfies the single-crossing assumption of Section 2.4 after the following change in variables: $\widetilde{r} = -r$. Thus, all results of Section 2.4 apply. With this change of variables, $\widehat{r}(s) = -s$, and $\overline{G}_{\widetilde{r}}(x) \equiv \Pr(\widetilde{r} > x)$ is equal to $G(-x)$. I prove each part in turn.

1. By Theorem 2.1 part 1, all mechanisms are equivalent if and only if $s - r = \frac{G(s)-G(r)}{b(r)}$ for some positive b and all $s \in S$, $r \in S$, which is equivalent to G being linear on R.

2. By Theorem 2.1 part 2, ϕ_{full} is optimal if and only if the sender prefers to reveal s_1 and s_2 than to pool them at $\widetilde{r}$ where $\widetilde{r} \in (-s_2, -s_1)$, which is equivalent to

$$G(r) \leq \frac{s_2 - r}{s_2 - s_1} G(s_1) + \frac{r - s_1}{s_2 - s_1} G(s_2).$$

Clearly, this inequality holds if and only if G is convex on S.

3. By Theorem 2.1 part 3, ϕ_{no} is optimal if and only if the sender prefers to pool s_1, s_2, s_3 at

$\widetilde{r}_{no}$ than to pool s_1, s_2 at $\widetilde{r}$ and to reveal s_3 for all possible s_1, s_2, s_3, $\widetilde{r}$, which is equivalent to

$$G(r_{no}) \geq \frac{(s_3 - r_{no})}{(s_3 - r)} G(r) + \frac{(r_{no} - r)}{(s_3 - r)} G(s_3)$$

for all $r \in S$ and $s_3 \in S$ such that $(r - r_{no})(s_3 - r_{no}) < 0$. The change of variables $r_2 = s_3$ and $r_1 = r$ completes the proof.

4. Lemma 2.1 implies that the described mechanism is optimal if there exists feasible (η, ν) for (2.9):

$$\eta(s) + (s - r)\nu(r) \geq G(r) \text{ for all } (r, s) \in R \times S \tag{2.16}$$

such that weak duality condition (2.11) holds with equality. I now construct (η, ν) that satisfies (2.16). Note that condition (2.16) bounds ν only from one side for $r \notin S$. In particular, $\nu(r) \geq \frac{G(r)-\eta(s)}{s-r}$ if $r < \underline{s}$ and $\nu(r) \leq \frac{\eta(s)-G(r)}{r-s}$ if $r > \overline{s}$. Thus, we can set $\nu(r) = 0$ if $r < \underline{s}$ and $v(t^R) = -K$ if $r > \overline{s}$ where K is sufficiently large.[23] For $(r, s) \in S \times S$, we can set:

$$\eta(s) = \begin{cases} G(s) \text{ if } s \in [\underline{s}, s_c], \\ G(r_c) + g(r_c)(s - r_c) \text{ if } s \in (s_c, \overline{s}], \end{cases}$$

$$\nu(r) = \begin{cases} -g(r) \text{ if } r \in [\underline{s}, s_c], \\ -g(r_c) \text{ if } r \in (s_c, \overline{s}]. \end{cases}$$

It is straightforward to verify that η is convex and greater than G and $-\nu$ is a subderivative of η. Thus, (2.16) holds. Further, (2.16) holds with equality if (r, s) lies in the support of the described mechanism. Thus, weak duality condition (2.11) holds. ■

Proof of Proposition 2.3. First, note that if for all s, $\frac{\widetilde{u}(r,s)}{b(r)}$ is strictly increasing in r for some positive function b, then $v_H(r)$ crosses the horizontal axis at most once and from below for all distributions H:

$$\frac{1}{b(r_2)} \int_S \widetilde{u}(r_2, s)\, dH(s) > \frac{1}{b(r_1)} \int_{TS} \widetilde{u}(r_1, s)\, dH(s).$$

To prove the proposition, I construct the required b. Define $\widehat{s}(r)$ as s that solves $u(r, s) = 0$. If there does not exist such s, set $\widehat{s}(r) = \underline{s}$ if $u(r, \underline{s}) > 0$, and set $\widehat{s}(r) = \overline{s}$ if $u(r, \overline{s}) < 0$. The required

[23] To see that $0 \geq \frac{G(r)-\eta(s)}{s-r}$ if $r < \underline{s}$, note that $\eta(s) \geq G(s)$ for all $s \in S$ as follows from (2.16) if $s = r$.

b is defined as a solution to the following differential equation:

$$\frac{d\ln b(r)}{dr} = \left.\frac{\partial \ln g(r|s)}{\partial r}\right|_{s=\widehat{s}(r)}.$$

Indeed,

$$\frac{\partial\left(\frac{\widetilde{u}(r,s)}{a(r)}\right)}{\partial r} = \frac{g(r|s)}{b(r)}\left[\frac{\partial u(r,s)}{\partial r} + u(r,s)\left(\frac{\partial \ln g(r|s)}{\partial r} - \frac{d\ln b(r)}{dr}\right)\right] > 0$$

because $\frac{g(r|s)}{b(r)}$, $\frac{\partial u(r,s)}{\partial r}$ are positive and $u(r,s)$, $\frac{\partial \ln g(r|s)}{\partial r} - \frac{d\ln b(r)}{dr}$ have the same sign since they are increasing in s and are equal to zero at $s = \widehat{s}(r)$. ■

Proof of Lemma 2.1. The proof of similar results can be found in Anderson and Nash (1987). However, to make the chapter self-contained, I prove this lemma.

Multiplying (2.7) by η and integrating them over S gives

$$\int_S \eta(s) f(s)\, ds = \int_{R\times S} \eta(s)\, d\phi(r,s).$$

Multiplying (2.8) by ν and integrating them over R gives

$$\int_{R\times S} \widetilde{u}(r,s)\,\nu(r)\, d\phi(r,s) = 0.$$

Summing up these two inequalities gives

$$\int_S \eta(s) f(s)\, ds = \int_{R\times S} (\eta(s) + \widetilde{u}(r,s)\,\nu(r))\, d\phi(r,s). \tag{2.17}$$

Integrating (2.10) over $R \times S$ gives

$$\int_{R\times S} \overline{G}(r|s)\, d\phi(r,s) \leq \int_{R\times S} (\eta(s) + \widetilde{u}(r,s)\,\nu(r))\, d\phi(r,s). \tag{2.18}$$

Conditions (2.17) and (2.18) yield (2.11).

Suppose that inequality (2.11) holds with equality for some feasible (η,ν) and ϕ:

$$\int_S \eta(s) f(s)\, ds = \int_{R\times S} \overline{G}(r|s)\, d\phi(r,s). \tag{2.19}$$

Consider any other feasible $\widetilde{\phi}$. Inequality (2.11) implies

$$\int_{R\times S} \overline{G}(r|s)\, d\widetilde{\phi}(r,s) \leq \int_S \eta(s) f(s)\, ds.$$

Combining this inequality with (2.19) gives

$$\int_{R\times S} \overline{G}(r|s)\, d\widetilde{\phi}(r,s) \leq \int_{R\times S} \overline{G}(r|s)\, d\phi(r,s),$$

showing that ϕ is an optimal solution to the primal problem (2.6). An analogous argument proves that (η,ν) is optimal solutions to (2.9). Finally, combining (2.17) and (2.19) for optimal ϕ and (η,ν) gives (2.12). ■

Proof of Lemma 2.2. The proof of this lemma is omitted because it essentially repeats that of Theorem 5.2 in Anderson and Nash (1987). ■

Proof of Proposition 2.4. We start by proving the first part. The receiver's expected utilities under any mechanism ϕ and the no revelation mechanism ϕ_{no} are:

$$\begin{aligned}
\mathbb{E}_{\phi}[u] &= \int_{R\times S}\left(\int_{r}^{\overline{r}} \widetilde{u}(\widetilde{r},s)\, d\widetilde{r}\right) d\phi(r,s),\\
\mathbb{E}_{\phi_{no}}[u] &= \int_{R\times S}\left(\int_{r_{no}}^{\overline{r}} \widetilde{u}(\widetilde{r},s)\, d\widetilde{r}\right) f(s)\, ds\\
&= \int_{R\times S}\left(\int_{r_{no}}^{\overline{r}} \widetilde{u}(\widetilde{r},s)\, d\widetilde{r}\right) d\phi(r,s).
\end{aligned}$$

The first two lines hold because a message m induces the receiver to act if $r \geq m$ and ϕ_{no} sends the message r_{no} regardless of s. The third line holds because the marginal distribution of s for any mechanism ϕ coincides with the prior distribution of s. For a mechanism ϕ, denote the conditional distribution of s given a message r by $\phi(s|r)$ and the marginal distribution of a message r by $\phi(r)$. Fubini's theorem gives

$$\begin{aligned}
\mathbb{E}_{\phi}[u] - \mathbb{E}_{\phi_{no}}[u] = {} & \textstyle\int_{\underline{r}}^{r_{no}} \left[\int_{r}^{r_{no}} \left(\int_S \widetilde{u}(\widetilde{r},s)\, d\phi(s|r)\right) d\widetilde{r}\right] d\phi(r)\\
& - \textstyle\int_{r_{no}}^{\overline{r}} \left[\int_{r_{no}}^{r} \left(\int_S \widetilde{u}(\widetilde{r},s)\, d\phi(s|r)\right) d\widetilde{r}\right] d\phi(r).
\end{aligned} \tag{2.20}$$

By the single-crossing assumption, $\int_S \widetilde{u}(\widetilde{r},s)\, d\phi(s|r) > 0$ for $\widetilde{r} > r$, so $\int_r^{r_{no}} \left(\int_S \widetilde{u}(\widetilde{r},s)\, d\phi(s|r)\right) d\widetilde{r} > 0$ for $r < r_{no}$. Since $\phi(r)$ of any mechanism ϕ that differs from ϕ_{no} puts strictly positive probability on messages in $[\underline{r}, r_{no})$, the first integral in (2.20) is strictly positive. The analogous argument shows that the second integral in (2.20) is strictly negative, so $\mathbb{E}_{\phi}[u] - \mathbb{E}_{\phi_{no}}[u] > 0$ for any ϕ that differs from ϕ_{no}, which proves the first part.

We now turn to the second part. The receiver's expected utility under ϕ_{full} is

$$\begin{aligned}\mathbb{E}_{\phi_{full}}[u] &= \int_S \left(\int_{\widehat{r}(t^S)}^{\overline{r}} \widetilde{u}(\widetilde{r},s)\,d\widetilde{r}\right) f(s)\,ds \\ &= \int_{R\times S} \left(\int_{\widehat{r}(s)}^{\overline{r}} \widetilde{u}(\widetilde{r},s)\,d\widetilde{r}\right) d\phi(r,s).\end{aligned}$$

Taking into account that $\widetilde{u}(\widehat{r}(s),s)=0$ gives

$$\begin{aligned}\mathbb{E}_{\phi_{full}}[u]-\mathbb{E}_{\phi}[u] = &\int_{(\widehat{r}(s),\overline{r}]\times S}\left(\int_{\widehat{r}(s)}^{r}\widetilde{u}(\widetilde{r},s)\,d\widetilde{r}\right)d\phi(r,s)\\ &-\int_{[\underline{r},\widehat{r}(s))\times S}\left(\int_{r}^{\widehat{r}(s)}\widetilde{u}(\widetilde{r},s)\,d\widetilde{r}\right)d\phi(r,s).\end{aligned}\tag{2.21}$$

By the single-crossing assumption, $\widetilde{u}(\widetilde{r},s)>0$ for $\widetilde{r}>\widehat{r}$, so $\int_{\widehat{r}(s)}^{r}\widetilde{u}(\widetilde{r},s)\,d\widetilde{r}>0$ for $r>\widehat{r}(s)$. Any ϕ that differs from ϕ_{full} puts strictly positive probability on $(\widehat{r}(s),\overline{r}]\times S$, otherwise $\int_{R\times S}\widetilde{u}(\widetilde{r},s)\,d\phi(r,s)$ is strictly negative. Therefore, the first integral in (2.21) is strictly positive. The analogous argument shows that the second integral in (2.21) is strictly negative, so $\mathbb{E}_{\phi_{full}}[u]-\mathbb{E}_{\phi}[u]>0$ for any ϕ that differs from ϕ_{full}, which proves the second part. ■

Proof of Theorem 2.1. I prove each part in turn.

The "only if" part of 1. Suppose to get a contradiction that there exist s_1, s_2, and r such that the sender is not indifferent to reveal s_1 and s_2 or to pool them at r. Consider two mechanisms that differ only in that one sends different messages for $s\in[s_1,s_1+\varepsilon_1]$ and $s\in[s_2-\varepsilon_2,s_2]$, and the other sends the same message for $s\in[s_1,s_1+\varepsilon_1]\cup[s_2-\varepsilon_2,s_2]$ where ε_1 and ε_2 are sufficiently small and satisfy

$$\int_{s_1}^{s_1+\varepsilon_1}\widetilde{u}(r,s)\,f(s)\,ds+\int_{s_2-\varepsilon_2}^{s_2}\widetilde{u}(r,s)\,f(s)\,ds=0.$$

Clearly, these two mechanisms are not equivalent. Therefore, if all mechanisms are equivalent, then the sender is indifferent to reveal s_1 and s_2 or to pool them at r for all $r\in(\widehat{r}(\overline{s}),\widehat{r}(\underline{s}))$ and all s_1, s_2 such that $u(r,s_1)<0<u(r,s_2)$:

$$\frac{\overline{G}(\widehat{r}(s_1)|s_1)-\overline{G}(r|s_1)}{\widetilde{u}(r,s_1)}=\frac{\overline{G}(\widehat{r}(s_2)|s_2)-\overline{G}(r|s_2)}{\widetilde{u}(r,s_2)}.$$

Therefore, we can define the required b as $\frac{\overline{G}(\widehat{r}(s)|s)-\overline{G}(r|s)}{\widetilde{u}(r,s)}$, which is positive and does not depend on s.

The "if" part of 1. Consider any mechanism ϕ. Substituting (2.15) into (2.8) gives

$$\int_{R\times S} \overline{G}(r|s)\, d\phi(r,s) = \int_{R\times S} \overline{G}(\widehat{r}(s)|s)\, d\phi(r,s).$$

Taking into account (2.7) gives

$$\int_{R\times S} \overline{G}(r|s)\, d\phi(r,s) = \int_S \overline{G}(\widehat{r}(s)|s)\, f(s)\, ds,$$

which implies that the probability that the receiver acts is the same for all mechanisms.

The "only if" part of 2. Suppose to get a contradiction that there exist s_1, s_2, and r such that it is strictly better to pool s_1 and s_2 at r than to reveal them. Consider the mechanism that differs from ϕ_{full} only in that it sends the same message if $s \in [s_1, s_1+\varepsilon_1] \cup [s_2-\varepsilon_2, s_2]$ where ε_1 and ε_2 are sufficiently small and satisfy

$$\int_{s_1}^{s_1+\varepsilon_1} \widetilde{u}(r,s)\, f(s)\, ds + \int_{s_2-\varepsilon_2}^{s_2} \widetilde{u}(r,s)\, f(s)\, ds = 0.$$

Clearly, this mechanism strictly dominates ϕ_{full}.

The "if" part of 2. The complementarity condition (2.12) suggests that

$$\eta(s) + \widetilde{u}(\widehat{r}(s), s)\, \nu(r) = \overline{G}(\widehat{r}(s)|s) \text{ for all } s \in S.$$

Taking into account that $\widetilde{u}(\widehat{r}(s), s) = 0$ gives $\eta(s) = \overline{G}(\widehat{r}(s)|s)$ for all $s \in S$. Note that the weak duality condition (2.11) is satisfied with equality for $\eta(s) = \overline{G}(\widehat{r}(s)|s)$. Therefore, by Lemma 2.1, ϕ_{full} is optimal if there exists ν such that

$$\overline{G}(\widehat{r}(s)|s) + \widetilde{u}(r,s)\, \nu(r) \geq \overline{G}(r|s) \text{ for all } (r,s) \in R \times S, \tag{2.22}$$

which is equivalent to

$$\frac{\overline{G}(r|s_2) - \overline{G}(\widehat{r}(s_2)|s_2)}{\widetilde{u}(r,s_2)} \leq \nu(r) \leq \frac{\overline{G}(\widehat{r}(s_1)|s_1) - \overline{G}(r|s_1)}{-\widetilde{u}(r,s_1)}$$

for all $r \in (\widehat{r}(\overline{s}), \widehat{r}(\underline{s}))$ and s_1, s_2 such that $r \in (\widehat{r}(s_2), \widehat{r}(s_1))$. For $r \notin (\widehat{r}(\overline{s}), \widehat{r}(\underline{s}))$, the existence of ν is obvious because (2.22) bounds ν only from one side.

The "only if" part of 3. Suppose to get a contradiction that there exist s_1, s_2, s_3, r such that it is strictly better to pool s_1, s_2 at r and to reveal s_3 than to pool s_1, s_2, s_3 at r_{no}. Consider the mechanism that differs from ϕ_{no} only in that it sends one message if $s \in [s_1, s_1+\varepsilon_1] \cup [s_2, s_2+\varepsilon_2]$ and another message if $s \in [s_3-\varepsilon_3, s_3]$ where ε_1, ε_2, and ε_3 are sufficiently small and satisfy

$$\sum_{i=1,2} \int_{s_i}^{s_i+\varepsilon_i} \widetilde{u}(r_{no}, s) f(s)\, ds + \int_{s_3-\varepsilon_3}^{s_3} \widetilde{u}(r_{no}, s) f(s)\, ds = 0.$$

Clearly, this mechanism strictly dominates ϕ_{no}.

The "if" part of 3. The complementarity condition (2.12) suggests that

$$\eta(s) + \widetilde{u}(r_{no}, s)\, \nu(r_{no}) = \overline{G}(r_{no}|s) \text{ for all } s \in S. \tag{2.23}$$

Note that the weak duality condition (2.11) is satisfied with equality for $\eta(s) = \overline{G}(r_{no}|t^S) - \widetilde{u}(r_{no}, s)\, \nu(r_{no})$. Therefore, by Lemma 2.1, ϕ_{no} is optimal if there exists ν such that

$$\overline{G}(r_{no}|s) - \widetilde{u}(r_{no}, s)\, \nu(r_{no}) + \widetilde{u}(r, s)\, \nu(r) \geq \overline{G}(r|s) \text{ for all } (r, s) \in R \times S. \tag{2.24}$$

First, for $(r, s) \in R \times S$ such that $u(r, s) = 0$, (2.24) is equivalent to

$$\frac{\overline{G}(\widehat{r}(s_1)|s_1) - \overline{G}(r_{no}|s_1)}{-\widetilde{u}(r_{no}, s_1)} \leq \nu(r_{no}) \leq \frac{\overline{G}(r_{no}|s_3) - \overline{G}(\widehat{r}(s_3)|s_3)}{\widetilde{u}(r_{no}, s_3)} \tag{2.25}$$

for all s_1,s_3 such that $r_{no} \in (\widehat{r}(s_3), \widehat{r}(s_1))$. Taking the limit $s_1, s_3 \to s_{no}$ where s_{no} is the unique s that solves $u(r_{no}, s_{no}) = 0$, inequalities (2.25) imply $\nu(r_{no}) = -\frac{g(r_{no}|s_{no})}{\partial \widetilde{u}(r_{no}, s_{no})/\partial r}$.

Second, for $(r, s) \in R \times S$ such that $u(r, s) \neq 0$, (2.24) is equivalent to

$$\frac{\overline{G}(r|s_2) - \left(\overline{G}(r_{no}|s_2) + \frac{\widetilde{u}(r_{no}, s_2) g(r_{no}|s_{no})}{\frac{\partial \widetilde{u}(r_{no}, s_{no})}{\partial r}}\right)}{\widetilde{u}(r, s_2)} \leq \nu(r) \leq \frac{\left(\overline{G}(r_{no}|s_1) + \frac{\widetilde{u}(r_{no}, s_1) g(r_{no}|s_{no})}{\frac{\partial \widetilde{u}(r_{no}, s_{no})}{\partial r}}\right) - \overline{G}(r|s_1)}{-\widetilde{u}(r, s_1)} \tag{2.26}$$

for all $r \in (\widehat{r}(\overline{s}), \widehat{r}(\underline{s}))$, and $s_1, s_2 \in S$ such that $r \in (\widehat{r}(s_2), \widehat{r}(s_1))$. For $r \notin (\widehat{r}(\overline{s}), \widehat{r}(\underline{s}))$, the existence of ν is obvious because (2.24) bounds ν only from one side.

