Free resolutions, combinatorics, and geometry

by

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Abstract

Boij–Söderberg theory is the study of two cones: the first is the cone of graded Betti tables over a polynomial ring, and the second is the cone of cohomology tables of coherent sheaves over projective space. Each cone has a triangulation induced from a certain partial order. Our first result gives a module-theoretic interpretation of this poset structure. The study of the cone of cohomology tables over an arbitrary polarized projective variety is closely related to the existence of an Ulrich sheaf, and our second result shows that such sheaves exist on the class of Schubert degeneracy loci. Finally, we consider the problem of classifying the possible ranks of Betti numbers for modules over a regular local ring.

Thesis Supervisor: Richard P. Stanley
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There are many other people who should appear here, so I apologize for omissions.

Contents

Chapter 1

Introduction.

This thesis contains results that are related to some recent theorems of Eisenbud and Schreyer. To state the results, we set some notation. Let K be a field and let $A = K[x_1, \dots, x_n]$ be a polynomial ring with the standard grading. Let M be a finitely generated graded A-module. Then the tor modules $\mathrm{Tor}_i^A(M, K)$ are naturally graded vector spaces, and we define the **graded Betti table**

$$\beta_{i,j}(M) = \dim \mathrm{Tor}_i^A(M, K)_j.$$

This can equivalently be phrased in terms of the number of generators of degree j in the ith term of a minimal free resolution of M. By the Hilbert syzygy theorem, $\beta_{i,j}(M) = 0$ if $i > n$. A module is said to have a **pure resolution** if for all i, we have that $\beta_{i,j}(M) \neq 0$ for at most one value of j. If this is the case, we set d_i to be this value of j and define $d(M) = (d_0, d_1, \dots)$ to be the **degree sequence** of M. Note that $d_0 < d_1 < \cdots$.

If M is Cohen–Macaulay and has a pure resolution, then the degree sequence determines the Betti numbers up to simultaneous scalar multiple. In other words, if $d_0 < d_1 < \cdots$ is the degree sequence, then there exists a rational number c (depending on M) such that

$$\beta_{i,d_i}(M) = c \prod_{j \neq i} \frac{1}{|d_j - d_i|}$$

(see [HK, Theorem 1]). Define $\beta(d)$ to be the table of numbers obtained by setting $c = 1$ above.

The first fundamental result about Betti tables is that given any degree sequence $d_0 < d_1 < \cdots < d_r$ with $r \leq n$, there exists a Cohen–Macaulay module M with pure resolution with the given degree sequence [EFW, ES2]. The second fundamental about Betti tables is that the pure resolutions are the building blocks for all Betti tables. More precisely, given any Cohen–Macaulay module M, there exist degree sequences $d^1, \dots, d^N$ and positive rational numbers $a_1, \dots, a_N$ such that

$$\beta(M) = a_1 \beta(d^1) + \cdots + a_N \beta(d^N) \tag{1.0.1}$$

[ES2, Theorem 0.2]. See [BS2] for the case of arbitrary modules M.

We can phrase this in more geometric terms as follows. First note that $\beta(M \oplus N) = \beta(M) + \beta(N)$, so that the set of Betti tables forms a semigroup. Then the result above says that the extremal rays of the convex cone spanned by these Betti tables are given by the $\beta(d)$. This convex cone has a natural triangulation given as follows. First, given two degree sequences d and d' (of the same length for simplicity), define a poset structure by setting $d \leq d'$ if and only if $d_i \leq d'_i$ for all

i (for the full definition, see §2.2). Then the cone of Betti tables is a geometric realization of this poset, and one can make the expression (1.0.1) unique by requiring that $d^1 < d^2 < \cdots < d^N$.

One of the results in Chapter 2 (which represents joint work with Berkesch, Erman, and Kummini [BEKS1]) is to give a module-theoretic interpretation of this poset structure. Namely, given two degree sequences d and d', we have that $d \le d'$ if and only if there exist Cohen–Macaulay modules M and M' which have pure resolutions of the respective types such that $\mathrm{Hom}_A(M', M)_{\le 0} \neq 0$.

Given the convex-geometric interpretation above, a natural question becomes: how does one interpret the linear inequalities that define the cone of Betti tables? The answer to this question comes from the study of cohomology tables of vector bundles on the projective space $\mathbf{P}^{n-1}_K = \mathrm{Proj}\, A$. Given a vector bundle $\mathcal{E}$ on $\mathbf{P}^{n-1}$, we define its **cohomology table**

$$\gamma(\mathcal{E})_{i,j} = \mathrm{h}^i(\mathbf{P}^{n-1}; \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^{n-1}}(j)).$$

Eisenbud and Schreyer defined a nonnegative bilinear pairing between Betti tables and cohomology tables of vector bundles, which we will not repeat here. Hence each vector bundle $\mathcal{E}$ gives a nonnegative linear functional on the set of all Betti tables. The defining linear equalities are given by the supernatural vector bundles: a vector bundle $\mathcal{E}$ is **supernatural** if for all j, there exists at most one i such that $\gamma_{i,j}(\mathcal{E}) \neq 0$, and if the Hilbert polynomial of $\mathcal{E}$ has distinct integral roots. In other words, for all twists $\mathcal{E}(j)$, we want that there is at most one nonzero cohomology group, and that the cohomology of $\mathcal{E}(j)$ vanishes completely for exactly $n-1$ distinct values of j. In this case, the increasing sequence of the roots is called the **root sequence** of $\mathcal{E}$.

There are many analogies between Betti tables and cohomology tables. First, if $\mathcal{E}$ is supernatural, then the ranks of its cohomology groups are determined, up to simultaneous scalar multiple, by its root sequence (and all possible root sequences are realizable by vector bundles [ES2, Theorem 0.4]). Second, for an arbitrary cohomology table can be written as a positive rational linear combination of the cohomology tables of supernatural vector bundles [ES2, Theorem 0.5]. This linear combination can be made unique if we introduce a poset structure on root sequences analogous to before and require that they form a chain. In a similar way as for modules, Chapter 2 gives a sheaf-theoretic interpretation of this poset structure. The description of the space of cohomology tables of all coherent sheaves can also be given in terms of supernatural vector bundles [ES3].

Given an arbitrary projective variety X equipped with a very ample line bundle $\mathcal{O}(1)$, it is natural to ask about the set of cohomology tables of coherent sheaves on X. It can be shown that the cone coincides with that of $\mathbf{P}^{\dim X}$ (with the standard very ample line bundle $\mathcal{O}(1)$) if and only if X has an Ulrich sheaf [ES5, Theorem 4.2]: a coherent sheaf $\mathcal{U}$ is **Ulrich** if $\mathrm{h}^i(X; \mathcal{U}(j)) = 0$ in the following cases:

1. $j \ge 0$ and $i > 0$
2. $-\dim X \le j \le -1$
3. $j < -\dim X$ and $i < \dim X$.

It is an open question whether or not every embedded projective variety has an Ulrich sheaf. Translating to the module $\bigoplus_{d \in \mathbf{Z}} \mathrm{H}^0(X; \mathcal{U}(d))$, the Ulrich property is also known as being a **maximally generated Cohen–Macaulay module** (MGMCM). It is shown in [BHU, Proposition 1.4] that a graded module is MGMCM if and only if it has a finite linear free resolution (over the homogeneous coordinate ring of X).

In Chapter 3, which is based on [Sam], we construct MGMCM on the class of projective **Schubert degeneracy loci**. These are defined as follows. Let X be a Cohen–Macaulay variety and choose two vector bundles $\mathcal{E}$ and $\mathcal{F}$ of the same rank. Suppose