Inequality (2.25) requires that the sender prefers to pool s_1 and s_3 at r_{no} than to reveal them, which is satisfied because we can set $s_1 = s_2$. Inequality (2.26) requires that the sender prefers to pool s_1, s_2, s_3 at r_{no} than to pool s_1 and s_2 at r and to reveal s_3 where $s_3 \to s_{no}$, which is satisfied because it is satisfied for all $s_3 \neq s_{no}$ and all functions are smooth. ■

Proof of Lemma 2.3. Because $\int_S \widetilde{u}(r, s)\, d\phi(s|r) = 0$ for each r, there exists a vector function

(v_1, v_2, q_1, q_2) from $R \times [0,1]$ to $[\min_{s\in S} \widetilde{u}(r,s), 0] \times [0, \max_{s\in S} \widetilde{u}(r,s)] \times [0,1] \times [0,1]$ such that

$$\begin{aligned}
&\int_{v_1(\widetilde{r},e)<\widetilde{u}(\widetilde{r},s)<v_2(\widetilde{r},e)} \widetilde{u}(\widetilde{r},s)\, d\phi(s|\widetilde{r}) \\
&+ \sum_{i=1,2} v_i(\widetilde{r},e)\, q_i(\widetilde{r},e) \Pr{}_\phi (\widetilde{u}(\widetilde{r},s) = v_i(\widetilde{r},e) \,|r = \widetilde{r}) = 0, \\
&\Pr{}_\phi (v_1(\widetilde{r},e) < \widetilde{u}(\widetilde{r},s) < v_2(\widetilde{r},e) \,|r = \widetilde{r}) \\
&+ \sum_{i=1,2} q_i(\widetilde{r},e) \Pr{}_\phi (\widetilde{u}(\widetilde{r},s) = v_i(\widetilde{r},e) \,|r = \widetilde{r}) = e
\end{aligned}$$

for all $(\widetilde{r}, e) \in R \times [0,1]$. Define distribution φ of $(\widetilde{r}, e, s)$ as follows. The marginal distribution of $\widetilde{r}$ for φ coincides with the marginal distribution of $\widetilde{r}$ for ϕ. The conditional distribution of e given $\widetilde{r}$ is uniform on the unit interval $[0,1]$. The conditional distribution of s given $\widetilde{r}$ and e puts probability p_1 on s_1 and probability $1-p_1$ on s_2 where s_1, s_2 satisfy $\widetilde{u}(\widetilde{r}, s_1) = v_1(\widetilde{r}, e)$, $\widetilde{u}(\widetilde{r}, s_2) = v_2(\widetilde{r}, e)$, and p_1 solves $p_1 v_1(\widetilde{r}, e) + (1-p_1) v_2(\widetilde{r}, e) = 0$. Clearly, φ satisfies the two properties. ■

Proof of Proposition 2.5. I prove this proposition under weaker assumptions imposed in Proposition 2.6. Denote $\mu_{ij} = \gamma(m_1|r_i, s_j)\, p_j$. A mechanism γ has to satisfy the receiver's incentive constraints for reporting his true r and for acting whenever he receives m_1:

$$\begin{aligned}
&\widetilde{u}_{11}\mu_{11} + \widetilde{u}_{12}\mu_{12} \geq \widetilde{u}_{11}\mu_{21} + \widetilde{u}_{12}\mu_{22} \\
&\widetilde{u}_{11}\mu_{11} + \widetilde{u}_{12}\mu_{12} \geq \widetilde{u}_{11}(p_1 - \mu_{21}) + \widetilde{u}_{12}(p_2 - \mu_{22}) \\
&\widetilde{u}_{11}\mu_{11} + \widetilde{u}_{12}\mu_{12} \geq 0 \\
&\widetilde{u}_{11}\mu_{11} + \widetilde{u}_{12}\mu_{12} \geq \widetilde{u}_{11}p_1 + \widetilde{u}_{12}p_2 \\
&\widetilde{u}_{21}\mu_{21} + \widetilde{u}_{22}\mu_{22} \geq \widetilde{u}_{21}\mu_{11} + \widetilde{u}_{22}\mu_{12} \\
&\widetilde{u}_{21}\mu_{21} + \widetilde{u}_{22}\mu_{22} \geq \widetilde{u}_{21}(p_1 - \mu_{11}) + \widetilde{u}_{22}(p_2 - \mu_{12}) \\
&\widetilde{u}_{21}\mu_{21} + \widetilde{u}_{22}\mu_{22} \geq 0 \\
&\widetilde{u}_{21}\mu_{21} + \widetilde{u}_{22}\mu_{22} \geq \widetilde{u}_{21}p_1 + \widetilde{u}_{22}p_2.
\end{aligned} \tag{2.27}$$

Note that μ and ϕ are related by:

$$\begin{aligned}
\mu_{1j}p_{1|j} &= \Pr(a=1, r=r_1, s=s_j) = \phi^j_{1,2}p_{1|j} \text{ for } j=1,2, \\
\mu_{2j}p_{2|j} &= \Pr(a=1, r=r_2, s=s_j) = \phi^j_2 p_{2|j} + \phi^j_{1,2}p_{2|j} \text{ for } j=1,2,
\end{aligned}$$

which implies that $\mu_{1j} = \phi^j_{1,2}$ and $\mu_{2j} = \phi^j_{1,2} + \phi^j_2$. Rewriting (2.27) in variables ϕ gives:

$$
\begin{aligned}
&\widetilde{u}_{11}\phi^1_2 + \widetilde{u}_{12}\phi^2_2 \le 0\\
&\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} \ge \widetilde{u}_{11}\left(p_1 - \phi^1_2 - \phi^1_{1,2}\right) + \widetilde{u}_{12}\left(p_2 - \phi^2_2 - \phi^2_{1,2}\right)\\
&\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} \ge 0\\
&\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} \ge \widetilde{u}_{11}p_1 + \widetilde{u}_{12}p_2\\
&\widetilde{u}_{21}\phi^1_2 + \widetilde{u}_{22}\phi^2_2 \ge 0\\
&\widetilde{u}_{21}\left(\phi^1_2 + \phi^1_{1,2}\right) + \widetilde{u}_{22}\left(\phi^2_2 + \phi^2_{1,2}\right) \ge \widetilde{u}_{21}\left(p_1 - \phi^1_{1,2}\right) + \widetilde{u}_{22}\left(p_2 - \phi^2_{1,2}\right)\\
&\widetilde{u}_{21}\left(\phi^1_2 + \phi^1_{1,2}\right) + \widetilde{u}_{22}\left(\phi^2_2 + \phi^2_{1,2}\right) \ge 0\\
&\widetilde{u}_{21}\left(\phi^1_2 + \phi^1_{1,2}\right) + \widetilde{u}_{22}\left(\phi^2_2 + \phi^2_{1,2}\right) \ge \widetilde{u}_{21}p_1 + \widetilde{u}_{22}p_2
\end{aligned}
$$

which immediately implies that all constraints (2.28) except possibly $\phi^j_2 \ge 0$ are satisfied. Finally, $\phi^j_2 \ge 0$ follows from $\widetilde{u}_{11}\phi^1_2 + \widetilde{u}_{12}\phi^2_2 \le 0$, $\widetilde{u}_{21}\phi^1_2 + \widetilde{u}_{22}\phi^2_2 \ge 0$, $\widetilde{u}_{12}, \widetilde{u}_{22} > 0$, and $k_2 < k_1$ $(\widetilde{u}_{22}\widetilde{u}_{11} < \widetilde{u}_{21}\widetilde{u}_{12})$. ∎

Proof of Proposition 2.6. The optimal mechanism ϕ maximizes

$$\Pr{}_\phi\,(a=1) = p_{2|1}\phi^1_2 + p_{2|2}\phi^2_2 + \phi^1_{1,2} + \phi^2_{1,2}$$

subject to

$$
\begin{aligned}
&\phi^j_K \ge 0 \text{ for } j = 1,2 \text{ and } K = \{\emptyset\}, \{2\}, \{1,2\},\\
&\phi^j_\emptyset + \phi^j_2 + \phi^j_{1,2} = p_j \text{ for } j = 1,2,\\
&\widetilde{u}_{21}\phi^1_2 + \widetilde{u}_{22}\phi^2_2 \ge 0,\\
&\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} \ge 0,\\
&\widetilde{u}_{21}\phi^1_\emptyset + \widetilde{u}_{22}\phi^2_\emptyset < 0 \text{ or } \phi^1_\emptyset = \phi^2_\emptyset = 0,\\
&\widetilde{u}_{11}\phi^1_2 + \widetilde{u}_{12}\phi^2_2 < 0 \text{ or } \phi^1_2 = \phi^2_2 = 0.
\end{aligned}
\tag{2.28}
$$

Consider the relaxed problem that omits the last two constraints with strict inequalities. The solution to the relaxed problem satisfies $\phi^2_\emptyset = 0$, $\widetilde{u}_{21}\phi^1_2 + \widetilde{u}_{22}\phi^2_2 = 0$, and $\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} = 0$, otherwise we can increase $\Pr_\phi(a=1)$ by the following changes to the mechanism. If $\phi^2_\emptyset \ne 0$, change $\widetilde{\phi}^2_{1,2} = \phi^2_{1,2} + \phi^2_\emptyset$ and $\widetilde{\phi}^2_\emptyset = 0$; if $\widetilde{u}_{11}\phi^1_{1,2} + \widetilde{u}_{12}\phi^2_{1,2} > 0$, change $\widetilde{\phi}^1_{1,2} = \phi^1_{1,2} + \varepsilon$ and either $\widetilde{\phi}^1_2 = \phi^1_2 - \varepsilon$ or $\widetilde{\phi}^1_\emptyset = \phi^1_\emptyset - \varepsilon$; if $\widetilde{u}_{21}\phi^1_2 + \widetilde{u}_{22}\phi^2_2 > 0$, change $\widetilde{\phi}^2_{1,2} = \phi^2_{1,2} + \varepsilon$ and $\widetilde{\phi}^2_2 = \phi^2_2 - \varepsilon$ where ε is a small positive number. These observations together with $k_2 < k_1$ imply that the solution to the relaxed problem satisfies the last two constraints and, therefore, it also solves the original problem. The original problem simplifies to the maximization of

$$\Pr_{\phi}(a=1) = \left(1 - \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}\right) p_2 - \left(p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} - \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}\right) \phi_2^2$$

over ϕ_2^2 subject to

$$\left(\tfrac{\widetilde{u}_{12}}{\widetilde{u}_{11}} - \tfrac{\widetilde{u}_{22}}{\widetilde{u}_{21}}\right) \phi_2^2 \le p_1 + \tfrac{\widetilde{u}_{12}}{\widetilde{u}_{11}} p_2.$$
$$0 \le \phi_2^2 \le p_2.$$

The solution to this problem is: (i) $\phi_2^2 = 0$ if $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} \ge \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$; (ii) $\phi_2^2 = p_2$ if $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} < \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$ and $\widetilde{u}_{21} p_1 + \widetilde{u}_{22} p_2 < 0$; (iii) $\phi_2^2 = \frac{p_1 + \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}} p_2}{\frac{\widetilde{u}_{12}}{\widetilde{u}_{11}} - \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}}} = \widetilde{u}_{21} \frac{\widetilde{u}_{11} p_1 + \widetilde{u}_{12} p_2}{\widetilde{u}_{12}\widetilde{u}_{21} - \widetilde{u}_{11}\widetilde{u}_{22}}$ if $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} < \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$ and $\widetilde{u}_{21} p_1 + \widetilde{u}_{22} p_2 \ge 0$. Finally, $\phi_{1,2}^2 = p_2 - \phi_2^2$, $\phi_2^1 = -\frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} \phi_2^2$, $\phi_{1,2}^1 = -\frac{\widetilde{u}_{12}}{\widetilde{u}_{11}} \phi_{1,2}^2$, $\phi_{\emptyset}^2 = 0$, $\phi_{\emptyset}^1 = p_1 - \phi_2^1 - \phi_{1,2}^1$.

Recall that the receiver's value of information is defined as the difference between the receiver's expected utilities under the optimal mechanism ϕ and the no revelation mechanism ϕ_{no}. Under ϕ_{no}, the receiver's expected utility is

$$\mathbb{E}\left[\max_a \mathbb{E}_{\phi_{no}}\left[au(r,s)\,|r\right]\right] = \max\left\{0, \widetilde{u}_{21} p_1 + \widetilde{u}_{22} p_2\right\}.$$

Under ϕ, the receiver's expected utility is

$$\begin{aligned}
\mathbb{E}\left[\max_a \mathbb{E}_{\phi}\left[au(r,s)\,|r,m\right]\right] &= \left(\widetilde{u}_{11}\phi_{1,2}^1 + \widetilde{u}_{12}\phi_{1,2}^2\right) + \left(\widetilde{u}_{21}\phi_{1,2}^1 + \widetilde{u}_{22}\phi_{1,2}^2\right) + \left(\widetilde{u}_{21}\phi_2^1 + \widetilde{u}_{22}\phi_2^2\right) \\
&= \widetilde{u}_{21}\phi_{1,2}^1 + \widetilde{u}_{22}\phi_{1,2}^2 \\
&= \begin{cases} \left(\frac{\widetilde{u}_{11}\widetilde{u}_{22} - \widetilde{u}_{12}\widetilde{u}_{21}}{\widetilde{u}_{11}}\right) p_2 \text{ in case (i)}, \\ 0 \text{ in case (ii)}, \\ \widetilde{u}_{21} p_1 + \widetilde{u}_{22} p_2 \text{ in case (iii)}. \end{cases}
\end{aligned}$$

The second equality holds because $\widetilde{u}_{11}\phi_{1,2}^1 + \widetilde{u}_{12}\phi_{1,2}^2 = \widetilde{u}_{21}\phi_2^1 + \widetilde{u}_{22}\phi_2^2 = 0$. The equality for (i) holds because $\phi_{1,2}^2 = p_2$ and $\phi_{1,2}^1 = -\frac{\widetilde{u}_{12}}{\widetilde{u}_{11}} p_2$. The equality for (ii) holds because $\phi_{1,2}^1 = \phi_{1,2}^2 = 0$. The equality for (iii) holds because $\phi_{1,2}^1 = p_1 - \phi_2^1$, $\phi_{1,2}^2 = p_2 - \phi_2^2$, and $\widetilde{u}_{21}\phi_2^1 + \widetilde{u}_{22}\phi_2^2 = 0$. Therefore, R's expected utilities under ϕ and ϕ_{no} differ if and only if $p_{1|2} + \frac{\widetilde{u}_{22}}{\widetilde{u}_{21}} p_{2|1} \ge \frac{\widetilde{u}_{12}}{\widetilde{u}_{11}}$. ■

Proof of Proposition 2.7. The optimal mechanism ϕ maximizes

$$\Pr_{\phi}(a=1) = p_{1|1}\phi_1^1 + p_{1|2}\phi_1^2 + p_{2|1}\phi_2^1 + p_{2|2}\phi_2^2 + \phi_{1,2}^1 + \phi_{1,2}^2$$

subject to

$$\begin{aligned}
&\phi_K^j \geq 0 \text{ for } j=1,2 \text{ and } K=\{\emptyset\},\{1\},\{2\},\{1,2\},\\
&\phi_\emptyset^j+\phi_1^j+\phi_2^j+\phi_{1,2}^j=p_j \text{ for } j=1,2,\\
&\widetilde{u}_{i1}\phi_i^1+\widetilde{u}_{i2}\phi_i^2 \geq 0 \text{ for } i=1,2,\\
&\widetilde{u}_{i1}\phi_{1,2}^1+\widetilde{u}_{i2}\phi_{1,2}^2 \geq 0 \text{ for } i=1,2,\\
&\widetilde{u}_{i1}\phi_{3-i}^1+\widetilde{u}_{i2}\phi_{3-i}^2 < 0 \text{ or } \phi_{3-i}^1=\phi_{3-i}^2=0 \text{ for } i=1,2,\\
&\widetilde{u}_{i1}\phi_\emptyset^1+\widetilde{u}_{i2}\phi_\emptyset^2 < 0 \text{ or } \phi_\emptyset^1=\phi_\emptyset^2=0 \text{ for } i=1,2.
\end{aligned}$$

Note that $\widetilde{u}_{11}\Pr_\phi(s_1|m)+\widetilde{u}_{12}\Pr_\phi(s_2|m)\geq 0$ is equivalent to $\Pr_\phi(s_2|m)\leq k_1$, and $\widetilde{u}_{21}\Pr_\phi(s_1|m)+\widetilde{u}_{22}\Pr_\phi(s_2|m)\geq 0$ is equivalent to $\Pr_\phi(s_2|m)\geq k_2$. Therefore, the receiver r_1 acts if $\Pr_\phi(s_2|m)\leq k_1$, and the receiver r_1 acts if $\Pr_\phi(s_2|m)\geq k_2$. If $k_2\leq k_1$, then no mechanism can send the message $m_\emptyset$ because $\Pr_\phi(s_2|m)<k_2$ and $\Pr_\phi(s_2|m)>k_1$ cannot both hold. On the contrary, if $k_2>k_1$, then no mechanism can send the message $m_{1,2}$ because $\Pr_\phi(s_2|m)\geq k_2$ and $\Pr_\phi(s_2|m)\leq k_1$ cannot both hold. Consider these two cases in turn.

Let $k_2\leq k_1$ and, thus, $\phi_\emptyset^1=\phi_\emptyset^2=0$. Consider the relaxed problem with the constraints $\phi_K^j\geq 0$, $\phi_1^j+\phi_2^j+\phi_{1,2}^j=p_j$, $\widetilde{u}_{11}\phi_1^1+\widetilde{u}_{12}\phi_1^2\geq 0$, and $\widetilde{u}_{11}\phi_{1,2}^1+\widetilde{u}_{12}\phi_{1,2}^2\geq 0$ for all K and j. Note that the last two constraints imply $\widetilde{u}_{11}\left(\phi_1^1+\phi_{1,2}^1\right)+\widetilde{u}_{12}\left(\phi_1^2+\phi_{1,2}^2\right)\geq 0$, so the solution to the relaxed problem satisfies $\phi_1^1=\phi_1^2=0$, otherwise we can increase $\Pr_\phi(a=1)$ by the following changes to the mechanism: $\widetilde{\phi}_{1,2}^j=\phi_{1,2}^j+\phi_1^j$ and $\widetilde{\phi}_1^j=0$ for $j=1,2$. Substituting $\phi_{1,2}^j=p_j-\phi_2^j$, the relaxed problem simplifies to: ϕ_2^1 and ϕ_2^2 maximize

$$\Pr{}_\phi(a=1)=1-p_{1|1}\phi_2^1-p_{1|2}\phi_2^2$$

subject to

$$\begin{aligned}
&\phi_2^j\in[0,p_j] \ \text{ for } j=1,2,\\
&\widetilde{u}_{11}\phi_2^1+\widetilde{u}_{12}\phi_2^2\leq \widetilde{u}_{11}p_1+\widetilde{u}_{12}p_2.
\end{aligned}$$

The solution to this problem is $\left(\phi_2^1,\phi_2^2\right)=\left(0,\frac{\widetilde{u}_{11}p_1+\widetilde{u}_{12}p_2}{\widetilde{u}_{12}}\right)$. It is also the solution to the original problem because it satisfies all constraints of the original problem.

Let $k_2>k_1$ and, thus, $\phi_{1,2}^1=\phi_{1,2}^2=0$. In the optimal mechanism, $\phi_\emptyset^1=\phi_\emptyset^2=0$, otherwise we can increase $\Pr_\phi(a=1)$ by the following changes to the mechanism: $\widetilde{\phi}_i^i=\phi_i^i+\phi_\emptyset^i$ and $\widetilde{\phi}_\emptyset^i=0$ for $i=1,2$. Consider the relaxed problem with the constraints $\phi_1^j,\phi_2^j\geq 0$, $\phi_1^j+\phi_2^j=p_j$, $\widetilde{u}_{11}\phi_1^1+\widetilde{u}_{12}\phi_1^2\geq 0$, and $\widetilde{u}_{21}\phi_2^1+\widetilde{u}_{22}\phi_2^2\geq 0$ for all $j=1,2$. Substituting $\phi_1^j=p_j-\phi_2^j$, the relaxed problem simplifies to: ϕ_2^1 and ϕ_2^2 maximize

$$\Pr_{\phi}(a=1) = p_{1|1}p_1 + p_{1|2}p_2 + \left(1-2p_{1|1}\right)\phi_2^1 + \left(1-2p_{1|2}\right)\phi_2^2$$

subject to

$$\begin{aligned}
&\phi_2^j \in [0,p_j] \text{ for } j=1,2,\\
&\widetilde{u}_{11}\phi_2^1 + \widetilde{u}_{12}\phi_2^2 \le \widetilde{u}_{11}p_1 + \widetilde{u}_{12}p_2,\\
&\widetilde{u}_{21}\phi_2^1 + \widetilde{u}_{22}\phi_2^2 \ge 0.
\end{aligned}$$

The coefficients $1-2p_{1|1}$ and $1-2p_{1|2}$ in the objective function can have any sign and, therefore, any extreme point of the constraints can be a solution to this problem. If $p_2 \ge k_2$, the extreme points of $\left(\phi_2^1,\phi_2^2\right)$ are $(0,p_2)$, $\left(0,\frac{\widetilde{u}_{11}p_1+\widetilde{u}_{12}p_2}{\widetilde{u}_{12}}\right)$, (p_1,p_2). If $p_2 < k_2$, the extreme points of $\left(\phi_2^1,\phi_2^2\right)$ are $(0,p_2)$, $\left(0,\frac{\widetilde{u}_{11}p_1+\widetilde{u}_{12}p_2}{\widetilde{u}_{12}}\right)$, $\left(-\frac{\widetilde{u}_{22}}{\widetilde{u}_{21}}p_2,p_2\right)$, $\frac{\widetilde{u}_{11}p_1+\widetilde{u}_{12}p_2}{\widetilde{u}_{11}\widetilde{u}_{22}-\widetilde{u}_{12}\widetilde{u}_{21}}\left(\widetilde{u}_{22},-\widetilde{u}_{21}\right)$. All these extreme points can be a solution to the original problem because they satisfy all constraints of the original problem.

■

2.9 Appendix C: Commitment Assumption

The assumption that the sender can commit to a mechanism is strong, but it allows us to abstract from the sender's incentive constraint and find the best possible information revelation from the sender's perspective, with the only constraint being that the receiver takes an action that maximizes his expected utility given his information. As I argue in the introduction, in some real-life examples, the commitment assumption is natural. Below, I provide three other kinds of situations that are formally equivalent to assuming that the sender has commitment power. These other situations are good approximations of many other real-life examples.

The first situation is based on public communication with a large population of heterogeneous receivers.[24] Suppose that instead of one receiver, there is a unit measure of receivers indexed by $i \in [0,1]$ for whom vectors (r_i, s_i) are independent and identically distributed. Assume that the sender's ex post utility is equal to the measure of receivers who act $\int_0^1 a_i di$.[25] Denote ϕ^* as the optimal mechanism in the original model with one receiver, and p^* as the probability that the receiver acts under ϕ^*. Define $\widetilde{\phi}^*$ as a mechanism that applies ϕ^* to each $i \in [0,1]$ independently.

[24] This situation is distantly related to the literature on cheap talk with multiple audiences (Farrell and Gibbons (1989)) and multidimensional cheap talk (Chakraborty and Harbaugh (2007)).

[25] A technical issue arises with the existence of such a measure and the validity of the law of large numbers. However, it can be bypassed using the appropriate interpretation of the integral (Uhlig (1996)).

The mechanism $\widetilde{\phi}^*$ is optimal in the environment with a unit measure of receivers because the sender's utility is additive in the receivers' actions and (r_i, s_i) are identically distributed.

The optimal mechanism $\widetilde{\phi}^*$ can be supported as an equilibrium outcome of a cheap talk game between the sender and the receivers if the receivers are allowed to communicate with each other. To see this, note that if the sender sends messages according to $\widetilde{\phi}^*$ and the receivers believe so, then the measure of receivers who act is fixed at p^* by the law of large numbers. The only reason why the sender may want to deviate from $\widetilde{\phi}^*$ is to increase the measure of receivers who act above p^*. However, this deviation can be detected and punished by the receivers if before taking their actions, they announce what actions they are planning to take and compare the measure of receivers who are going to act with p^*. Moreover, it can be shown that if r_i is independent of s_i, then $\widetilde{\phi}^*$ can be supported as an equilibrium outcome of a cheap talk game with public messages from the sender and without communication between receivers.

If (r_i, s_i) are dependent or there are finitely many receivers, then the optimal mechanism may be unachievable to the sender, but she can still do better than the no revelation mechanism. For example, suppose that the receivers are uninformed and suppose that $u(s_i) = s_i = \sigma_i + z$ where σ_i are independent and identically distributed, and z is independent of σ_i. We can interpret σ_i as a receiver-specific taste for acting and z as an aggregate taste for acting, where both σ_i and z are known only by the sender. It can be shown that in this example, the sender can support the optimal revelation of σ_i but not of z as an equilibrium outcome of a cheap talk game.

In the case with multiple receivers, a mechanism can be viewed as an abstraction for Bayesian updating by receivers. The sender does not literally send a separate message m_i to each receiver i. Instead, the sender makes a public announcement. If the receivers are not perfectly homogeneous ((r_i, s_i) are not perfectly correlated), they make different inferences (draw different messages m_i) from the same announcement. For example, an announcement that a political party supports tax reduction has different welfare implications for individuals with different income levels. Similarly, a particular advertisement of a product has different welfare implications for consumers who value different attributes of the product.

The second situation is based on reputation considerations and is used by Rayo and Segal (2010) to motivate their model. Consider a repeated game in which each period i, the sender plays a communication game with a new receiver i whose (r_i, s_i) is independent and identically distributed. If the history of play $\{a_i, m_i, r_i, s_i\}_i$ is observable, the sender is sufficiently patient, and receivers put non-trivial probability on the event that the sender is committed to the optimal mechanism ϕ, then the sender can essentially build the reputation for playing the mechanism ϕ in each period as Fudenberg and Levine (1992) show. Real-life examples with a long-run sender include grading policy, advertising, and credit agencies.

The third situation is based on verifiable communication with endogenous information structure and is used by Kamenica and Gentzkow (2011) to motivate their model. They show that the optimal mechanism can be supported as an equilibrium outcome of the following game. First, the sender chooses a mechanism ϕ. Second, a message m is drawn according to ϕ and is privately observed by the sender. Third, the sender makes a verifiable report (in Milgrom (1981) sense) about the chosen mechanism and the received message. Finally, the receiver forms a belief and takes an action. A prosecutor persuading a judge to convict a suspect is a good example example of this situation.

Chapter 3

Optimal Limited Authority for Principal[1]

3.1 Introduction

A principal needs to elicit information from an agent in order to make decisions, but their inherent conflict of interest makes truthful communication difficult. When the principal cannot credibly give up her authority to make the final decision, the seminal paper by Crawford and Sobel (1982) (hereafter CS) shows that the principal's decisions suffer from the agent's incentive to distort his information in favor of his bias. When the principal can credibly delegate her decision-making authority, the agent uses his information efficiently but his decision is biased. In reality, however, the principal may be able to give up certain aspects of her decision-making authority, but not all, due to institutional or technological reasons.

This chapter presents a model of *limited authority*: ex ante, the principal can credibly rule out certain decisions as infeasible; but for the remaining decisions, she cannot commit to any particular decision rule such as adopting the agent's recommendation without change ex post. Real life examples of this type of limited authority abound. For instance, in a typical university tenure system, the university has only two decisions given a department's recommendation on a tenure case: promote the assistant professor or fire him. In the US House of Representatives, the Rules Committee can establish a set of special rules to limit the amendment process when a bill is introduced. In particular, the committee can adopt a structured rule that specifies the amendments

[1] **This chapter is co-authored with Li Hao and Wei Li.**

to be considered and the time for debate. Finally, in a factory setting, the owner's choice of one type of an assembly line may make the production of certain products impossible, but she can still choose the final product within the capacity of the assembly line after hearing the manager's recommendation.

Our model of limited authority presents an ex ante tradeoff for the principal in deciding how much ex post authority to retain. On one hand, by retaining more decisions the principal can make better use of the agent's reported information. On the other hand, more retained decisions creates a bigger credibility problem: the information content of the report is lower because the agent anticipates the principal's incentive to exploit it. Using the same general framework as CS, we show that under the optimal limited authority, finitely many decisions are retained. The agent partitions the state space and makes a recommendation from the set of retained decisions for each partition element, and the principal always follows the recommendation and never randomizes. Moreover, the principal is strictly better off under the optimal limited authority than in any CS equilibrium. Intuitively, by ruling out some decisions, the principal reduces her incentives to distort decisions recommended by the agent, which allows the biased agent to make more precise recommendations than in the cheap talk game.

To better understand the tradeoff for the principal under limited authority, in particular the properties of the retained decisions, we turn to the example with uniformly distributed state and convex loss function, which is a slight generalization of the uniform-quadratic example commonly used in the communication literature. We fully characterize the principal's optimal limited authority in this case. To begin with, in the optimal limited authority, all the retained decisions are above the principal's ex post optimal decisions—in the direction of the agent's bias—given that she learns the partitional elements. Second, retained decisions are more evenly distributed under the optimal limited authority than the induced decisions in a CS equilibrium. Intuitively, in a CS equilibrium each induced decision is ex post optimal because the principal has no commitment power, and thus the agent induces decisions that grow in distance between each other in the direction of his bias. In contrast, under the optimal limited authority the principal restricts the set of decisions she can choose from, which reduces the agent's incentive to distort his recommendations due to their conflict of interest. This increases the possible number of decisions that can be credibly retained, and decreases the distance between them. Third, we show that, contrary to the predictions of both the cheap talk and delegation models, the principal does not necessarily maximize the number of decisions that can be credibly retained. Intuitively, allowing more decisions ex ante can make the principal worse off by reducing the communication quality because the credibility problem distorts the choices of the decisions. In particular, some decisions may be used with almost zero probability, but their presence still forces the principal to move other retained decisions away from the ex ante optimal ones.

This chapter is directly related to the literature on delegation initiated by Holmstrom (1984). He shows that the optimal outcome under full commitment of the principal is achieved by restricting the set of decisions and delegating decision-making authority to the agent. This chapter analyzes the environment in which the principal cannot delegate authority to the agent, but can restrict the set of decisions. Closely related are Dessein (2002) and Marino (2007), who study the optimal delegation problem where the principal can veto the agent's decision and replace it with some default decision, and Mylovanov (2008), who instead assumes that the principal can choose the default decision ex ante. Less related to our work, Milgrom and Roberts (1988) and Szalay (2005) analyze how restricting the set of decisions affects influence activities and information acquisition respectively. Sections 3.2.3 and 3.5.1 discuss the related literature in greater detail.

The rest of this chapter is organized as follows. Section 3.2 sets up the limited authority model by adding to the cheap talk game a first move by the principal choosing the set of retained decisions. Section 3.2.3 provides detailed motivations for our limited authority assumption. Section 3.3 derives general properties about the optimal limited authority by first characterizing it as a solution to a constrained maximization problem. This characterization provides an equivalent interpretation of our limited authority model as a delegation game in which the principal chooses the delegation set but cannot commit to not changing the agent's decision within the set. Section 3.4 provides a full characterization of the example with uniformly distributed state, convex loss functions for both the principal and the agent, and a state-independent bias for the agent. Section 3.5 compares the principal's welfare under optimal limited authority to various organizational forms studied in the existing literature. We find that the principal's ex ante expected payoffs are similar under the optimal limited authority and optimal delegation, and both are significantly higher than that under the most informative cheap talk equilibrium. Section 3.6 briefly discusses extensions of the model. All proofs can be found in the appendix.

3.2 Model

3.2.1 Setup

This chapter analyzes the CS model with one modification. In CS the set of decisions is a real line; we instead assume that, ex ante, the principal can credibly restrict the set of decisions from which she makes decisions ex post. The model specified in this section is called a model of *limited authority* throughout the chapter. It has two natural interpretations suitable for different environments, namely, the cheap talk game and the delegation game. We first present the model using the cheap talk game, and then comment briefly on the delegation game.

Formally, there is an informed agent A (he) and an uninformed principal P (she). Payoffs of A and P, denoted by $u^A(y,\theta)$ and $u^P(y,\theta)$, are both functions of the decision y and the state of the world θ. The timing of the cheap talk game is as follows:

1. P chooses a decision set Y, a compact subset of the real line.
2. A observes Y and privately learns θ, drawn from the interval $(0,1]$ according to a positive probability density function $f(\theta)$.
3. A sends a cheap talk message m from the interval $[0,1]$.
4. P receives m and makes a decision $y \in Y$.

All aspects of the game are common knowledge. We make the CS assumptions on functions $u^A(y,\theta)$ and $u^P(y,\theta)$, which are maintained throughout the chapter:

Assumption 3.1 *There exists a function u and a scalar $b > 0$ such that $u^A(y,\theta) = u(y,\theta,b)$ and $u^P(y,\theta) = u(y,\theta,0)$. Moreover,*

1. *u is twice continuously differentiable in all variables.*
2. *$u_{yy}(y,\theta,\beta) < 0$ for all $y \in \mathbb{R}$, $\theta \in [0,1]$, and $\beta \in [0,b]$.*
3. *$u_y(y^*(\theta),\theta,\beta) = 0$ for some function $y^*(\theta)$, and for all $\theta \in [0,1]$ and $\beta \in [0,b]$.*
4. *$u_{y\theta}(y,\theta,\beta) > 0$ for all $y \in \mathbb{R}$, $\theta \in [0,1]$, and $\beta \in [0,b]$.*
5. *$u_{y\beta}(y,\theta,\beta) > 0$ for all $y \in \mathbb{R}$, $\theta \in [0,1]$, and $\beta \in [0,b]$.*

Parts 2 and 3 imply that both A and P's preferences are single-peaked. Parts 1-3 together imply that $y^i(\theta) \equiv \arg\max_{y\in\mathbb{R}} u^i(y,\theta)$ is well defined and continuous in θ for all $\theta \in [0,1]$ and $i = A, P$. Part 4 is a sorting condition, which ensures that both $y^A(\theta)$ and $y^P(\theta)$ are increasing in θ for all $\theta \in [0,1]$. Finally, part 5 guarantees that $y^P(\theta) < y^A(\theta)$ for all $\theta \in [0,1]$.

In the delegation game interpretation of the model, first P chooses a delegation set Y, and then A chooses some y from Y, which P can approve or change to some other $\tilde{y}$ in Y. The only formal difference between this interpretation and the above cheap talk interpretation is that in the delegation game A makes a choice y from Y, instead of sending a cheap talk message. The reduction in A's strategy space, from the set of messages $[0,1]$ in the cheap talk game to the set Y in the delegation game, turns out to be immaterial to our characterization of the optimal limited authority. This claim will be formally established as part of the proof of Proposition 1 in the next section.[2] As a result, the delegation game and the cheap talk game are two interpretations of the same limited authority model.

3.2.2 Solution Concept and Definitions

The solution concept we use is Perfect Bayesian Equilibria (hereafter PBE). A PBE is P's choice of Y, A's report strategy $\sigma : 2^{\mathbb{R}} \times (0,1] \to \Delta[0,1]$, P's decision strategy $\rho : 2^{\mathbb{R}} \times [0,1] \to \Delta\widetilde{Y}$, and P's belief $p : 2^{\mathbb{R}} \times [0,1] \to \Delta\,(0,1]$, such that strategies are optimal given players' beliefs, and beliefs are derived from Bayes' rule whenever possible.[3] Formally, the equilibrium conditions are, for all $\widetilde{Y} \subset \mathbb{R}$, $m \in [0,1]$, for m^* in the support of $\sigma(\cdot|\widetilde{Y},\theta)$, and for y^* in the support of $\rho\left(\cdot|\widetilde{Y},m\right)$:

$$
\begin{aligned}
Y &\in \arg\max_{\widetilde{Y}\subset\mathbb{R}} \int_{\widetilde{Y}\times[0,1]\times[0,1]} u^P(y,\theta)\,\rho\left(y|\widetilde{Y},\tilde{m}\right)\sigma(\tilde{m}|\widetilde{Y},\theta)f(\theta)\,dy\,d\theta\,d\tilde{m},\\
m^* &\in \arg\max_{\tilde{m}\in[0,1]} \int_{\widetilde{Y}} u^A(y,\theta)\,\rho\left(y|\widetilde{Y},\tilde{m}\right)dy,\\
y^* &\in \arg\max_{y\in\widetilde{Y}} \int_0^1 u^P(y,\theta)\,p\left(\theta|\widetilde{Y},m\right)d\theta,\\
p\left(\theta|\widetilde{Y},m\right) &= \frac{\sigma(m|\widetilde{Y},\theta)f(\theta)}{\int_0^1 \sigma\left(m|\widetilde{Y},\theta\right)f(\theta)d\theta}.
\end{aligned}
$$

[2] The claim is obviously true if we restrict the attention to equilibria where P uses a pure strategy on the equilibrium path. The proof of Proposition 1 establishes the claim allowing for the possibility of random decisions by P.

[3] A technical issue arises with the existence of the conditional distribution function, $p(\theta|Y,m)$, which can be bypassed using the notion of distributional strategies (see Milgrom and Weber (1985)) and Theorem 33.3 of Billingsley (1995).

A PBE of the delegation game is defined analogously, with the only difference being that A's mixed strategy is a mapping from the set of states $[0,1]$ to the set of probability distributions over the set Y chosen by P, instead of to the set of distributions over the message space $[0,1]$.[4]

Regardless of the interpretation of the limited authority model, we adopt the following definitions. The decision y is *induced by* θ (or equivalently θ *induces* y) in a PBE if y is chosen by P with positive probability when the state is θ in this PBE, or

$$\int_{\{m:\rho(y|Y,m)>0\}} \sigma(m|Y,\theta)dm > 0.$$

The decision y is *induced in a PBE* if y is induced in at least one state. A PBE is *informative* if there are at least two induced decisions, and *uninformative* otherwise. The *uninformative decision* y^P is defined as $y^P \equiv \arg\max_{y\in\mathbb{R}} \int_0^1 u^P(y,\theta) f(\theta) d\theta$. Finally, a PBE is a *partition equilibrium* $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ if $\{\theta_i\}_{i=0}^n$ is a partition of $(0,1]$, and $\{y_i\}_{i=1}^n \subset Y$ is a set of induced decisions where

$$\begin{aligned} &0 = \theta_0 < \theta_1 < \ldots < \theta_n = 1, \\ &y_1 < \ldots < y_n, \end{aligned} \tag{3.1}$$

such that any $\theta \in (\theta_{i-1}, \theta_i]$ induces decision y_i for all $i = 1, \ldots, n$. Condition (3.1) is called the *partition* condition. Clearly, a partition equilibrium can be supported as a PBE of the delegation game where the delegation set Y chosen by P on the equilibrium path has the following properties. First, it is *minimal*, in that each decision $y \in Y$ is induced; and second, it is *veto-free*, in that P chooses the same y chosen by A.

Two remarks are in order. First, all CS equilibria can be supported as a PBE in this framework. Indeed, consider any CS equilibrium. The following strategies and beliefs constitute a PBE. If P chooses $\widetilde{Y} = \mathbb{R}$, then both A and P's strategies and beliefs are given by the CS equilibrium. If P chooses $\widetilde{Y} \neq \mathbb{R}$, then A sends uninformative messages; P believes so and makes the best decision out of $\widetilde{Y}$ based on her prior belief. This observation implies that a PBE always exists.

Second, similar to the CS model, for each PBE there exists an outcome equivalent PBE in which all messages in $[0,1]$ are sent on the equilibrium path. Therefore, we cannot refine the set of PBE using standard equilibrium refinements such as those of Cho and Kreps (1987) which restrict out-

[4]In the cheap talk game, it is without loss of generality to restrict the set of messages to $[0,1]$, as in CS. In the delegation game, we have implicitly assumed that A cannot choose lotteries. This assumption is non-consequential to our analysis, as established later in Proposition 1.

of-equilibrium beliefs.[5] This chapter mostly focuses on PBE that maximizes P's expected payoff, which we refer to as the *optimal* PBE. Such a refinement is natural if P does not only choose Y at the first stage, but also announces the outcome she plans to implement with the chosen decision set Y.

3.2.3 Discussion

The imperfect commitment assumption that ex ante the principal can credibly restrict the set of decisions available to her ex post, but she cannot commit to any specific decision rule deserves further discussion. Below we provide three motivations. Our first motivation is formal, and is based on the incomplete contracting approach initiated by Grossman and Hart (1986) and Hart and Moore (1988). Our limited authority model describes a contracting environment in which the authority to make final (irreversible) decisions resides with the principal, but *only* these final decisions are verifiable.[6] In this environment, all the principal can do is to ex ante exclude some decisions from what she may choose ex post. In the same spirit of the observable-but-not-verifiable assumption in the incomplete contracting literature, our restrictions on contractibility are certainly severe. In particular, in our limited authority model neither communication by the agent to the principal, such as reports on his information or recommendations to the principal, nor decision rights is verifiable. Allowing reports or recommendations by the agent to be verifiable would of course turn our model into an exercise in mechanism design without transfers; likewise, allowing the decision rights to be contractible would change our model into an optimal delegation problem. Both these problems have been extensively studied in the literature; see for example the more recent works by Kovac and Mylovanov (2009) and Alonso and Matouschek (2008). The innovation in this chapter is instead to study a more primitive contracting environment than the full-commitment framework initiated by Holmstrom (1984), while at the same time demonstrate what "simple" contracts can achieve relative to the no-contracting, cheap talk framework of CS. Furthermore, from an applied point of view, there are contracting situations for which our limited authority model is appropriate. For example,

[5]Some refinements for cheap talk games have been proposed in the literature but they do not generally select a unique equilibrium. A notable exception is due to Chen et al. (2008), which selects the most informative equilibrium under some regularity conditions.

[6]Hart and Moore (2004) impose a similar contractibility assumption. They assume that ex ante the parties can restrict the set of outcomes over which they bargain ex post. However, the parties cannot commit to any specific mechanism according to which the outcome from this restricted set is chosen ex post. Also, Hermalin et al. (2007) propose a similar approach to model situations in which a contract has ambiguous provisions. That is, each contingency in a contract is associated with a set of outcomes from which the final outcome is chosen. In this context, the imperfect commitment assumption requires that the same set of possible outcomes should be associated with each contingency.

it may be prohibitively costly for the agent to present physical evidence of his communication with the principal in the court. Similarly, to delegate formal authority to the agent, the principal may need to sell relevant productive assets to the agent, which may be impractical because the same assets are used by the principal for other purposes.

The second motivation for the imperfect commitment assumption is technological, and presumes a contracting environment where verifiability of any reports or decisions is completely absent. For example, making certain decisions may require a specialized equipment which is prohibitively expensive to procure ex post. In this case, not procuring the equipment ex ante commits the principal to not making the decisions that use the equipment. Similarly, in many organizations, managers make decisions using software packages, such as SAP ERP. This software is typically adjusted to the specific needs of each organization so that certain decisions are made unavailable, such as trading of some products at certain prices in a financial company. In addition, it may be impractical to give control over this software to those who have relevant information for decision making in an organization. This technology thus allows the principal to restrict her decision set without making it possible for her to commit to decisions based on the agent's reports. It is worth noting that between the two aspects in the standard full-commitment model, the principal's ability to make certain decisions infeasible and her ability to commit to not changing the decision made by the agent, the first one may be accomplished through some technology while the second one is often technologically harder or even impossible.

The third and final motivation for the imperfect commitment assumption is institutional, and is based on realistic assumptions about how decision rights are allocated in organizations. In many organizations, managers typically make critical decisions based on information supplied by their subordinates, but are held accountable for the final decisions. Organization rules often prohibit managers from delegating their decisions to their subordinates, but allow managers to credibly commit to not taking certain decisions. This is the kind of organizational setting that makes our limited authority model applicable. The same situation may also arise in a multi-level hierarchy in which contracts can be written only among certain parties. For instance, our limited authority model applies in a multidivisional organization with the headquarters, a division manager, and the manager's subordinate, where enforceable contracts can be written only between the headquarters and the manager.

3.3 General Analysis

In this section we provide a general analysis of the optimal PBE in our limited authority model. We start by characterizing the optimal PBE as a solution to a constrained maximization problem in Proposition 3.1. This is a useful result that we exploit further in the uniform-convex loss setup in Section 3.4 to completely characterize the optimal PBE. Here we use it to establish the main result of the section, Proposition 3.2, that the optimal PBE strictly improves the principal's welfare relative to the most informative equilibrium of CS. Under further assumptions on the preference functions u^A and u^P, Proposition 3.3 provides a tight upper bound on the agent's bias parameter b for the principal to benefit from limited authority relative to the CS model.

Our first result establishes the existence of optimal PBE under limited authority and characterizes its basic properties. In particular, it shows that the optimal PBE is a partition equilibrium with a finite number of induced decisions.

Proposition 3.1 *An optimal PBE exists and is a partition equilibrium with a finite number of elements. Moreover, among all partition equilibria* $(\{\theta_i\}_{i=0}^{n}, \{y_i\}_{i=1}^{n})$ *with a finite* n*, it maximizes* $\sum_{i=1}^{n} \int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta$ *subject to, for each* $i = 2, \ldots, n$*,*

$$u^A(y_i, \theta_{i-1}) = u^A(y_{i-1}, \theta_{i-1}), \tag{3.2}$$

$$\int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta \geq \int_{\theta_{i-1}}^{\theta_i} u^P(y_{i-1}, \theta) f(\theta) d\theta. \tag{3.3}$$

We start the proof of Proposition 1 by characterizing all PBE. First, we exploit the assumptions on the payoff functions u^A and u^P to show that any PBE in the cheap talk game can be supported as one in the delegation game, meaning that we can restrict attention to PBE's in which P's equilibrium decision set Y is minimal and veto-free. Second, we prove that any PBE is a partition equilibrium with a finite number of elements. The proof of the finiteness is quite standard, except that the distance between three rather than two adjacent induced lotteries is bounded away from zero. Third, in any PBE, the adjacent incentive conditions are sufficient for all incentive conditions of P because P's payoff function is strictly concave. Further, in any PBE in which P does not randomize over the set of decisions upon receiving a message, P never has incentives to deviate to higher decisions, because A has an upward bias and even he weakly prefers not to deviate to higher decisions.

To complete the proof of Proposition 1, we characterize optimal PBE. First, we show that in any optimal PBE, P never randomizes. The proof is involved because of the finiteness of decision set Y.[7] We show that for any PBE with non-degenerate lotteries, P can increase her payoff by replacing each non-degenerate lottery with the higher decision in the lottery. The observation above that in any non-degenerate lottery $y < y'$, A strictly prefers the higher decision y' implies that P can improve the quality of communication by A by increasing the probability weight on y'. By putting all the weight on y' instead, she makes the (now degenerate) lottery more attractive so that A recommends y' for a set of higher states than in the old PBE. Since P prefers a lower decision for the same set of states, she is strictly better off. Finally, the problem of finding the optimal PBE reduces to a constrained maximization problem where the set of feasible choices is all partition equilibria with a finite number of elements that satisfy A's indifference conditions (3.2) and P's adjacent downward incentive conditions (3.3). The solution to this problem exists by the maximum theorem.

Clearly, P cannot do worse than in any CS equilibrium, as she can replicate any CS equilibrium outcome by restricting the set of decisions to those induced in the CS equilibrium. Our second result shows that P can in fact do strictly better.

Proposition 3.2 *P's expected payoff is strictly higher in the optimal PBE than in any informative CS equilibrium.*

In a CS equilibrium, each induced decision is ex post optimal for P in that it maximizes P's expected payoff over all possible decisions $y \in \mathbb{R}$ given P's belief about the state after receiving a message from A. Therefore, P's incentive conditions (3.3) are not binding in an informative CS equilibrium, and she can marginally increase any induced decision y_i without violating (3.3). As P increases y_i, by the envelope theorem, her expected payoff is unaffected by the introduction of ex post inefficiencies, but it increases marginally due to an increase in the partition thresholds θ_{i-1} and θ_i. For example, as θ_{i-1} increases to θ'_{i-1}, an upwardly biased A induces y_{i-1} instead of a higher decision y_i for states $\theta \in (\theta_{i-1}, \theta'_{i-1}]$, which increases P's expected payoff.

We can strengthen Proposition 3.2 by showing that P can still strictly improve her expected payoff by restricting the set of decisions even when there does not exist an informative CS equilibrium. More formally, suppose that an informative CS equilibrium exists whenever b is less than b^*,

[7] In CS P's payoff function is strictly convex in a decision and the set of decisions is convex. Thus, upon receiving a message, P has a unique optimal decision.

with two decisions y_1 and y_2. Then there exists ε such that for all b less than $b^* + \varepsilon$, ε sufficiently small, P's expected payoff is strictly higher in the optimal PBE than in any CS equilibrium. By Proposition 3.2, for b less than b^*, P can increase either y_1 or y_2 to achieve the desired PBE. By continuity of u, these new induced decisions still constitute a PBE in which P's expected payoff is strictly higher than that in the uninformative CS equilibrium at $b = b^* + \varepsilon$.

Under additional assumptions on the function u, we can further strengthen Proposition 3.2. We show that P's expected payoff is strictly higher in the optimal PBE than a babbling equilibrium if and only if delegation is valuable under full commitment. Adopting a definition from Alonso and Matouschek (2008), we say that *delegation is valuable* if P can improve on the uninformed decision y^P by committing to letting A choose between at least two decisions.

Proposition 3.3 *Let $u^P(y,\theta) = -(y - y^P(\theta))^2$ and $u^A(.,\theta)$ be symmetric around $y^A(\theta)$. P's expected payoff is strictly higher in the optimal PBE than in any CS equilibrium if and only if delegation is valuable.*

The "only if" part is immediate, because by Proposition 3.1 the optimal PBE is a partition equilibrium and any partition equilibrium can be implemented through delegation as the incentive conditions (3.3) for P are absent in delegation under full commitment. The proof of the "if" part is based on a result due to Alonso and Matouschek (2008). They show that if delegation is valuable, then P can improve on implementing the uninformed decision y^P by letting A choose between exactly two decisions. We show that these two decisions satisfy P's incentive condition (3.3), and thus can be induced in a PBE.

3.4 Uniform-Convex Loss Example

This section focuses on a slight generalization of the leading example of CS. In particular, we assume:

Assumption 3.2 *$f(\theta) = 1$ for $\theta \in (0,1]$, $u(y,\theta,\beta) = -l(|y - (\theta + \beta)|)$, where l is increasing and convex with $l(0) = l'(0) = 0$.*

Assumption 3.2 includes the leading uniform-quadratic example as a special case with $l(z) = z^2$. Clearly, Assumption 3.2 satisfies Assumption 3.1, so Propositions 3.1 and 3.2 hold. We focus on this example because it is widely used in applications as a building block.[8]

This example is particularly well-behaved to apply the constrained maximization program given in Proposition 3.1. In leading to the main result of this section, a complete characterization of the optimal PBE in Proposition 3.5, we provide a few results that have independent interests and a solution approach that yields further insights about the optimal PBE. We first establish that in the optimal PBE each induced decision is higher than what is ex post optimal conditional on P learning the corresponding partition element. As a result, P's incentive conditions (3.3) take a particularly simple form. In Proposition 3.4, we show that binding these conditions yields both an upper bound on the number of induced decision in the optimal PBE, and a PBE that achieves this upper bound. We then solve the full-commitment problem of maximizing P's expected payoff with a fixed number of induced decisions, subject only to the partition conditions (3.1) and the agent's indifference conditions (3.2). The solution provides a lower bound on the number of induced decisions in the optimal PBE when it satisfies P's incentive conditions (3.3). The optimal PBE can be then characterized by considering all partition equilibria with a number of induced decisions between the lower bound from the full-commitment problem and the upper bound from Proposition 3.4.

3.4.1 Maximal Limited Authority

From Proposition 3.1, an optimal PBE exists and it is a partition equilibrium that satisfies A's indifference conditions (3.2) and P's adjacent downward incentive conditions (3.3). In the present uniform-convex loss model, these conditions can be rewritten as: for all $i = 2, \ldots, n$,

$$\theta_{i-1} + b - y_{i-1} = y_i - \theta_{i-1} - b; \tag{3.4}$$

$$|y_i - y_i^*| \leq |y_i^* - y_{i-1}|, \tag{3.5}$$

where $y_i^* = \frac{1}{2}(\theta_{i-1} + \theta_i)$ is P's *ex post optimal* decision conditional on the interval $(\theta_{i-1}, \theta_i]$. We now show that induced decisions y_i are higher than ex post optimal y_i^* for all $i = 1, \ldots, n$.

Lemma 3.1 *In any optimal PBE $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$, $y_i > y_i^*$ for each $i = 1, \ldots, n$.*

[8]There is another uniform-quadratic example that has been analyzed in recent papers including Gordon (2007) and Alonso et al. (2008). In this example, A has an outward rather than upward bias such that his payoff is given by $u^A(y, \theta) = -(y - b - c\theta)^2$ where $b < 0$ and $b + c > 1$. Intuitively, an outwardly biased A prefers extreme decisions when the state of the world is extreme. In the example with outwardly biased A, there exists an equilibrium with a countable number of induced decisions which eliminates an integer problem peculiar to the leading example of CS. Therefore, in some applications, an example with outwardly biased A is simpler to analyze.

The first part of the proof of the above result establishes that if $y_i \leq y_i^*$ for some $i = 1, \ldots, n$ in the optimal PBE, then P's $(i+1)$-th incentive condition binds. This holds for the general model set up in Section 3.2, not just the uniform-convex loss model here. The intuition is that if $y_i \leq y_i^*$ and the $(i+1)$-th condition is slack, P can obtain a greater expected payoff by marginally increasing y_i without affecting any incentive condition. Her payoff gain is clear: a higher y_i moves her closer to her ex post optimal decision given the same belief of the states; and the resulting increases in thresholds θ_{i-1} and θ_i mean that A now recommends y_i for a set of higher states, making P better off by the same argument as in Proposition 3.2.[9] The second part of the proof of Lemma 3.1 uses the special structure of the uniform-convex loss model to show that $y_i \leq y_i^*$ is incompatible with binding P's $(i+1)$-th incentive condition.

We now restate the constrained maximization problem of Proposition 3.1 for the uniform-convex loss setup by substituting out A's indifference conditions (3.4). The choice variables are $\{y_i\}_{i=1}^n$. By partition condition (3.1), $\theta_0 = 0$ and $\theta_n = 1$, so the objective function becomes

$$U_n^P = -\sum_{i=1}^{n} \int_{\theta_{i-1}}^{\theta_i} l\left(|\theta - y_i|\right) d\theta \tag{3.6}$$

The constraints are

$$y_1 < \ldots < y_n, \tag{3.7}$$

$$y_1 + y_2 > 2b, \tag{3.8}$$

$$y_{i+1} - y_{i-1} \geq 4b, \text{ for each } i = 2, \ldots, n-1, \tag{3.9}$$

$$y_n + y_{n-1} \leq 2\left(1 - b\right). \tag{3.10}$$

Conditions (3.7) are part of the partition condition (3.1). Condition (3.8) ensures that $\theta_1 > 0$; $\theta_{n-1} < 1$ is implied by condition (3.10); and $\theta_1 < \theta_2 < \ldots \theta_{n-2} < \theta_{n-1}$ follow from (3.7). Conditions (3.9) and (3.10) are equivalent to (3.5) for $i = 2, \ldots, n$ by Lemma 3.1. Condition (3.10) takes a different form because $\theta_n = 1$ by partition condition (3.1), instead of being determined by A's indifference condition in (3.4).

[9] In the general model, this result implies that in an optimal PBE the highest decisions y_n and y_{n-1} satisfy $y_n > y_n^*$ and $y_{n-1} > y_{n-1}^*$, and that no two adjacent decisions y_i and y_{i+1} are below y_i^* and y_{i+1}^* respectively. Further, in the hypothetical problem of full commitment with a fixed number of decisions introduced in Section 3.4.2, every decision y_i is strictly higher than y_i^*.

Observe that conditions (3.9) and (3.10) place constraints on the distance between decisions, and thus the possible number of decisions, in the optimal PBE. We now proceed to characterize the maximum number of decisions that P can possibly use in an optimal PBE for a given bias b. More importantly, we show that there always exists a PBE in which the maximum number of decisions is induced.

Proposition 3.4 *The number of decisions induced in an optimal PBE is strictly less than* $1/(2b)+1$. *Conversely, there exists a PBE with* n *induced decisions for any positive integer* $n < 1/(2b)+1$.

Note that conditions (3.8) and (3.10) require that $b < \frac{1}{2}$. This is consistent with the above proposition, as the maximum number of decisions that P can possibly use in any PBE is 1 if $b \geq \frac{1}{2}$. The second part of Proposition 3.4 is established by construction. A PBE that achieves the upper bound on the number of decisions in an optimal PBE in the present uniform-convex loss model is called *maximal limited authority.* The construction binds all P's incentive conditions (3.9), with symmetric and equidistant decisions.[10] It is instructive to compare the maximal limited authority with the most informative CS equilibrium. In a CS equilibrium, the distance between subsequent induced decisions grows at the rate $4b$, that is, $y_{i+1} - y_i = y_i - y_{i-1} + 4b$ for $i = 2, ..., n-1$. Therefore, the number of induced decisions n in any CS equilibrium has to satisfy $2n(n-1)b < 1$. In contrast, under limited authority, the distance between two subsequent induced decisions does not grow. In fact, under the maximal limited authority with N induced decisions, binding P's i-th incentive condition we have $y_{i+1} - y_{i-1} = 4b$ for all $i = 2, \ldots, N-1$. As a result, N is the largest integer n satisfying $2(n-1)b < 1$, which is greater than the number of induced decisions in the most informative CS equilibrium.

Although both the upper bound on the number of decisions in an optimal PBE and the construction of a PBE that achieves the upper bound are specific to the uniform-convex loss setup, there is a more general logic behind the result that the number of decisions under maximal limited authority is larger than that in the most informative CS equilibrium. In a CS equilibrium, each induced decision y_i is ex post optimal conditional on the corresponding interval $(\theta_{i-1}, \theta_i]$ because P has no commitment power. In contrast, because our limited authority model gives P some commitment power, in a partition equilibrium, P's incentive conditions are given by (3.3), requiring

[10] When $n = 2$, the maximum limited authority coincides with the full commitment solution introduced in Lemma 3.2, so P's incentive condition does not bind. See the proof of the proposition in the appendix.

only that P prefer y_i to the adjacent lower decision y_{i-1} conditional on $(\theta_{i-1}, \theta_i]$. Therefore the partitioning of the state space $(0,1]$ under limited authority need not be as rightward skewed as in a CS equilibrium, and more decisions can be induced as a result.

3.4.2 Optimal Limited Authority

Before characterizing the optimal PBE, it is useful to consider the hypothetical problem of maximizing (3.6) subject to constraints (3.7), (3.8), and

$$y_{n-1} + y_n < 2(1+b), \tag{3.11}$$

where we have dropped P's incentive conditions (3.9) and (3.10), but added (3.11) to ensure that $\theta_{n-1} < 1$. This hypothetical problem has the interpretation of the case of full commitment, with the restriction to a finite number n of decisions, and is informative about the limited authority model.[11] The solution also provides an upper bound on what P can achieve in an optimal PBE if it has n decisions. Denote the solution as $Y^{FC}(n) = \left\{y_i^{FC}\right\}_{i=1}^n$. (We write y_i^{FC} instead of $y_i^{FC}(n;b)$ whenever it can be done without loss of clarity.) The following lemma provides a characterization of $Y^{FC}(n)$.

Lemma 3.2 *Suppose that $b < \frac{1}{2}$. For any natural n, $Y^{FC}(n)$ is given by $y_i^{FC} = \frac{1}{2} + \Delta\left(i - \frac{n+1}{2}\right)$, $i = 1, \ldots, n$, where $\Delta > 0$ is uniquely determined by*

$$2l(y_1^{FC}) = l\left(b + \Delta/2\right) + l\left(|b - \Delta/2|\right).$$

For $b \geq \frac{1}{2}$, the optimal full commitment solution has the single decision of $\frac{1}{2}$. In contrast, for $b < \frac{1}{2}$, under full commitment for any n optimal decisions y_i^{FC} are equidistant, that is, all decisions $i = 1, \ldots, n$ are Δ apart and symmetric around $\frac{1}{2}$. In order to minimize P's expected payoff given by (3.6), the decisions need to be equidistant to make the partition of the state space uniform in that $\theta_i - \theta_{i-1}$ is the same for all $i = 2, \ldots, n-1$. The uniform partition in turn minimizes the loss of information, which can be loosely understood as the average residual uncertainty of the state of the world provided that P learns the partition elements (see Section 3.5 for more details when the loss

[11] Melumad and Shibano (1991) show that the optimal decision set is equal to $[b, 1-b]$ in the full commitment model in which P can commit to not to change A's recommendation without the restriction to a finite number of decisions. The solution given in Lemma 3.2 becomes arbitrarily close to $[b, 1-b]$ as $n \to \infty$. To see this, note that as $n \to \infty$, $\Delta \to 0$ and $(n-1)\Delta \to 1 - 2b$.

function is quadratic). That decisions y_i^{FC} are symmetric around $\frac{1}{2}$ is also intuitive, because P is unbiased, that is, $y^P(\theta) = \theta$.[12] Finally, Lemma 3.2 implies that as n increases, the maximized value of U_n^P in the hypothetical full-commitment problem strictly increases. In fact, a robust feature of models with full commitment is that more decisions can only improve A's communication quality because P commits to not using the information A revealed strategically.

Solution $Y^{FC}(n)$, however, may violate P's incentive conditions (3.9) and thus become infeasible under limited authority. We now turn to the problem of maximizing P's expected payoff (3.6) by choosing a set of a fixed finite number n of decisions, subject to all constraints (3.7)-(3.10). Denote the solution to this problem as $Y^{LC}(n) = \{y_i^{LC}(n)\}_{i=1}^n$, and we refer to it as the *n-optimal limited authority* as it takes n as given. A few observations are immediate. First, obviously, $Y^{LC}(1) = Y^{FC}(1)$ for any b, with $y_1^{LC}(1) = \frac{1}{2}$. Second, for each $n \geq 2$, by Proposition 3.4, $Y^{LC}(n)$ exists only if and only if

$$b < b^{LC}(n) \equiv \frac{1}{2(n-1)}. \tag{3.12}$$

Third, for $n \geq 3$, the solution $Y^{FC}(n)$ to the full commitment problem satisfies P's incentive condition (3.9), and hence $Y^{LC}(n) = Y^{FC}(n)$, if and only if $b \leq b^{FC}(n)$, where $b^{FC}(n)$ is uniquely determined by

$$2l(1/2 - (n-1)b^{FC}(n)) = l(2b^{FC}(n)). \tag{3.13}$$

This follows because the above definition of b^{FC} is simply the evaluation of the condition in Lemma 3.2 at $\Delta = 2b$, so the distance Δ between two adjacent decisions y_{i+1}^{FC} and y_i^{FC} is greater than or equal to $2b$ if and only if $b \leq b^{FC}(n)$.[13] Intuitively, when b or n is small, the decisions under $Y^{FC}(n)$ are sufficiently far apart from each other so that P would not want to deviate to the lower decision y_{i-1} when A recommends y_i. For $n = 2$, it turns out that $Y^{FC}(2)$ given in Lemma 3.2 is always incentive compatible for P, and thus $Y^{LC}(2) = Y^{FC}(2)$.[14] Since $Y^{FC}(2)$ exists if and only if $b \leq \frac{1}{2}$, we write $b^{FC}(2) = \frac{1}{2}$.

[12] If we focused on PBE that maximized A's expected payoff instead of P's, then the optimal decisions would tend to be symmetric around $\frac{1}{2} + b$ because A has the upward bias $b > 0$.

[13] Since l is convex, the right-hand side of the condition in Lemma 3.2 is increasing in Δ regardless of whether $\Delta \geq 2b$ or $\Delta < 2b$. The remaining incentive condition (3.10) of P can be shown to be equivalent to $\theta_1 \geq 0$, which is always satisfied under $Y^{FC}(n)$. For details see the proof of Lemma 3.2 in the appendix.

[14] The only incentive condition of P is (3.10). This is equivalent to $\theta_1 \geq 0$, which is satisfied because in this case $\theta_1 = \frac{1}{2} - b$ and $b \leq \frac{1}{2}$. See the proof of Proposition 3.4 in the appendix for details.

The following lemma characterizes the n-optimal limited authority $Y^{LC}(n)$ for $n \geq 3$ and $b \in (b^{FC}(n), b^{LC}(n))$. The crucial feature is that P's incentive conditions (3.9) all bind under $Y^{LC}(n)$. That is, under any $Y^{LC}(n)$, whenever some of P's incentive conditions must bind because $b > b^{FC}(n)$, P is indifferent between implementing each recommended decision y_i^{LC} and replacing it with the adjacent lower decision y_{i-1}^{LC} for each $i = 2, \ldots, n-1$. Otherwise, if some, but not all, incentive conditions (3.9) bind, then it would be possible to modify the decisions to make them more equidistant and P better off. For example, if $y_{i+1} - y_{i-1} > 4b$ for some i, then we could increase y_{i-1} or decrease y_{i+1} without violating any incentive condition of P. That all incentive conditions of P must bind if any of them binds is an intuitive result due to the assumption of uniform distribution of the state, and this result is what makes the characterization of $Y^{LC}(n)$ relatively straightforward. It turns out that the characterization of $Y^{LC}(n)$ depends on whether n is odd or even. In both cases, the decisions $\{y_i^{LC}\}_{i=1}^{n}$ are symmetric around $\frac{1}{2}$, which is intuitive because the state is uniformly distributed and the loss function l is convex. When n is odd, the decisions are all equidistant at $2b$. When n is even, the decisions are equidistant in an alternating manner, with $y_{i+1}^{LC} - y_i^{LC}$ equal for all odd i and for all even i respectively but strictly smaller for odd i.[15]

Lemma 3.3 *Fix any $n \geq 3$ and $b \in (b^{FC}(n), b^{LC}(n))$. The n-optimal limited authority $Y^{LC}(n)$ is given by $y_i^{LC} = \frac{1}{2} + 2b\left(i - \frac{n+1}{2}\right) + \left(b - \frac{1}{2}\Delta_1\right)$ for odd i, and $y_i^{LC} = \frac{1}{2} + 2b\left(i - \frac{n+1}{2}\right) - \left(b - \frac{1}{2}\Delta_1\right)$ for even i, where $\Delta_1 = 2b$ if n is odd and $\Delta_1 < 2b$ determined by*

$$2l(y_1^{LC}) = l(b + \Delta_1/2) + l(b - \Delta_1/2) - \frac{n-2}{2}\left[l(3b - \Delta_1/2) - l(b + \Delta_1/2)\right] \tag{3.14}$$

if n is even.

Given the above characterization of the n-optimal limited authority decision set $Y^{LC}(n)$ for each fixed n and for all $b \in \left(b^{FC}(n), b^{LC}(n)\right)$, we can now present the main result of this section. This is a characterization of the optimal PBE, achieved by comparing P's expected payoffs under

[15] Having all $y_{i+1}^{LC} - y_i^{LC}$ equal to $2b$ is not optimal when n is even, because the number of such differences is not a multiple of the number of incentive conditions in (3.9). Starting from a set of decisions $\{y_i\}_{i=1}^{n}$ that are equidistant at $2b$ and symmetric around $\frac{1}{2}$, we can increase P's expected payoff by increasing y_i for all odd i and decreasing it for even i by the same amount.

all feasible decision sets. By (3.12) and (3.13),[16]

$$b^{FC}(n) < b^{LC}(n+1) < b^{FC}(n-1).$$

Since P prefers $Y^{FC}(n-1)$ to $Y^{FC}(i)$ and $Y^{LC}(i)$ for all $i < n-1$, we can restrict the search for the optimal PBE in the interval $[b^{FC}(n), b^{FC}(n-1))$ to $Y^{FC}(n-1)$, $Y^{LC}(n)$ and $Y^{LC}(n+1)$.

Proposition 3.5 *Suppose that $l(z) = z^2$. For each $n \geq 3$, there exists $b(n, n-1) \in (b^{LC}(n+1), b^{FC}(n-1))$ such that the induced decisions in the optimal PBE are given by $Y^{LC}(n)$ for all $b \in [b^{FC}(n), b(n, n-1))$, and by $Y^{FC}(n-1)$ for all $b \in [b(n, n-1), b^{FC}(n-1))$.*

The logic behind the comparison among $Y^{FC}(n-1)$, $Y^{LC}(n)$ and $Y^{LC}(n+1)$, which holds for all loss function l that satisfies Assumption 3.2, can be seen as follows. First, observe that at $b = b^{FC}(n)$, the optimal decision sets under full commitment and limited authority are identical: $Y^{FC}(n) = Y^{LC}(n)$. P's expected payoff jumps down discontinuously at $b^{FC}(n)$ if decisions change from $Y^{FC}(n)$ to $Y^{FC}(n-1)$. In contrast, $Y^{LC}(n)$ changes continuously with b at $b^{FC}(n)$, consequently P is strictly better off with $Y^{LC}(n)$ than with $Y^{FC}(n-1)$ if b is sufficiently close to and greater than $b^{FC}(n)$. Second, at b just below the cutoff value $b^{LC}(n+1)$, P strictly prefers $Y^{LC}(n)$ to $Y^{LC}(n+1)$. This follows because under $Y^{LC}(n+1)$, the lowest partition threshold $\theta_1^{LC}(n+1)$ equals 0 at $b^{LC}(n+1)$, so effectively only n decisions are recommended by A and approved by P. Since $Y^{LC}(n)$ is available at $b^{LC}(n+1)$, $Y^{LC}(n+1)$ is dominated for P: the additional decision in $Y^{LC}(n+1)$ does nothing to improve her expected payoff, but distorts the quality of her decisions, making her worse off. Therefore, the cutoff value $b^{LC}(n+1)$ is not relevant for P's search for optimal PBE in the interval $[b^{FC}(n), b^{FC}(n-1))$.

As b increases in the interval $(b^{FC}(n), b^{FC}(n-1))$, P's expected payoff decreases under each of $Y^{FC}(n-1)$, $Y^{LC}(n)$ and $Y^{LC}(n+1)$. Under the assumption of $l(z) = z^2$, the proof of Proposition 3.5 in the appendix ranks the rate of decrease for the three sets of decisions. In particular, we show that P's expected payoff decreases slower under $Y^{FC}(n-1)$ than under $Y^{LC}(n)$ for any $b \in (b^{FC}(n), b^{FC}(n-1))$, and in turn slower under $Y^{LC}(n)$ than under $Y^{LC}(n+1)$ for any $b \in (b^{FC}(n), b^{LC}(n+1))$. We then show that at $b^{FC}(n-1)$, P strictly prefers $Y^{FC}(n-1)$ to $Y^{LC}(n)$.

[16] Note that the function $g(k, b) \equiv 2l(1/2 - (k-1)b) - l(2b)$ is decreasing in b for $b \in (0, b^{LC}(k))$ and is equal to 0 at $b^{FC}(k)$ for all k. The first inequality holds because $g(n, b^{LC}(n+1)) = 2l\left(\frac{1}{2n}\right) - l\left(\frac{1}{n}\right) < 0$ for any convex loss function l. The second inequality holds because $g(n-1, b^{LC}(n+1)) = l\left(\frac{1}{n}\right) > 0$.

Shifting the indices forward by 1 and noting that $Y^{LC}(n) = Y^{FC}(n)$ at $b^{FC}(n)$, we have that P strictly prefers $Y^{LC}(n)$ to $Y^{LC}(n+1)$. This evaluation then allows us to establish the proposition.

Proposition 3.5 makes it clear that the optimal limited authority does not generally coincide with the maximal limited authority. This is reflected in two ways. First, when b falls in $[b^{FC}(n), b^{LC}(n+1))$, $Y^{LC}(n+1)$ is available but is never optimal. Indeed, as observed above, for any loss function l that satisfies Assumption 3.2, P strictly prefers $Y^{LC}(n)$ to $Y^{LC}(n+1)$ for b sufficiently close to and lower than $b^{LC}(n+1)$. Second, when b falls in $(b(n, n-1), b^{FC}(n-1))$, $Y^{LC}(n)$ is available but P strictly prefers $Y^{FC}(n-1)$. Thus, unlike in the solution to the hypothetical full-commitment problem, P's payoff does not necessarily increase with the number of decisions under limited authority. Intuitively, in an optimal PBE, P retains fewer decisions in order to relax the incentive conditions due to limited authority.[17]

Our result that the optimal PBE under limited authority does not always maximize the number of induced decisions contrasts strongly with CS and models with full commitment. This may be counterintuitive, but recall that we have restricted the search for the optimal PBE under limited authority to decision sets that are minimal and veto-free. Thus, in characterizing the decision set $Y^{LC}(n)$ for each fixed n, we have imposed the condition that all n decisions are induced in some states, and precluded the standard reasoning that adding a decision cannot make P worse off. Instead, each additional decision in our model presents P with a credibility problem. In a PBE with a larger number of induced decisions there is better information transmission, which would benefit P, everything else being equal. This better information transmission, however, is achieved due to P's commitment to ex-post suboptimal decisions. As a result, P may prefer a PBE with worse information transmission, but better decision-making.

3.5 Welfare Comparison across Organizational Forms

Many existing papers have analyzed extensions of the CS model that improve communication quality and thus P's welfare. We categorize these papers into six organizational forms: cheap talk, delegation, persuasion, informational control, noisy talk, and limited authority. Then we compare P's ex ante expected payoffs under these organizational forms. All the payoff comparisons are based on specializing the uniform-convex loss example of Section 3.4.2 to the quadratic loss function.

[17] By Proposition 3.5, in the familiar uniform-quadratic case, the number of induced decisions under the maximal and optimal limited authority is the same only for $b \in [b^{LC}(n+1), b(n-1, n))$.

3.5.1 Organizational Forms

Our first organizational form is CS's *cheap talk* model in which neither P nor A has any commitment power. In the CS model, there are essentially three players: nature, A, and P. Nature draws the state of the world $\theta \in \Theta$. A privately observes the state θ and sends a message $m \in M$ to P, who then makes a decision $y \in Y$.

Let us introduce a fourth non-strategic player to the CS model who takes an input $i \in I$ and returns a possibly stochastic output $o \in O$ according to some prespecified mapping $\phi : I \to \Delta O$, where ΔO denotes the set of lotteries over outcomes O. The fourth player can either replace one of the players or be a mediator. Each possible way that a fourth player could be introduced into the game corresponds to a different *organizational form*. Note that a strategic fourth player is just a particular non-strategic player who uses a certain (equilibrium) mapping. Thus we can analyze either a fourth player designed by P, whose mapping maximizes P's expected payoff; or a fourth player with an equilibrium mapping, which corresponds to a strategic player or to a modification of the CS environment.

We first discuss organizational forms where the fourth player replaces another player (Figure 3-1, left panel). There are two such forms: *delegation* and *persuasion*, which correspond to the replacement of P and A respectively by a fourth player. Delegation refers to P's commitment power: under delegation, A sends a message m to the fourth player instead of to P, and the fourth player then makes a decision y according to $\phi_D : M \to \Delta Y$. Such delegation encompasses both optimal delegation and full delegation: the former corresponds to to a fourth player designed by P and the latter corresponds to A being the fourth player respectively.[18] Persuasion refers to A's commitment power: under persuasion, the fourth player observes the state θ and sends a message m to P according to $\phi_{PM} : \Theta \to \Delta M$.[19]

We turn next to organizational forms where the fourth player is a mediator (Figure 3-1, right panel). There are three such forms: *informational control*, *noisy talk*, and *limited authority*, which correspond to information, message, and decision mediation, respectively. Informational control refers to A's information structure: under informational control, the fourth player privately observes

[18] Holmstrom (1984), Melumad and Shibano (1991), Martimort and Semenov (2006), and Alonso and Matouschek (2008) study optimal delegation in which a third player is restricted to deterministic decision rules. Goltsman et al. (2009), and Kovac and Mylovanov (2009) study optimal stochastic delegation. Dessein (2002) studies full delegation and analyzes how it compares to cheap talk. Gilligan and Krehbiel (1987), Krishna and Morgan (2001), and Mylovanov (2008) study veto-delegation.

[19] Kamenica and Gentzkow (2011) analyze optimal persuasion that maximizes A's expected payoff. Clearly, full information transmission maximizes P's expected payoff.

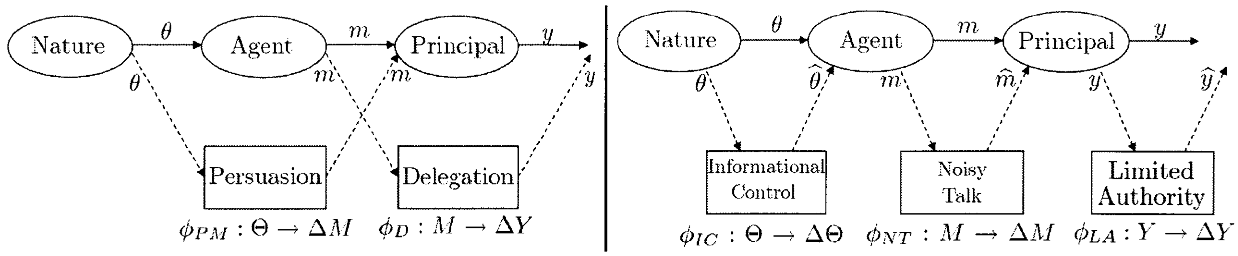

Figure 3-1: Classification of organizational forms

the state θ and returns a signal $\widehat{\theta} \in \Theta$ according to $\phi_{IC} : \Theta \to \Delta\Theta$; A privately observes the signal $\widehat{\theta}$ and sends a message to P, who makes a decision.[20] Noisy talk refers to the quality of the communication channel between A and P: under noisy talk, the fourth player receives a message m from A and sends a perturbed message $\widehat{m} \in M$ to P according to $\phi_{NT} : M \to \Delta M$.[21]

Finally, limited authority refers to the set of decisions available to P: under limited authority, P receives a message from A and then recommends a decision y to the fourth player, who implements a decision $\widehat{y} \in Y$ according to $\phi_{RA} : Y \to \Delta Y$. This chapter focuses on P's optimal deterministic limited authority (deterministic in that a decision mediator is bound to use a deterministic mapping $\phi_{RA} : Y \to Y$). Note that limited authority is equivalent to letting P make any decision from the restricted set $\phi_{RA}(Y)$, which justifies the setup of our model.

3.5.2 Payoff Comparisons

In this section, we compare P's highest ex ante expected payoffs within each organizational form. To begin with, Proposition 3.6 calculates P and A's expected payoffs for a small bias b under optimal limited authority.

Proposition 3.6 *The first term in Taylor expansion around $b = 0$ of P and A's expected payoffs under optimal limited authority is equal to $-\frac{4}{3}b^2$ and $-\frac{1}{3}b^2$ respectively.*

[20] Ivanov (2010b) studies optimal informational control, while Austen-Smith (1994), and Fischer and Stocken (2001) study a non-optimal informational control. The extensive literature on information acquisition and reputational cheap talk is also related to informational control, but there are additional elements added to the CS model.

[21] Goltsman et al. (2009) characterize optimal noisy talk. Furthermore, there are a number of papers that can be represented as a particular type of a message mediator: Krishna and Morgan (2004) analyze back and forth communication between the agent and the principal; Blume et al. (2007) study communication with perturbed messages; Ivanov (2010a), Li (2010), and Ambrus et al. (2010) study communication with a strategic message mediator and a sequence of strategic message mediators, respectively.

To understand the intuition behind Proposition 3.6, it is useful to decompose P and A's expected payoffs as the sum of the loss of information and the loss of control:

$$U^P = -\mathbb{E}\left[(y-\theta)^2\right] = \underbrace{-\mathbb{E}\left[(y_m - \mathbb{E}[\theta|m])^2\right]}_{P\text{'s Loss of Control}} - \underbrace{\mathbb{E}\left[Var(\theta|m)\right]}_{\text{Loss of Information}},$$

$$U^A = -\mathbb{E}\left[(y-(\theta+b))^2\right] = \underbrace{-\mathbb{E}\left[(y_m - \mathbb{E}[\theta|m] - b)^2\right]}_{A\text{'s Loss of Control}} - \underbrace{\mathbb{E}\left[Var(\theta|m)\right]}_{\text{Loss of Information}},$$

where y_m, $\mathbb{E}[\theta|m]$, and $Var(\theta|m)$ are the decision taken, the expectation, and the variance of the state θ given a message m (under P's beliefs), respectively. The loss of information is defined as the expected conditional variance of the state given P's belief at the time of decision making. Therefore, the loss of information captures the residual uncertainty that P has after communication took place. P and A' loss of control is defined as the expected loss from making decision y_m instead of the ex post optimal decision given the message m.

Now we calculate the loss of information and the loss of control under optimal limited authority. By Proposition 3.5, adjacent induced decisions are symmetric around $\frac{1}{2}$ and approximately $2b$ apart from each other ($y_i \approx \frac{1}{2} + \left(i - \frac{n+1}{2}\right)2b$). Therefore, the loss of information can be approximated as

$$\sum_{i=1}^{n} \Pr\left((\theta_{i-1},\theta_i]\right) Var\left(\theta|\left(\theta_{i-1},\theta_i\right]\right) \approx Var\left(\theta|\left(\theta_{i-1},\theta_i\right]\right) \approx \frac{(2b)^2}{12} = \frac{1}{3}b^2,$$

and P's loss of control can be approximated as

$$\sum_{i=1}^{n} \Pr\left((\theta_{i-1},\theta_i]\right)\left(y_i - \mathbb{E}\left[\theta|\left(\theta_{i-1},\theta_i\right]\right]\right)^2 \approx \left(y_i - \frac{\theta_{i-1}+\theta_i}{2}\right)^2 \approx b^2$$

where the last equality follows from A's indifference conditions (3.4). Analogous calculations show that A has essentially no loss of control. Summing up the loss of information and the loss of control yields Proposition 3.6.

Next we compare P and A's expected payoffs for a small bias b under all organizational forms (see Table 1).[22]

[22] We believe that these results hold more generally with a caveat that each row of Table 1 is multiplied by some constant. In particular, we expect them to hold if P's and A's payoffs are given by arbitrary smooth loss functions $u^P(y,\theta) = -l^P(|y-\theta|)$, $u^A(y,\theta) = -l^A(|y-(\theta+b)|)$. Intuitively, as the bias goes to zero, the distance between any two subsequent decisions also goes to zero. Therefore, the loss functions can be approximated by quadratic functions, and the distribution function can be approximated by a piecewise uniform distribution.

	Persuasion	Informational Control	Optimal Delegation	Full Delegation	Limited Authority	Noisy Talk	Cheap Talk
U^P	0	$-\frac{1}{3}b^2$	$-b^2$	$-b^2$	$-\frac{4}{3}b^2$	$-\frac{1}{3}b$	$-\frac{1}{3}b$
U^A	$-b^2$	$-\frac{4}{3}b^2$	$-\frac{8}{3}b^3$	0	$-\frac{1}{3}b^2$	$-\frac{1}{3}b$	$-\frac{1}{3}b$

Table 1. P and A's expected payoffs under all organizational forms.

To understand these payoff comparisons, we decompose P's expected payoff into the loss of information and the loss of control. In terms of loss of control, in all organizational forms, either P or A has essentially no loss of control, and thus the other party has a loss of control equal to b^2. Under delegation and limited authority, P has commitment power and effectively delegates authority to A to improve information transmission, and her loss of control b^2 is simply due to A's bias. Next, we turn to the loss of information. There is essentially no loss of information under delegation and persuasion because the state is almost fully revealed. The loss of information is approximately $\frac{1}{3}b^2$ under informational control and limited authority because induced decisions are approximately $2b$ apart from each other.[23] Under cheap talk and noisy talk, however, the partition is coarse such that the distance between adjacent decisions grows at the approximate rate $4b$, leading to a much larger loss of information of approximately $\frac{1}{3}b$. Combining these two parts lead to the payoff comparisons in Table 1.

The above intuition suggests that P and A's expected payoffs are still given by Table 1 under A's optimal organizational forms, except for A's optimal delegation, which coincides with full delegation. It is also straightforward to characterize P and A's expected payoffs under combinations of different organizational forms. However, the first term in Taylor expansion of P and A's expected payoffs will depend on whether we are looking at P or A's optimal combination of organizational forms. For example, if both P and A have full-commitment power (a combination of persuasion and delegation), then they can eliminate the loss of information and split the loss of control arbitrarily such that P and A's expected payoffs on the Pareto frontier are given by $U^A = -(\alpha b)^2$ and $U^P = -((1-\alpha)b)^2$, where $\alpha \in [0,1]$.

[23] This connection between limited authority and informational control is due to limited commitment power of both A and P. Under limited authority, P makes the decision space discrete to relax her incentive conditions, whereas under informational control, P makes the state space discrete to relax A's incentive conditions.

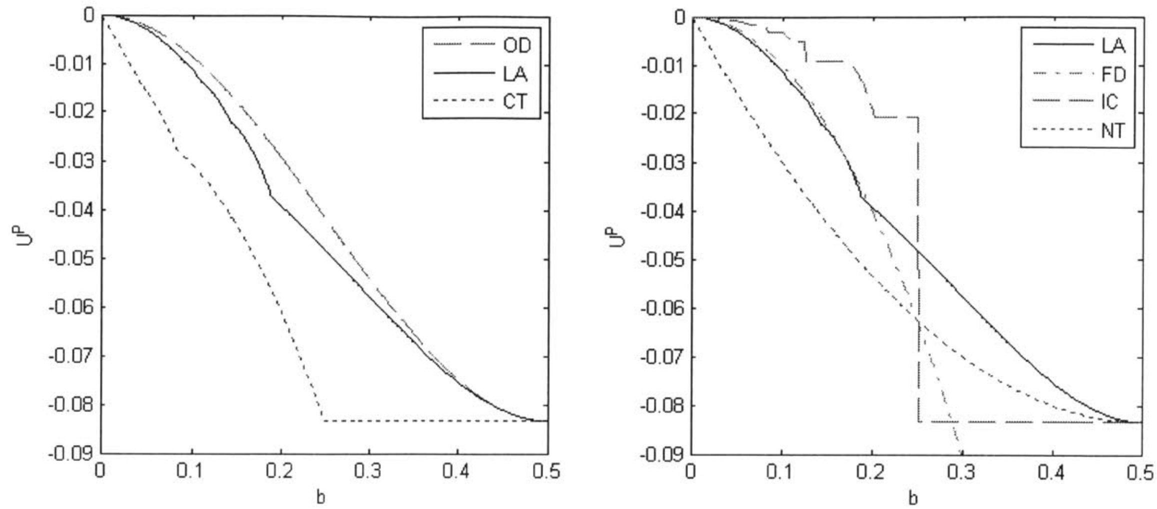

Figure 3-2: P's expected payoff under Optimal delegation (OD), Limited authority (LA), Cheap talk (CT), Full delegation (FD), Information control (IC) and Noisy talk (NT)

Figure 3-2 illustrates P's expected payoff under all organizational forms for all possible values of the bias $b \in \left(0, \frac{1}{2}\right)$. As we can see from the figure, for the whole range of A's bias, P's expected payoffs are considerably lower under cheap talk and noisy talk than under all other organizational forms. P's payoff is of the same order of the bias under informational control, delegation, and limited authority. In particular, P's payoff under limited authority is almost as high as P's payoff under optimal delegation despite the fact that only finite decisions can be induced under limited authority. Moreover, P receives a higher payoff under informational control than under limited authority when A's bias is small whereas the opposite is true when A's bias is large. Although these payoff comparisons alone do not indicate which organizational form should be chosen due to differences in environment, they do suggest that organizations can potentially benefit from credibly ruling out some decisions ex ante even if it cannot fully delegate the decision-making authority.

3.6 Concluding Remarks

Our model of limited authority aims to explore and understand the environment in which the principal has some degree of commitment power, but not all. The remainder of this section contains a discussion of how the optimal equilibrium may be affected by different assumptions about the communication process and contracting environment as well as some thoughts for further research.

Because only finitely many decisions can be induced in any optimal equilibrium under limited authority, one may wonder whether the principal benefits from a finite decision set per se, that is, when the principal's decision space is discrete. It can be shown that in the uniform-quadratic setup in Section 3.4, the equilibria in the CS model with a discrete set of decisions are similar to equilibria in the CS model with a continuous set of decisions, but with a modified smaller bias. Therefore, by discretizing the set of available decisions, the principal can effectively decrease the agent's bias and thereby improve communication.[24] In fact, under the optimal limited authority given in Proposition 3.5, the agent's effective bias disappears such that a uniform partition becomes feasible.

One possible critique of the limited authority model is that parties can renegotiate, after the agent's recommendation, to a decision not in the prespecified set if both prefer to do so. To address this issue, we can strengthen our solution concept and look for the optimal renegotiation-proof equilibrium. In our model the scope for renegotiation is limited for two reasons. First, because transfers between the parties are not allowed, both the principal and the agent must prefer the renegotiated decision to the optimal equilibrium decision. Second, in any PBE there is unresolved uncertainty about the state of the world after the agent's recommendation, and thus the principal does not know to which decision, if any, she should renegotiate. In the uniform-quadratic setup, we can show that a renegotiation-proof equilibrium exists, and has a maximal possible number of induced decisions that can be supported as an equilibrium.

Another critique is that if the principal can credibly restrict the set of decisions ex ante, then she may also be able to commit to transfers contingent on decisions. Following the literature on communication and delegation, we rule out such transfers in the limited authority model.[25] Note that with transfers, the principal can do at least as well as in the limited authority model by committing to no transfers for decisions in the prespecified set and very large transfers for decisions outside the set. In fact, with transfers, the principal can achieve the first best outcome, which maximizes the sum of the principal and agent's expected payoffs, if there are no monetary frictions. However, not all decision rules can be supported with transfers. For example, the principal's and the agent's optimal decision rules are not achievable. Further, if there are frictions, such as when

[24]This result resembles that of Alonso and Matouschek (2007) who show that the principal's commitment power reduces the agent's "effective" bias.

[25]Monetary transfers may be explicitly ruled out by law, or implicitly ruled out if the parties involved are very risk averse with respect to money. However, the assumption that there are no transfers is strong. Even though explicit transfers between parties may be ruled out, the parties can effectively "burn" money, which generally improves their welfare (see, for example, Austen-Smith and Banks, 2000).

the agent is protected by limited liability or if the principal can "burn" money, then there is a tradeoff between making efficient decisions and leaving a quasi rent to the agent. Due to this tradeoff, there is incomplete information revelation for high states of the world, even though full information revelation is feasible.[26]

In the current model, the principal never vetoes the agent's recommendation in equilibrium. A further topic of research is to extend our model to allow veto to happen in equilibrium. One way to model this is to imagine that there is some small, exogenous probability that the principal can observe the true state after hearing the agent's recommendation, and may consequently desire to change her decision (still within the prespecified decision set) given this information. In this case, it is without loss of generality to restrict to equilibria in which the principal follows the agent's recommendation when she does not learn the state and otherwise makes a choice independent of the recommendation. Thus, any equilibrium characterized in Proposition 3.1 remains feasible, and further, the principal can do better by adding any decision that is chosen with zero probability in equilibrium so long as her incentive conditions (when she does not learn the state) are unaffected. The interesting question is how the principal optimally modifies the decisions that are used with positive probabilities when she does not learn the state, in order to retain more decisions that she will use when she does. Answering this question can further our understanding of the principal's tradeoff between maintaining the flexibility of responding to new information and establishing the credibility of letting the agent best use his private information.

3.7 Appendix: Proofs

Proof of Proposition 3.1: We prove the proposition through a series of lemmas.

First, we establish that it is without loss of generality to restrict attention to PBE's in which all decisions in P's equilibrium choice Y are induced, each message from A's is a *recommendation* of some probability distribution over Y, and no recommendation is vetoed by P. This is a version of the revelation principle adapted to our setting. Fix any PBE and the subgame after P has made the equilibrium choice Y. We refer to any response by P to a message m from A as a *lottery*, and a particular choice from Y as a *degenerate lottery*. We say that two PBE's are *outcome equivalent* if

[26] This result is analogous to that of Krishna and Morgan (2008) who consider a communication game in which the principal can commit to transfers contingent on messages and the agent is protected by limited liability. Kartik (2009) obtains a somewhat similar result. He shows that in a model of communication with lying costs, there is pooling for high states of the world.

they both result in the same (random) mapping from states to decisions on the equilibrium path.

Lemma 3.4 *Consider a PBE with P's equilibrium choice Y. There exists an outcome-equivalent PBE with P's equilibrium choice $\tilde{Y} \subseteq Y$, where $\tilde{Y}$ is the union of the supports of all induced lotteries and for any induced lottery there is a unique y in its support chosen by A as a message.*

Proof: Fix any PBE and the subgame after P has chosen the equilibrium Y. Since $u^P(\cdot, \theta)$ is strictly concave, there are at most two decisions y and y' in Y that are optimal given the equilibrium belief of P conditional on any m. Thus, a non-degenerate lottery has exactly two decisions. Moreover, if y and y' in Y satisfying $y < y'$ are in the support of some lottery, then $(y, y') \cap Y = \emptyset$; otherwise, strict concavity of $u^P(\cdot, \theta)$ implies that the lottery would not be optimal for P. Finally, no two induced lotteries have the same support. Otherwise, if $y, y' \in Y$ with $y < y'$ are in the common support of two distinct lotteries induced after A chooses m and m' respectively, then one of them, say the lottery following m', first order stochastically dominates the other. Since $u_{y\beta} > 0$, P being indifferent between y and y' given the belief conditional on m implies that A strictly prefers y' to y given the same belief. Thus, there are states in which A is supposed to choose m but would find it profitable to deviate to m' to induce the lottery following m', a contradiction. By the same argument, if $y, y' \in Y$ with $y < y'$ are the support of some induced lottery, y' is not induced as a degenerate lottery.

Let $\tilde{Y}$ be the union of the supports of all induced lotteries following Y. We construct an outcome-equivalent PBE where P chooses $\tilde{Y}$ instead of Y on the equilibrium path and A's message space is restricted to P's choice of set of decisions on and off the equilibrium path. For any choice of P that is not $\tilde{Y}$, including Y, let the continuation in the new PBE be such that A chooses the lowest decision in the set chosen by P regardless of realized θ and P chooses a decision that is optimal in the set given her prior belief. It remains to specify the continuation equilibrium in the new PBE following $\tilde{Y}$ that is outcome-equivalent to the continuation equilibrium in the original PBE following Y. For each degenerate lottery $y \in Y$ induced in the continuation equilibrium following Y after A chooses some message m, let A choose y in the subgame following $\tilde{Y}$ and let P's belief be the same as in the original PBE conditional on m; and for each non-degenerate lottery where P randomizes between y and y' with $y < y'$ following Y after A chooses some message m', let A choose y' in the subgame following $\tilde{Y}$ and let P's belief be the same as in the original PBE conditional on m'. All equilibrium conditions are satisfied in the new PBE following $\tilde{Y}$ as they are a subset of the equilibrium conditions in the original PBE following Y. Further, by construction $\tilde{Y}$

is part of the new PBE, because Y is part of the original PBE, and the equilibrium payoff for P is greater than or equal to the payoff from choosing y^P. **QED**

Second, we show that in any PBE the number of induced lotteries is finite. Denote $\{y, y'; w\}$ as a lottery induced in some continuation game after P has chosen Y, with P choosing y with probability $w \in (0,1)$ and $y' \geq y$ with probability $1-w$. We adopt the convention that a degenerate lottery is represented by $y' = y$. The proof of Lemma 3.4 implies that any two distinct lotteries $\{y_1, y_1'; w_1\}$ and $\{y_2, y_2'; w_2\}$ can be ordered, with the first *lower* than the latter, such that $y_1 \leq y_1' \leq y_2 \leq y_2'$, with at least one strict inequality and $y_1' = y_2$ implying that $y_2' > y_2$.

Lemma 3.5 *The number of decisions induced in any PBE is finite.*

Proof: Fix some PBE and the subgame after P has chosen the equilibrium Y. Let $\{y_i, y_i'; w_i\}$, $i = 1, 2, 3$, be three distinct induced lotteries, in increasing order. Since both $\{y_2, y_2'; w_2\}$ and $\{y_3, y_3'; w_3\}$ are induced, there is a state $\hat{\theta}$ such that

$$w_2 u^A(y_2, \hat{\theta}) + (1-w_2) u^A(y_2', \hat{\theta}) = w_3 u^A(y_3, \hat{\theta}) + (1-w_3) u^A(y_3', \hat{\theta}).$$

Since $u^A(\cdot, \hat{\theta})$ is strictly concave, $y^A(\hat{\theta}) \in (y_2, y_3')$. Further, since $u^A_{y\theta} > 0$, the lottery $\{y_2, y_2'; w_2\}$ is not induced for any $\theta > \hat{\theta}$, as

$$\begin{aligned}
&w_3(u^A(y_3, \theta) - u^A(y_3, \hat{\theta})) + (1-w_3)(u^A(y_3', \theta) - u^A(y_3', \hat{\theta})) \\
&\geq u^A(y_3, \theta) - u^A(y_3, \hat{\theta}) \\
&\geq u^A(y_2', \theta) - u^A(y_2', \hat{\theta}) \\
&\geq w_2(u^A(y_2, \theta) - u^A(y_2, \hat{\theta})) + (1-w_2)(u^A(y_2', \theta) - u^A(y_2', \hat{\theta})),
\end{aligned}$$

with at least one inequality being strict. This implies that $\{y_2, y_2'; w_2\}$ can only be induced if the state θ is smaller than $\hat{\theta}$. As a result, we have $y^P(\hat{\theta}) > y_1$; otherwise, since $u^P_{y\theta} > 0$, a similar argument as above would imply that P prefers the lottery $\{y_1, y_1'; w_1\}$ to $\{y_2, y_2'; w_2\}$ for all $\theta < \hat{\theta}$ but then $\{y_2, y_2'; w_2\}$ would never be induced. It then follows that $y_1 < y^P(\hat{\theta}) < y^A(\hat{\theta}) < y_3'$. Since $y^P(\theta) < y^A(\theta)$ for all $\theta \in [0,1]$ and are both continuous, there exists $\varepsilon > 0$ such that $y^A(\theta) - y^P(\theta) \geq \varepsilon$ for all $\theta \in [0,1]$. There can be at most one induced decision greater than $y^P(1)$ and one lower than $y^P(0)$. The lemma then follows immediately. **QED**

By the first two lemmas, for any PBE, it is without loss of generality to assume that the equilibrium Y has a finite number of decisions, and each decision $y \in Y$ is induced either in a degenerate lottery or in a lottery with another decision $y' \in Y$. Denote the induced lotteries as $\{y_i, y_i'; w_i\}$, $i = 1, \ldots, n$, in increasing order. Since $u^A_{y\theta} > 0$, there is a partition $\{\theta_i\}_{i=0}^n$ of the state space $[0,1]$, with $\theta_0 = 0$ and $\theta_n = 1$, such that each $\{y_i, y_i'; w_i\}$, $i = 1, \ldots, n$, is induced in state $\theta \in (\theta_{i-1}, \theta_i]$. The necessary equilibrium conditions are A's indifference conditions: for each partition threshold θ_i, $i = 1, \ldots, n-1$,

$$w_i u^A(y_i, \theta_i) + (1 - w_i) u^A(y_i', \theta_i) = w_{i+1} u^A(y_{i+1}, \theta_i) + (1 - w_{i+1}) u^A(y_{i+1}', \theta_i); \tag{3.15}$$

and P's incentive condition for each lottery $\{y_i, y_i'; w_i\}$, $i = 1, \ldots, n$,

$$\int_{\theta_{i-1}}^{\theta_i} (w_i u^P(y_i, \theta) + (1 - w_i) u^P(y_i', \theta)) f(\theta) d\theta \geq \int_{\theta_{i-1}}^{\theta_i} u^P(\widetilde{y}, \theta) f(\theta) d\theta \tag{3.16}$$

for $\widetilde{y} = y_j, y_j'$ and all $j = 1, \ldots, n$. If in addition,

$$\sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} (w_i u^P(y_i, \theta) + (1 - w_i) u^P(y_i', \theta)) f(\theta) d\theta \geq \int_0^1 u^P(y^P, \theta) f(\theta) d\theta$$

so that P's expected payoff is greater than that from making an uninformed decision, then the above necessary conditions are also sufficient for PBE.

Third, we simplify the incentive conditions for P.

Lemma 3.6 *In any PBE, P's incentive conditions (3.16) must all be slack except for $\widetilde{y} = y_{i-1}', y_i, y_i', y_{i+1}$. Further, if $y_i = y_i'$ for all i, then P's incentive conditions except for (3.3) are all slack.*

Proof: We first argue that adjacent incentive conditions are sufficient for all incentive conditions. Consider all P's incentive conditions for $\{y_i, y_i'; w_i\}$. Since P prefers $\{y_i, y_i'; w_i\}$ to y_{i-1}' conditional on $(\theta_{i-1}, \theta_i]$, the most preferred decision conditional on the interval is higher than y_{i-1}'. By the strict concavity of u^P, P strictly prefers y_{i-1}', and hence $\{y_i, y_i'; w_i\}$, to all decisions lower than y_{i-1}' conditional on $(\theta_{i-1}, \theta_i]$. By the same argument, P strictly prefers $\{y_i, y_i'; w_i\}$ to all decisions higher than y_{i+1} conditional on $(\theta_{i-1}, \theta_i]$.

Next, we argue that the adjacent upward incentive condition is satisfied if all induced lotteries are degenerate. To see this, note that in any partition equilibrium A's indifference conditions (3.2) hold. Since $u_{y\beta} > 0$, A being indifferent between y_i and y_{i+1} in state θ_i implies that P strictly prefers y_i to y_{i+1} in the same state. By $u^P_{y\theta} > 0$, P then prefers y_i to y_{i+1} for all $\theta < \theta_i$, and in particular, for any $\theta \in (\theta_{i-1}, \theta_i]$. **QED**

Fourth, we show that if an optimal PBE exists, then on the equilibrium path, P never randomizes over the set of decisions.

Lemma 3.7 *For each PBE in which lotteries are induced, there exists another PBE in which only degenerate lotteries are induced and P obtains a higher expected payoff.*

Proof: Fix some PBE with induced lotteries $\{y_i, y'_i; w_i\}$, $i = 1, \ldots, n$, in increasing order. We prove this lemma in two steps. First, we show that there is another PBE in which P's equilibrium choice of Y contains only y'_i, $i = 1, ..., n$, and each decision y'_i is induced in a degenerate lottery. Each new threshold $\widehat{\theta}_i$ is given by (3.15) where w_i and w_{i+1} are set to 0. Since $y_i \leq y'_i \leq y_{i+1} \leq y'_{i+1}$, the concavity of $u^A(\cdot, \widehat{\theta}_i)$ and A's indifference condition at $\widehat{\theta}_i$ between y'_i and y'_{i+1} imply that $u^A(y_i, \widehat{\theta}_i) \leq u^A(y'_i, \widehat{\theta}_i)$ and $u^A(y_{i+1}, \widehat{\theta}_i) \geq u^A(y'_{i+1}, \widehat{\theta}_i)$. Then, since $u^A_{y\theta} > 0$, using the implicit function theorem applied to (3.15) gives that the solution to (3.15) decreases in w_i and w_{i+1}, which implies that each new threshold $\widehat{\theta}_i$ is higher than the original threshold θ_i. The distribution function of the state θ conditional on $[\widehat{\theta}_{i-1}, \widehat{\theta}_i]$, given by $(F(\theta) - F(\widehat{\theta}_{i-1}))/(F(\widehat{\theta}_i) - F(\widehat{\theta}_{i-1}))$, first-order stochastically dominates the distribution function of the state θ conditional on $[\theta_{i-1}, \theta_i]$, because it is decreasing in $\widehat{\theta}_{i-1}$ and $\widehat{\theta}_i$. Since the difference $u^P(y'_i, \theta) - u^P(y'_{i-1}, \theta)$ is increasing in θ by the assumption of $u^P_{y\theta} > 0$, P prefers y'_i to y'_{i-1} conditional on $[\widehat{\theta}_{i-1}, \widehat{\theta}_i]$, because in the original PBE, P prefers y'_i to y'_{i-1} conditional on $[\theta_{i-1}, \theta_i]$. Since the downward incentive conditions are satisfied, Lemma 3.6 implies that we indeed constructed a new PBE.

Second, we show that P obtains a higher expected payoff in the new PBE than in the original PBE by transforming the original PBE into the new PBE in such a way that P's expected payoff continuously increases. We continuously decrease each lottery weight $\tilde{w}_i$ from w_i to 0, one lottery at a time starting at $i = 1$ and ending at $i = n$, while increasing thresholds $\tilde{\theta}_i$ and $\tilde{\theta}_{i-1}$ to always satisfy A's indifference conditions (3.15). Note that all other partition thresholds are unchanged when we continuously decrease $\tilde{w}_i$ alone. The partial derivative of P's expected payoff with respect to $\tilde{w}_i$ is given by

$$\int_{\tilde{\theta}_{i-1}}^{\tilde{\theta}_i} (u^P(y_i, \theta) - u^P(y'_i, \theta)) f(\theta) d\theta,$$

which is negative because P prefers y'_i to y_i conditional on $\left(\tilde{\theta}_{i-1}, \tilde{\theta}_i\right]$ (recall that P is indifferent between y'_i and y_i conditional on $(\theta_{i-1}, \theta_i]$, so the argument from the previous paragraph applies).

Thus, as we decrease $\tilde{w}_i$ continuously, the direct effect on P's expected payoff is positive. The partial derivative of P's expected payoff with respect to $\tilde{\theta}_i$ is equal to $f\left(\tilde{\theta}_i\right)$ multiplied by

$$\begin{aligned}
&\tilde{w}_i u^P(y_i,\tilde{\theta}_i) + (1-\tilde{w}_i)u^P(y_i',\tilde{\theta}_i) - (w_{i+1}u^P(y_{i+1},\tilde{\theta}_i) + (1-w_{i+1})u^P(y_{i+1}',\tilde{\theta}_i)) \\
&= \tilde{w}_i(w_{i+1}(u^P(y_i,\tilde{\theta}_i) - u^P(y_{i+1},\tilde{\theta}_i)) + (1-w_{i+1})(u^P(y_i,\tilde{\theta}_i) - u^P(y_{i+1}',\tilde{\theta}_i))) \\
&\quad + (1-\tilde{w}_i)(w_{i+1}(u^P(y_i',\tilde{\theta}_i) - u^P(y_{i+1},\tilde{\theta}_i)) + (1-w_{i+1})(u^P(y_i',\tilde{\theta}_i) - u^P(y_{i+1}',\tilde{\theta}_i))) \\
&> \tilde{w}_i(w_{i+1}(u^A(y_i,\tilde{\theta}_i) - u^A(y_{i+1},\tilde{\theta}_i)) + (1-w_{i+1})(u^A(y_i,\tilde{\theta}_i) - u^A(y_{i+1}',\tilde{\theta}_i))) \\
&\quad + (1-\tilde{w}_i)(w_{i+1}(u^A(y_i',\tilde{\theta}_i) - u^A(y_{i+1},\tilde{\theta}_i)) + (1-w_{i+1})(u^A(y_i',\tilde{\theta}_i) - u^A(y_{i+1}',\tilde{\theta}_i))) \\
&= \tilde{w}_i u^A(y_i,\tilde{\theta}_i) + (1-\tilde{w}_i)u^A(y_i',\tilde{\theta}_i) - (w_{i+1}u^A(y_{i+1},\tilde{\theta}_i) + (1-w_{i+1})u^A(y_{i+1}',\tilde{\theta}_i)) \\
&= 0,
\end{aligned}$$

where the inequality follows from $u_{y\beta} > 0$, and the last equality follows from A's indifference condition between $\{y_i, y_i'; \tilde{w}_i\}$ and $\{y_{i+1}, y_{i+1}'; w_{i+1}\}$ in state $\tilde{\theta}_i$. Because we replace one lottery at a time starting at $i = 1$, the lottery $\{y_{i-1}, y_{i-1}'; w_{i-1}\}$ must be degenerate. By construction $\widetilde{w}_{i-1} = 0$ when we decrease $\widetilde{w}_i$, so analagously the partial derivative of P's expected payoff with respect to $\tilde{\theta}_{i-1}$ is equal to $f\left(\tilde{\theta}_{i-1}\right)$ multiplied by

$$\begin{aligned}
&u^P(y_{i-1}',\tilde{\theta}_{i-1}) - (\tilde{w}_i u^P(y_i,\tilde{\theta}_{i-1}) + (1-\tilde{w}_i)u^P(y_i',\tilde{\theta}_{i-1})) \\
&= \tilde{w}_i(u^P(y_{i-1}',\tilde{\theta}_{i-1}) - u^P(y_i,\tilde{\theta}_{i-1})) + (1-\tilde{w}_i)(u^P(y_{i-1}',\tilde{\theta}_{i-1}) - u^P(y_i',\tilde{\theta}_{i-1})) \\
&> \tilde{w}_i(u^A(y_{i-1}',\tilde{\theta}_{i-1}) - u^A(y_i,\tilde{\theta}_{i-1})) + (1-\tilde{w}_i)(u^A(y_{i-1}',\tilde{\theta}_{i-1}) - u^A(y_i',\tilde{\theta}_{i-1})) \\
&= u^A(y_{i-1}',\tilde{\theta}_{i-1}) - (\tilde{w}_i u^A(y_i,\tilde{\theta}_{i-1}) + (1-\tilde{w}_i)u^A(y_i',\tilde{\theta}_{i-1})) \\
&= 0.
\end{aligned}$$

Thus, as we decrease $\tilde{w}_i$ continuously, the indirect effects of increased $\tilde{\theta}_{i-1}$ and $\tilde{\theta}_i$ on P's expected payoff are also positive. Finally, if we suppose that at least one induced lottery in the original PBE is non-degenerate, then the direct effect will be strictly positive, which implies that P's expected payoff is strictly higher in the new PBE. **QED**

Fifth and last, we show that an optimal PBE exists. Combining the above lemmas, we have already established that an optimal PBE, if one exists, is a solution to the constrained maximization problem where the objective is P's expected payoff and the feasible choices are all partition equilibria with a finite number of elements.

Lemma 3.8 *An optimal PBE exists.*

Proof: Let us consider a relaxed problem in which strict inequalities of the partition condition (3.1) are replaced with weak inequalities. By Lemma 3.5, the number of induced decisions n is uniformly bounded. Thus, the relaxed problem is a constrained maximization problem with finitely many variables. There exists $\overline{y}$ such that we can impose $|y_i| \leq \overline{y}$ for all $i = 1, \ldots, n$ without affecting the maximization problem.[27] These constraints, $|y_i| \leq \overline{y}$, together with a finite number of constraints (3.2) and (3.3) determine the compact set for variables $\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n$ over which the continuous function $\sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta$ is maximized. Clearly, there exists a solution to this relaxed problem. Finally, we need to show that the value of the relaxed problem is achievable with strict inequalities (3.1), which will prove the existence of an optimal PBE. If some of θ_i or y_i coincide, we can take the maximal subset $\{\theta_i'\}_{i=0}^{n'} \subset \{\theta_i\}_{i=0}^n$ and a corresponding subset of induced decisions $\{y_i'\}_{i=1}^{n'} \subset \{y_i\}_{i=1}^n$ such that all θ_i' and y_i' are distinct. These $\{\theta_i'\}_{i=0}^{n'}$ and $\{y_i'\}_{i=1}^{n'}$ will satisfy (3.2)-(3.3) and strict inequalities of the partition condition (3.1). Moreover, this modification does not change P's expected payoff. **QED**

This concludes the proof of Proposition 3.1. **QED**

Proof of Proposition 3.2: Consider a CS equilibrium $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ with $n \geq 2$. We prove that for any sufficiently small δ, there exists a PBE with P's equilibrium choice $\left\{y_i^{\delta,j}\right\}_{i=1}^n \equiv \{y_1, ..., y_j + \delta, ..., y_n\}$, and the corresponding partition $\left\{\theta_i^{\delta,j}\right\}_{i=0}^n \equiv \{\theta_0, ..., \theta_{j-1}(\delta), \theta_j(\delta), ..., \theta_n\}$. Moreover, we prove that P's expected payoff in this PBE is strictly higher than in the CS equilibrium. By the implicit function theorem applied to A's indifference condition (3.2), $\theta_{j-1}(\delta)$ and $\theta_j(\delta)$ are continuous functions in a neighborhood of $\delta = 0$ with

$$\left.\frac{d\theta_{j-1}(\delta)}{d\delta}\right|_{\delta=0} = \frac{-u_y^A(y_j, \theta_{j-1})}{u_\theta^A(y_j, \theta_{j-1}) - u_\theta^A(y_{j-1}, \theta_{j-1})} \text{ for } j \neq 1,$$

$$\left.\frac{d\theta_j(\delta)}{d\delta}\right|_{\delta=0} = \frac{u_y^A(y_j, \theta_j)}{u_\theta^A(y_{j+1}, \theta_j) - u_\theta^A(y_j, \theta_j)} \text{ for } j \neq n.$$

For $j = 1$ and $j = n$ we have $\left.\frac{d\theta_0(\delta)}{d\delta}\right|_{\delta=0} = \left.\frac{d\theta_n(\delta)}{d\delta}\right|_{\delta=0} = 0$ because $\theta_0(\delta) = 1 - \theta_n(\delta) = 0$.

[27] There can be at most one induced decision above $y(1)$ and one induced decision below $y(0)$. Moreover, there is at least one induced decision in $[y^P(0), y^P(1)]$. Let us define $g_1(y_2, \theta_1)$ as y_1 that solves $u^A(y_1, \theta_1) = u^A(y_2, \theta_1)$ and $g_n(y_{n-1}, \theta_{n-1})$ as y_n that solves $u^A(y_{n-1}, \theta_{n-1}) = u^A(y_n, \theta_{n-1})$. The functions g_1 and g_n are decreasing in the first argument and increasing in the second argument which implies that $y_1 \geq g_1(y^P(1), 0)$ and $y_n \leq g_n(y^P(0), 1)$. Therefore, $|y_i| \leq \max\{|g_1(y^P(1), 0)|, |g_n(y^P(0), 1)|\} \equiv \overline{y}$ for all i.

In the CS equilibrium, $\int_{\theta_{i-1}}^{\theta_i} u^P(y_i,\theta)f(\theta)d\theta > \int_{\theta_{i-1}}^{\theta_i} u^P(y_{i-1},\theta)f(\theta)d\theta$ for all i because $y_i = \arg\max_{y\in\mathbb{R}} \int_{\theta_{i-1}}^{\theta_i} u^P(y,\theta)f(\theta)d\theta$. Therefore, incentive conditions (3.3), $\int_{\theta_{i-1}^{\delta,j}}^{\theta_i^{\delta,j}} u^P(y_i^{\delta,j},\theta)f(\theta)d\theta > \int_{\theta_{i-1}^{\delta,j}}^{\theta_i^{\delta,j}} u^P(y_{i-1}^{\delta,j},\theta)f(\theta)d\theta$, hold for all i because functions $u^P(y,\theta)$, $\theta_{j-1}(\delta)$, $\theta_j(\delta)$ are continuous, and δ is sufficiently small.

The derivative of P's expected payoff with respect to δ at $\delta = 0$ is given by

$$\left(u^P(y_{j-1},\theta_{j-1}) - u^P(y_j,\theta_{j-1})\right) f(\theta_{j-1}) \left.\frac{d\theta_{j-1}(\delta)}{d\delta}\right|_{\delta=0} +$$
$$\left(u^P(y_j,\theta_j) - u^P(y_{j+1},\theta_j)\right) f(\theta_j) \left.\frac{d\theta_j(\delta)}{d\delta}\right|_{\delta=0} + \int_{\theta_{j-1}}^{\theta_j} u_y^P(y_j,\theta)f(\theta)d\theta$$

The last term in the above expression is 0 because $y_j = \arg\max_{y\in\mathbb{R}} \int_{\theta_{j-1}}^{\theta_j} u^P(y,\theta)f(\theta)d\theta$. The second term is positive for $j \neq n$ because $u^P(y_j,\theta_j) - u^P(y_{j+1},\theta_j) > 0$ holds by (3.2) and $u_{y\beta} > 0$; and $\left.\frac{d\theta_j(\delta)}{d\delta}\right|_{\delta=0} > 0$ holds by $u_y^A(y_j,\theta_j) > 0$ and $u_\theta^A(y_{j+1},\theta_j) - u_\theta^A(y_j,\theta_j) > 0$. More specifically, $u_y^A(y_j,\theta_j) > 0$ holds by (3.2) and $u_{yy}^A > 0$, whereas $u_\theta^A(y_{j+1},\theta_j) - u_\theta^A(y_j,\theta_j) > 0$ holds by $u_{y\theta}^A > 0$. Analogously, the first term is positive for $j \neq 1$. To sum up, the above expression is positive for all j. **QED**

Proof of Proposition 3.3: For the "only if" part, recall from Proposition 3.1 that any optimal PBE is a partition equilibrium. The claim then follows immediately, because any partition equilibrium can be implemented by delegation under full commitment.

For the "if" part, we use a result obtained by Alonso and Matouschek (2008). They show that if delegation is valuable then there exists $\theta^* \in (0,1)$ such that $S(\theta^*) < 0 < T(\theta^*)$, where $T(\theta^*) \equiv \int_0^{\theta^*} \left(y^A(\theta^*) - y^P(\theta)\right) f(\theta)\, d\theta$ and $S(\theta^*) \equiv \int_{\theta^*}^1 \left(y^A(\theta^*) - y^P(\theta)\right) f(\theta)\, d\theta$. The decision rule $\widetilde{y}(\theta)$ given by

$$\widetilde{y}(\theta) = \begin{cases} y^A(\theta^*) + T(\theta^*) - S(\theta^*) & \text{if } \theta > \theta^*, \\ y^A(\theta^*) + S(\theta^*) - T(\theta^*) & \text{if } \theta \leq \theta^*, \end{cases}$$

satisfies A's indifference condition (3.2) at θ^* because A's payoff function is symmetric around $y^A(\theta)$. Moreover, the difference in P's expected payoff under $\widetilde{y}(\theta)$ and the uninformative decision y^P is equal to $-4T(\theta^*)S(\theta^*)$, and is positive. The decision rule $\widetilde{y}(\theta)$ satisfies P's incentive condition (3.3) that P prefers decision $y^A(\theta^*) + T(\theta^*) - S(\theta^*)$ to $y^A(\theta^*) + S(\theta^*) - T(\theta^*)$, as otherwise she would prefer decision $y^A(\theta^*) + S(\theta^*) - T(\theta^*)$ to $\widetilde{y}(\theta)$. Thus, $\widetilde{y}(\theta)$ can be supported as a PBE. **QED**

Proof of Lemma 3.1: Suppose that in the optimal PBE, $y_i \leq y_i^*$ for some $i = 2, \ldots, n$. We first show by contradiction that P's $(i+1)$-th incentive condition binds. Suppose not. Consider marginally increasing y_i, keeping all other decisions unchanged. We know from the proof of Proposition 3.2 that θ_{i-1} and θ_i both increase, with all other thresholds unaffected. By the concavity of $u^P(\cdot, \theta)$, P's i-th incentive condition is slack and hence unaffected because $y_{i-1} < y_i \leq y_i^*$. Similarly, her $(i-1)$-th incentive condition remains satisfied because θ_{i-1} increases as y_i increases. The proof of Proposition 3.2 has already established that P's expected payoff is increased when either θ_i or θ_{i-1} increases. Her expected payoff is further increased because y_i moves closer to her ex post optimal decision y_i^* on $(\theta_i, \theta_{i+1}]$. A contradiction.

The above result immediately implies that $y_n > y_n^*$. For each $i = 2, ..., n-2$, note that P's $(i+1)$-th incentive condition binding implies that $y_i < y_{i+1}^* < y_{i+1}$, so we can rewrite it as $y_{i+1} + y_i = \theta_{i+1} + \theta_i$. Using (3.4) for θ_i and θ_{i+1}, we then have

$$y_{i+2} - y_i = 4b.$$

However, if $y_i \leq y_i^*$, from A's indifference conditions (3.4) we would have

$$y_{i+1} - y_i \geq 4b + (y_i - y_{i-1}) > 4b,$$

which contradicts P's binding $(i+1)$-th incentive condition. This establishes the lemma for $i = 2, \ldots, n-2$.

Next, we show that $y_{n-1} > y_{n-1}^*$. Suppose not. Consider marginally increasing y_{n-1} and decreasing y_n in such a way that θ_{n-1} remains unchanged. Then P's n-th incentive condition is unaffected. However, this increases P's expected payoff because by assumption $y_{n-1} \leq y_{n-1}^*$, $y_n > y_n^*$, and because θ_{n-2} increases as a result of increasing y_{n-1}. A contradiction.

Finally, it can be verified using the proof of Proposition 3.5 that $y_1 > y_1^*$ at the optimal PBE. Note that this result is not needed for rewriting P's incentive conditions (3.5). **QED**

Proof of Proposition 3.4: Adding up condition (3.9) for $i = 2, \ldots, n-1$, we have

$$y_n + y_{n-1} - (y_1 + y_2) \geq 4b(n-2).$$

Also, using conditions (3.8) and (3.10), we have $2b(n-1) < 1$, or $n < 1/(2b) + 1$.

For the converse, let n be a positive integer strictly less than $1/(2b) + 1$. By definition of N,

$1/(2N) \le b < 1/(2(N-1))$. Note that $N \ge 2$ if and only $b < \frac{1}{2}$.

If $n = 1$, then there exists a babbling equilibrium with the induced decision $\overline{y}_1(1) = \frac{1}{2}$.

If $n = 2$, suppose that $b < \frac{1}{2}$ and consider the "full commitment" problem of choosing two decisions y_1 and y_2 with $0 \le y_1 \le y_2 \le 1$ that maximizes P's expected payoff

$$U(2) = -\int_0^{\theta_1} l(|y_1 - \theta|)\ d\theta - \int_{\theta_1}^1 l(|y_2 - \theta|)\ d\theta,$$

subject only to A's indifference condition $\theta_1 = \frac{1}{2}(y_1 + y_2) - b$. The first order conditions with respect to y_1 and y_2 are

$$\frac{\partial U(n)}{\partial y_1} = \frac{1}{2}(l(y_2 - \theta_1) + l(|y_1 - \theta_1|)) - l(y_1) = 0;$$
$$\frac{\partial U(n)}{\partial y_2} = -\frac{1}{2}(l(y_2 - \theta_1) + l(|y_1 - \theta_1|)) + l(1 - y_2) = 0.$$

The above conditions imply that $y_1 = 1 - y_2$. It is straightforward to verify that the second order condition is satisfied. The above first order conditions become identical, and we can rewrite it as

$$2l((1-\Delta)/2) = l(b + \Delta/2) + l(|b - \Delta/2|),$$

where $\Delta = y_2 - y_1$. By the convexity of l, the right hand side is increasing in Δ, so there is a unique $\Delta \in (0,1)$ satisfying the above condition. The solution to the full commitment problem is then given by $\overline{y}_1(2) = \frac{1}{2}(1-\Delta)$ and $\overline{y}_2(2) = \frac{1}{2}(1+\Delta)$, with $\theta_1 = \frac{1}{2} - b$. Note that $b < \frac{1}{2}$ implies that $\theta_1 > 0$. The incentive condition of P is satisfied at this solution, because $\overline{y}_1(2) + \overline{y}_2(2) = 1 < 1 + \theta_1$. We can thus take the solution to the full commitment problem to be a limited authority PBE. Since the solution has $\overline{y}_2(2) - \overline{y}_1(2) = \Delta > 0$, it gives P a strictly higher payoff than making the uninformed decision of $\frac{1}{2}$.

Finally, for any $n \ge 3$, consider the set of n decisions $\overline{Y}(n)$ given by $\overline{y}_i(n) = \frac{1}{2} + 2b\left(i - \frac{n+1}{2}\right)$ for each $i = 1, \ldots, n$. Then, by A's indifference condition, $\theta_i = \overline{y}_i(n)$ for each $i = 1, \ldots, n-1$. It is straightforward to verify that conditions (3.8), (3.9) and (3.10) are all satisfied. The expected payoff for P under this construction is given by

$$\overline{U}(n) = -2\int_0^{\overline{y}_1(n)} l(\overline{y}_1(n) - \theta)\ d\theta - (n-1)\int_{\overline{y}_1(n)}^{\overline{y}_2(n)} l(\overline{y}_2(n) - \theta)\ d\theta. \tag{3.17}$$

It is straightforward to show that $U(n) > U(n-2)$: the difference is given by

$$\overline{U}(n) - \overline{U}(n-2) = 2\int_0^{\overline{y}_1(n)} (l(\overline{y}_2(n) - \theta) - l(\overline{y}_1(n) - \theta))\, d\theta > 0,$$

where $\overline{y}_1(n)$ is the smallest decision under the construction for n, and $\overline{y}_2(n)$ the second smallest decision for n and the smallest for $n-2$ (that is, $\overline{y}_2(n) = \overline{y}_1(n-2)$).

The above argument immediately implies that the expected payoff to P under the above construction with n decisions is greater than making the uninformed decision of $\frac{1}{2}$ for all $n \geq 3$ and odd. To complete the proof of the proposition, we only need to show that the above construction $\overline{Y}(4)$ for $n = 4$ dominates making the uninformed decision of $\frac{1}{2}$ for P. (This step is necessary because the payoff formula (3.17) does not apply to the case of $n = 2$.) It is straightforward to show that

$$\begin{aligned}
&\int_0^1 l(|1/2 - \theta|)\, d\theta - \sum_{i=1}^{4} \int_{\theta_{i-1}}^{\theta_i} l(\overline{y}_i - \theta)\, d\theta \\
> &\left[\int_{\overline{y}_3}^{\overline{y}_3+b} l(\theta - 1/2)\, d\theta - \int_{\overline{y}_2}^{1/2} l(\overline{y}_3 - \theta)\, d\theta\right] + \left[\int_{1/2}^{\overline{y}_3} (l(\theta - 1/2) - l(\overline{y}_3 - \theta))\, d\theta\right] \\
&\quad + \left[\int_{\overline{y}_2}^{1/2} l(1/2 - \theta)\, d\theta - \int_{\overline{y}_3}^{\overline{y}_3+b} l(\theta - \overline{y}_3)\, d\theta\right] \\
= &\ 0,
\end{aligned}$$

where the first line follows because $\frac{1}{2}$ is a more extreme decision than the corresponding decisions $\overline{y}_1$, $\overline{y}_2$ and $\overline{y}_4$ outside the interval $[\overline{y}_2, \overline{y}_3 + b]$, and the second line follows because each term in the bracket is zero.[28] **QED**

Proof of Lemma 3.2: For $n = 1$, it is trivially true that $y_1^{FC}(1) = \frac{1}{2}$. For $n = 2$, $Y^{FC}(2)$ is derived in the proof of Proposition 3.4. Fix any $n \geq 3$. Arguments similar to the proof of Lemma 3.8 in Proposition 3.1 can show that $Y^{FC}(n)$ exists. We will guess and verify later that conditions (3.7), (3.8), and (3.11) are not binding at $Y^{FC}(n)$. Denote $\Delta_i = y_{i+1}^{FC} - y_i^{FC}$.

To get a contradiction, suppose that there is i such that $\Delta_i \neq \Delta_{i-1}$. The derivative of P's

[28] One integral that appears in $\overline{U}(4)$ is $\int_{\overline{y}_3}^{\overline{y}_4} l(\overline{y}_4 - \theta)\, d\theta$. It is equal to $\int_{\overline{y}_3}^{\overline{y}_4} l(\theta - \overline{y}_3)\, d\theta$ by a change of variables. The first part of the latter integral, from $\overline{y}_3$ to $\overline{y}_3 + b$, is the integral that appears in the last bracket.

expected payoff with respect to y_i is

$$\begin{aligned}\frac{\partial U(n)}{\partial y_i} &= \frac{1}{2}\left[l(y_{i+1}-\theta_i)-l(y_i-\theta_{i-1})\right]-\frac{1}{2}\left[l(|y_{i-1}-\theta_{i-1}|)-l(|y_i-\theta_i|)\right]\\ &= \frac{1}{2}l\left(\frac{\Delta_i}{2}+b\right)+\frac{1}{2}l\left(\left|\frac{\Delta_i}{2}-b\right|\right)-\frac{1}{2}l\left(\frac{\Delta_{i-1}}{2}+b\right)-\frac{1}{2}l\left(\left|\frac{\Delta_{i-1}}{2}-b\right|\right) \quad (3.18)\end{aligned}$$

where we used A's indifference conditions. Since l is convex, $\frac{\partial U(n)}{\partial y_i}$ has the same sign as $\Delta_i - \Delta_{i-1}$ regardless of whether $\Delta \geq 2b$ or $\Delta < 2b$. Thus, P's expected payoff can be increased by changing y_i to decrease $|\Delta_i - \Delta_{i-1}|$. A contradiction.

Thus, the optimal decisions satisfy $y_i^{FC} - y_{i-1}^{FC} = \Delta > 0$ for all $i = 2, \ldots, n$, so the optimum is interior. From A's indifference conditions, we have $\theta_i - \theta_{i-1} = \Delta$ for all $i = 2, \ldots, n-1$. Since the state is uniformly distributed, we can rewrite P's expected payoff as

$$U(n) = -\int_0^{\theta_1} l(|y_1-\theta|)d\theta - (n-2)\int_{\theta_1}^{\theta_2} l(|y_2-\theta|)d\theta - \int_{\theta_{n-1}}^{1} l(|y_n-\theta|)d\theta. \quad (3.19)$$

To find Δ, we differentiate (3.19) with respect to y_1 and y_n. From the two first order conditions we immediately have $l(y_1) = l(1-y_n)$, and thus $y_1 = 1 - y_n = (1-(n-1)\Delta)/2$. The two conditions then become identical, and are given by

$$\frac{\partial U(n)}{\partial y_1} = \frac{1}{2}\left(l(b+\Delta/2)+l(|b-\Delta/2|)\right) - l((1-(n-1)\Delta)/2) = 0. \quad (3.20)$$

We claim that there exists a unique $\Delta \in (0, 1/(n-1))$ that solves (3.20). Since l is convex, $\partial U(n)/\partial y_1$ is strictly increasing in Δ regardless of whether $\Delta \geq 2b$ or $\Delta < 2b$, so there can be at most one value of Δ that solves (3.20). At $\Delta = 0$, we have $\partial U(n)/\partial y_1 < 0$ because by assumption $b < \frac{1}{2}$; and at $\Delta = 1/(n-1)$, we have $\partial U(n)/\partial y_1 > 0$. Thus, a unique $\Delta \in (0, 1/(n-1))$ exists that solves (3.20). Condition (3.20) is a necessary condition for Δ to be optimal. Since there exists a unique solution Δ, (3.20) is also sufficient.

To complete the derivation of $Y^{FC}(n)$, we verify that the dropped constraints are satisfied. Condition (3.7) is satisfied because $\Delta > 0$. Condition (3.8) is equivalent to $y_1 > b - \frac{1}{2}\Delta$. This is satisfied if $\Delta \geq 2b$ since $\Delta < 1/(n-1)$ implies that $y_1 > 0$; it also holds if $\Delta < 2b$, because in that case it is implied by (3.20). Finally, condition (3.11) is satisfied because given $y_n = 1 - y_1$ it is implied by (3.7). **QED**

Proof of Lemma 3.3: The lemma follows immediately from the three claims below.

Claim 3.1 *P's incentive conditions (3.9) bind at* $Y^{LC}(n)$ *for* $b \in \left[b^{FC}(n), b^{LC}(n)\right)$.

Proof: To get a contradiction, without loss of generality suppose that there exists i, $i = 2, ..., n-2$, such that $y_{i+2} - y_i = 4b$ and $y_{i+1} - y_{i-1} > 4b$ at $Y^{LC}(n)$. Denote $\Delta_i = y_{i+1} - y_i$. Below we will change one decision y_k in such a way that all conditions (3.7)-(3.10) are still satisfied and $|\Delta_k - \Delta_{k-1}|$ is decreased. Condition (3.18) then implies that P's expected payoff increases with this change, leading to a contradiction. If $\Delta_i \geq \Delta_{i-1}$ (and thus $\Delta_i > 2b > \Delta_{i+1} = 4b - \Delta_i$), then decrease y_{i+1} slightly. If $\Delta_i > \Delta_{i-1}$, then there are two cases. If $i - 1 = 1$ or $y_i - y_{i-2} > 4b$, then decrease y_i slightly, otherwise ($y_i - y_{i-2} = 4b$), increase y_{i-1} slightly. **QED**

Claim 3.2 *For any* $n \geq 3$ *and odd, and* $b \in \left(b^{FC}(n), b^{LC}(n)\right)$, $Y^{LC}(n)$ *is given by* $y_i^{LC} = \frac{1}{2} + 2b\left(i - \frac{n+1}{2}\right)$ *for all* $i = 1, ..., n$.

Proof: By Claim 3.1, $y_{i+2} - y_i = 4b$ for all $i = 1, \ldots, n-2$. Then, $y_i = y_1 + 2b(i-1)$ for i odd, and $y_i = y_2 + 2b(i-2)$ for i even. Further, $\theta_i - \theta_{i-1} = 2b$ for all $i = 2, \ldots, n-1$. Using the assumption that the state is uniformly distributed, we can rewrite P's expected payoff (3.6) as

$$\begin{aligned} U(n) &= -\int_0^{\theta_1} l(|y_1 - \theta|)d\theta - \frac{n-1}{2}\int_{\theta_1}^{\theta_2} l(|y_2 - \theta|)d\theta \\ &\quad -\frac{n-3}{2}\int_{\theta_1}^{\theta_2} l(|y_1 + 2b - \theta|)d\theta - \int_{\theta_{n-1}}^{1} l(|y_n - \theta|)d\theta. \end{aligned} \tag{3.21}$$

The first order conditions with respect to y_1 and y_2 are

$$\begin{aligned} \frac{\partial U(n)}{\partial y_1} &= -l(y_1) + l(1 - y_n) - \frac{n-1}{4}[l(3b - \Delta_1/2) - l(b + \Delta_1/2)] = 0; \\ \frac{\partial U(n)}{\partial y_2} &= \frac{n-1}{4}[l(3b - \Delta_1/2) - l(b + \Delta_1/2)] = 0 \end{aligned}$$

where $\Delta_1 = y_2 - y_1$. It follows immediately that $y_1 = 1 - y_n$ and $\Delta_1 = 2b$. Furthermore, it is straightforward to verify that the second order condition with respect to y_1 and y_2 are satisfied at $y_1 = 1 - y_n$ and $\Delta_1 = 2b$. Finally, (3.7) is satisfied because $\Delta_1 \in (0, 4b)$, and (3.8) and (3.10) are equivalent to $2b(n-1) < 1$, and thus are satisfied because $b < b^{LC}(n)$. **QED**

Claim 3.3 *For any* $n \geq 2$ *and even, and* $b \in \left(b^{FC}(n), b^{LC}(n)\right)$, $Y^{LC}(n)$ *is given by* $y_i^{LC} = \frac{1-\Delta_1}{2} + 2b\left(i - \frac{n}{2}\right)$ *for odd* i, *and* $y_i = \frac{1+\Delta_1}{2} + 2b\left(i - \frac{n+2}{2}\right)$ *for even* i, *where* $\Delta_1 < 2b$ *is uniquely determined by (3.14).*

Proof: Similar to Claim 3.2, we can rewrite P's expected payoff (3.6) as:

$$\begin{aligned} U(n) &= -\int_0^{\theta_1} l(|y_1-\theta|)d\theta - \frac{n-2}{2}\int_{\theta_1}^{\theta_2} l(|y_2-\theta|)d\theta \\ &\quad -\frac{n-2}{2}\int_{\theta_1}^{\theta_2} l(|y_1+2b-\theta|)d\theta - \int_{\theta_{n-1}}^{1} l(|y_n-\theta|)d\theta. \end{aligned} \tag{3.22}$$

The first order conditions with respect to y_1 and y_2 are

$$\begin{aligned} -l(y_1) + \frac{1}{2}[l(b+\Delta_1/2) + l(|b-\Delta_1/2)] - \frac{n-2}{4}[l(3b-\Delta_1/2) - l(b+\Delta_1/2)] = 0; \\ l(1-y_n) - \frac{1}{2}[l(b+\Delta_1/2) + l(|b-\Delta_1/2)] + \frac{n-2}{4}[l(3b-\Delta_1/2) - l(b+\Delta_1/2)] = 0 \end{aligned}$$

where $\Delta_1 = y_2 - y_1$. It follows immediately that $y_1 = 1 - y_n$ and Δ_1 satisfies (3.14). Furthermore, we can easily verify that the second order condition with respect to y_1 and y_2 are satisfied. Finally, (3.7) is satisfied because $\Delta_1 \in (0, 4b)$, and (3.8) and (3.10) are equivalent to $2b(n-1) < 1$, and thus are satisfied because $b < b^{LC}(n)$.

To see that there is a unique $\Delta_1 \in (0, 2b)$ that satisfies (3.14), note that since $b > b^{FC}(n)$, the left-hand side of (3.14) is strictly smaller than the right-hand side at $\Delta_1 = 2b$. As Δ decreases, the left-hand side of (3.14) increases while the right-hand side decreases because l is convex. At $\Delta = 0$, the left-hand side of (3.14) is strictly greater than the right-hand side because $b < b^{LC}(n)$. It follows that there exists a unique $\Delta_1 \in (0, 2b)$ that satisfies condition (3.14). **QED**

Proof of Proposition 3.5: First we establish a series of claims.

Claim 3.4 *Suppose that $l(z) = z^2$. For each $n \geq 3$, $dU^{FC}(n-1)/db > dU^{LC}(n)/db$ for all $b \in (b^{FC}(n), b^{FC}(n-1))$, where $U^{FC}(n-1)$ and $U^{LC}(n)$ are P's expected payoff under $Y^{FC}(n-1)$ and under $Y^{LC}(n)$ respectively.*

Proof: Consider $Y^{FC}(n-1)$. For $b < b^{FC}(n-1)$, from condition (3.13) for $n-1$ we have Δ given in Lemma 3.2 is strictly greater than $2b$. From (3.19) for $n-1$, using the Envelope Theorem we have

$$\frac{dU^{FC}(n-1)}{db} = -(n-2)[l(\Delta/2+b) - l(\Delta/2-b)].$$

It is straightforward to see from the first order condition (3.20) that $y_1^{FC}(n)$ is decreasing in n for fixed b and increasing in b for fixed n. Since $y_1^{FC}(n) = \frac{1}{2}(1-2b(n-1))$ at $b = b^{FC}(n)$, we have

$y_1^{FC}(n-1) > \frac{1}{2}(1-2b(n-1))$ for all $b > b^{FC}(n)$. It then follows from the convexity of l that

$$\frac{dU^{FC}(n-1)}{db} > -(n-2)[l(2b+b/(n-2)) - l(b/(n-2))].$$

Using the assumption of $l(z) = z^2$, we immediately have

$$\frac{dU^{FC}(n-1)}{db} > -(n-1)l(2b).$$

Next, suppose that n is odd and consider $Y^{LC}(n)$. From (3.21), using the Envelope Theorem we have

$$\frac{dU^{LC}(n)}{db} = -2(n-1)[l(2b) - l(1/2-(n-1)b)]. \tag{3.23}$$

Since $b > b^{FC}(n)$, from condition (3.13) we have $2l(1/2-(n-1)b) < l(2b)$, and thus

$$\frac{dU^{LC}(n)}{db} < -(n-1)l(2b),$$

establishing the claim for the case of n odd.

Lastly, suppose that n is even and consider $Y^{LC}(n)$. From (3.22), using the Envelope Theorem we have

$$\frac{dU^{LC}(n)}{db} = -\frac{1}{2}(n-2)(n+1)l(3b-\Delta_1/2) + \frac{1}{2}n(n-3)l(b+\Delta_1/2) + (n-1)l(b-\Delta_1/2). \tag{3.24}$$

For fixed b, the above is clearly increasing in Δ_1. Evaluating the above at $\Delta_1 = 2b$, we then have

$$\frac{dU^{LC}(n)}{db} < -(n-1)l(2b),$$

establishing the claim for the case of n even. **QED**

Claim 3.5 *For any l that satisfies Assumption 3.2, and for each $n \geq 3$, $dU^{LC}(n)/db > dU^{LC}(n+1)/db$ for all $b \in (b^{FC}(n), b^{LC}(n+1))$.*

Proof: First, suppose that n is odd. Using the first order condition (3.14) for $n+1$ we can rewrite (3.24) for $n+1$ as

$$\frac{dU^{LC}(n+1)}{db} = -(n-1)[l(3b-\Delta_1/2) + l(b+\Delta_1/2)] - 2l(b+\Delta_1/2) + 2nl(y_1^{LC}(n+1)).$$

Since l is convex, we have

$$l(3b - \Delta_1/2) + l(b + \Delta_1/2) > 2l(2b).$$

Further, (3.14) implies that $l(b + \Delta_1/2) > l(y_1^{LC}(n + 1))$. Thus,

$$\frac{dU^{LC}(n+1)}{db} < -2(n-1)[l(2b) - l(y_1^{LC}(n+1))].$$

The lemma then follows from $y_1^{LC}(n + 1) < \frac{1}{2} - (n - 1)b$ and (3.23).

Second, suppose that n is even. Using the first order condition (3.14) we can rewrite (3.24) as

$$\frac{dU^{LC}(n)}{db} = -(n+1)[l(b+\Delta_1/2) + l(b-\Delta_1/2) - 2l(y_1^{LC}(n))] - (n-1)[l(b+\Delta_1/2) - l(b-\Delta_1/2)].$$

Since $\Delta_1 < 2b$, from (3.14) we have

$$l(b + \Delta_1/2) + l(b - \Delta_1/2) - 2l(y_1^{LC}(n)) < l(2b) - 2l(1/2 - (n-1)b).$$

Thus

$$\frac{dU^{LC}(n)}{db} > -(n+1)[l(2b) - 2l(1/2 - (n-1)b)] - (n-1)[l(b+\Delta_1/2) - l(b-\Delta_1/2)].$$

The lemma then follows from $\Delta_1 < 2b$ and (3.23) for $n + 1$. **QED**

Claim 3.6 *Suppose that $l(z) = z^2$. Then, for any $n \geq 3$, $U^{FC}(n-1) > U^{LC}(n)$ at $b = b^{FC}(n-1)$.*

Proof: From Lemma 3.2 and condition (3.13), at $b^{FC}(n - 1)$ all $n - 1$ decisions in $Y^{FC}(n - 1)$ are $2b^{FC}(n - 1)$ apart, that is, Δ given in Lemma 3.2 is equal to $2b^{FC}(n - 1)$. Further, from A's indifference conditions we have $\theta_i = y_i^{FC}(n - 1)$ for all $i = 1, \ldots, n - 2$. We distinguish two cases.

First, suppose that n is odd. By Lemma 3.3, all n decisions in $Y^{LC}(n)$ are also $2b^{FC}(n - 1)$ apart, with $\theta_i = y_i^{LC}(n)$ for all $i = 1, \ldots, n - 1$. Note that $y_1^{FC}(n - 1) - y_1^{LC}(n) = b^{FC}(n - 1)$. Using (3.19) and (3.21), we can show that the difference between P's expected payoff $U^{FC}(n - 1)$ under $Y^{FC}(n - 1)$ and $U^{LC}(n)$ under $Y^{LC}(n)$ is given by

$$U^{FC}(n-1) - U^{LC}(n) = \int_0^{2b^{FC}(n-1)} l(\theta)d\theta - 2\int_{y_1^{FC}(n-1)-b^{FC}(n-1)}^{y_1^{FC}(n-1)} l(\theta)d\theta.$$

Using the assumption of $l(z) = z^2$, we can explicitly compute $y_1^{FC}(n - 1)$ in terms of $b^{FC}(n - 1)$

and use it to show that the above is strictly positive.

Second, suppose that n is even. In this case, under $Y^{LC}(n)$ the thresholds θ_i remain evenly spaced, with $\theta_{i+1} - \theta_i = 2b^{FC}(n-1)$. Note that $y_1^{FC}(n-1) - y^{LC}(n) = \frac{1}{2}\Delta_1$ where Δ_1 as defined by condition (3.14). As in the case of odd n, using (3.19) and (3.22) we can show that the difference between P's expected payoff $U^{FC}(n-1)$ under $Y^{FC}(n-1)$ and $U^{LC}(n)$ under $Y^{LC}(n)$ is given by

$$\begin{aligned} U^{FC}(n-1) - U^{LC}(n) &= \int_{b^{FC}(n-1)-\Delta_1/2}^{b^{FC}(n-1)+\Delta_1/2} l(\theta)d\theta - 2\int_{y_1^{LC}(n)}^{y_1^{LC}(n)+\Delta_1/2} l(\theta)d\theta \\ &\quad +(n/2-1)\left[\int_{2b^{FC}(n-1)}^{3b^{FC}(n-1)-\Delta_1/2} l(\theta)d\theta - \int_{b^{FC}(n-1)+\Delta_1/2}^{2b^{FC}(n-1)} l(\theta)d\theta\right]. \end{aligned}$$

Using the assumption of $l(z) = z^2$, we can explicitly compute Δ_1 from equation (3.14), and use it to show that the above is strictly positive. **QED**

Now, observe that $Y^{LC}(n) = Y^{FC}(n)$ at $b = b^{FC}(n)$, and recall from Lemma 3.2 that $Y^{FC}(n) > Y^{FC}(n-1)$. Then, from Claim 3.4 and Claim 3.6, we have that for any $n \geq 3$, there exists $b(n, n-1)$ such that $U^{FC}(n-1) < U^{LC}(n)$ for $b \in (b^{FC}(n), b(n, n-1))$ and $U^{FC}(n-1) > U^{LC}(n)$ for $b \in (b(n, n-1), b^{FC}(n-1)]$. Next, since Claim 3.6 implies that $U^{LC}(n) = U^{FC}(n) > U^{LC}(n+1)$ at $b = b^{FC}(n)$, it follows from Claim 3.5 that $U^{LC}(n) > U^{LC}(n+1)$ for all $b \in [b^{FC}(n), b^{LC}(n+1))$. Finally, using the assumption of $l(z) = z^2$, we can easily show that for any $n \geq 3$, $\overline{U}(n) > U^{FC}(n-1)$ at $b = b^{LC}(n+1)$ where $\overline{U}(n)$ is P's expected payoff under $\overline{Y}(n) = \left\{\frac{1}{2} + 2b\left(i - \frac{n+1}{2}\right)\right\}_{i=1}^{n}$. Since $U^{LC}(n) \geq \overline{U}(n)$ by the definition of $Y^{LC}(n)$, from Claim 3.4 we have $b(n, n-1) > b^{LC}(n+1)$. This concludes the proof of Proposition 3.5. **QED**

Proof of Proposition 3.6: As it is clear from the proof of Proposition 3.5, P's expected payoff (3.6) can be bounded below and above by substituting $Y^{FC}(n)$ and $Y^{FC}(n+1)$, respectively, where n is the largest integer smaller than $\frac{1}{2b} - \sqrt{2} + 1$. Since $\Delta(n; b)$ and $\Delta(n+1; b)$ given by Lemma 3.2 are equal to $2b + O(b^2)$, P's expected payoff for $k = n, n+1$ is

$$\begin{aligned} U_n^P &= -\frac{1}{12}(k-1)\Delta^3 - (k-1)b^2\Delta - \frac{1}{12}(1-(k-1)\Delta)^3 \\ &= -\frac{(2b)^2}{12} - b^2 - 0 + o(b^2) = -\frac{4}{3}b^2 + o(b^2). \end{aligned}$$

Similarly, A's expected payoff is

$$U_n^A = -\frac{1}{12}(k-1)\Delta^3 - b^2(1-(k-1)\Delta) - \frac{1}{12}(1-(k-1)\Delta)^3 = -\frac{1}{3}b^2 + o\left(b^2\right).$$

QED

Bibliography

Alonso, Ricardo, Wouter Dessein, and Niko Matouschek (2008) "When Does Coordination Require Centralization?" *American Economic Review*, **98** (1), pp. 145–179.

Alonso, Ricardo and Niko Matouschek (2007) "Relational Delegation," *RAND Journal of Economics*, **38** (4), pp. 1070–1089.

——— (2008) "Optimal Delegation," *Review of Economic Studies*, **75** (1), pp. 259–293.

Ambrus, Attila, Eduardo Azevedo, and Yichiro Kamada (2010) "Hierarchical Cheap Talk." Mimeo, Harvard University.

Anderson, Edward J. and Peter Nash (1987) *Linear Programming in Infinite-Dimensional Spaces: Theory and Applications*, New York: John Wiley and Sons.

Arvey, Richard D. and James E. Campion (1982) "The Employment Interview: A Summary and Review of Recent Research," *Personnel Psychology*, **35** (2), pp. 281–322.

Austen-Smith, David (1994) "Strategic Transmission of Costly Information," *Econometrica*, **62** (4), pp. 955–963.

Benoit, Jean-Pierre and Juan Dubra (2011) "Apparent Overconfidence," *Econometrica*, **79** (5), pp. 1591–1625.

Bergemann, Dirk and Martin Pesendorfer (2007) "Information Structures in Optimal Auctions," *Journal of Economic Theory*, **137** (1), pp. 580–609.

Billingsley, Patrick (1995) *Probability and Measure*, New York: John Wiley and Sons.

Blackwell, David (1953) "Equivalent Comparisons of Experiments," *Annals of Mathematical Statistics*, **24** (2), pp. 265–272.

Blume, Andreas, Oliver Board, and Kohei Kawamura (2007) "Noisy Talk," *Theoretical Economics*, **2** (4), pp. 395–440.

Chakraborty, Archishman and Rick Harbaugh (2007) "Comparative Cheap Talk," *Journal of Economic Theory*, **132** (1), pp. 70–94.

Chen, Ying (2009) "Communication with Two-sided Asymmetric Information." Mimeo, Arizona State University.

Chen, Ying, Navin Kartik, and Joel Sobel (2008) "Selecting Cheap-Talk Equilibria," *Econometrica*, **76** (1), pp. 117–136.

Cho, In-Koo and David M. Kreps (1987) "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, **102** (2), pp. 179–221.

Crawford, Vincent P. and Joel Sobel (1982) "Strategic Information Transmission," *Econometrica*, **50** (6), pp. 1431–1451.

Dessein, Wouter (2002) "Authority and Communication in Organizations," *Review of Economic Studies*, **69** (4), pp. 811–838.

Eso, Peter and Balazs Szentes (2007) "Optimal information disclosure in auctions and the handicap auction," *Review of Economic Studies*, **74** (3), p. 705.

Farrell, Joseph and Robert Gibbons (1989) "Cheap Talk with Two Audiences," *American Economic Review*, **79** (5), pp. 1214–1223.

Fischer, Paul E. and Phillip C. Stocken (2001) "Imperfect Information and Credible Communication," *Journal of Accounting Research*, **39** (1), pp. 119–134.

Fudenberg, D. and D.K. Levine (1992) "Maintaining a Reputation when Strategies are Imperfectly Observed," *The Review of Economic Studies*, **59** (3), p. 561.

Gilligan, Thomas W. and Keith Krehbiel (1987) "Collective Decisionmaking and Standing Committees: An Informational Rationale for Restrictive Amendment Procedures," *Journal of Law, Economics, and Organization*, **3** (2), pp. 287–335.

Goltsman, Maria, Johannes Horner, Gregory Pavlov, and Francesco Squintani (2009) "Mediation, Arbitration and Negotiation," *Journal of Economic Theory*, **144** (4), pp. 1397–1420.

Gordon, Sidartha (2007) "Informative Cheap talk equilibria as fixed points." Mimeo, Universite de Montreal.

Green, Jerry R. and Nancy L. Stokey (2007) "A Two-Person Game of Information Transmission," *Journal of Economic Theory*, **135** (1), pp. 90–104.

Grossman, Sanford J. and Oliver D. Hart (1986) "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Ownership," *Journal of Political Economy*, **94** (4), pp. 691–719.

Hart, Oliver D. and John Moore (1988) "Incomplete Contracts and Renegotiation," *Econometrica*, **56** (4), pp. 755–785.

——— (2004) "Agreeing Now to Agree Later: Contracts That Rule Out but Do Not Rule In." Mimeo, Harvard University.

Hermalin, Benjamin E., Avery W. Katz, and Richard Craswell (2007) "Contract Law," in A. Mitchell Polinsky and Steven Shavell eds. *Handbook of Law and Economics*, **1** of Handbooks in Economics: North-Holland, pp. 3 – 138.

Holmstrom, Bengt (1984) "On the Theory of Delegation," in M Boyer and R. Kihlstrom eds. *Bayesian Models in Economic Theory*, New York: North-Holland.

Ivanov, Maxim (2010a) "Communication via a Strategic Mediator," *Journal of Economic Theory*, **145** (2), pp. 869–884.

——— (2010b) "Informational Control and Organization Design," *Journal of Economic Theory*, **145** (2), pp. 721–751.

Kamenica, Emir and Matthew Gentzkow (2011) "Bayesian Persuasion," *American Economic Review* (forthcoming).

Kartik, Navin (2009) "Strategic Communication with Lying Costs," *Review of Economic Studies*, **76** (4), pp. 1359–1395.

Kovac, Eugen and Tymofiy Mylovanov (2009) "Stochastic Mechanism in Settings without Monetary Transfers: The Regular Case," *Journal of Economic Theory*, **144** (4), pp. 1373–1395.

Krishna, Vijay and John Morgan (2001) "Asymmetric Information and Legislative Rules: Some Amendments," *American Political Science Review*, **95** (2), pp. 435–452.

——— (2004) "The Art of Conversation: Eliciting Information from Experts through Multi-Stage Communication," *Journal of Economic Theory*, **117** (2), pp. 147–179.

——— (2008) "Contracting for Information under Imperfect Commitment," *RAND Journal of Economics*, **39** (4), pp. 905–925.

Lerner, Josh and Jean Tirole (2006) "A Model of Forum Shopping," *American Economic Review*, **96** (4), pp. 1091–1113.

Li, Wei (2010) "Peddling Influence through Intermediaries," *American Economic Review*, **100** (3), pp. 1136–1162.

Lipsky, Martin S. and Lisa K. Sharp (2001) "From Idea to Market: The Drug Approval Process," *Journal of the American Board of Family Medicine*, **14** (5), pp. 362–367.

Marino, Anthony M. (2007) "Delegation versus Veto in Organizational Games of Strategic Communication," *Journal of Public Economic Theory*, **9** (6), pp. 979–992.

Martimort, David and Aggey Semenov (2006) "Continuity in Mechanism Design Without Transfers," *Economics Letters*, **93** (2), pp. 182–189.

Melumad, Nahum D. and Toshiyuki Shibano (1991) "Communication in Settings with No Transfers," *RAND Journal of Economics*, **22** (2), pp. 173–198.

Milgrom, Paul (1981) "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, **12** (2), pp. 380–391.

Milgrom, Paul and John Roberts (1988) "An Economic Approach to Influence Activities in Organizations," *American Journal of Sociology*, **94** (1), pp. 154–179.

Milgrom, Paul and Robert Weber (1985) "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, **10** (4), pp. 619–632.

Myerson, Roger B. (1982) "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, **10** (1), pp. 67–81.

Mylovanov, Timofiy (2008) "Veto-Based Delegation," *Journal of Economic Theory*, **138** (1), pp. 297–307.

Ostrovsky, Michael and Michael Schwarz (2010) "Information Disclosure and Unraveling in Matching Markets," *Journal of Mathematical Economics*, **2** (2), pp. 34–63.

Radner, Roy and Joseph E. Stiglitz (1984) "A Nonconcavity in the Value of Information," in M Boyer and R. Kihlstrom eds. *Bayesian Models in Economic Theory*, New York: North-Holland.

Rayo, Luis and Ilya Segal (2010) "Optimal Information Disclosure," *Journal of Political Economy*, **118** (5), pp. 949–987.

Shaked, Moshe and J. George Shanthikumar (2007) *Stochastic Orders*, New York: Springer.

Szalay, Dezso (2005) "The Economics of Clear Advice and Extreme Options," *Review of Economic Studies*, **72** (4), pp. 1173–1198.

Uhlig, Harald (1996) "A Law of Large Numbers for Large Economies," *Economic Theory*, **8** (1), pp. 41–50.

Watson, Joel (1996) "Information Transmission when the Informed Party is Confused," *Games and Economic Behavior*, **12** (1), pp. 143–161.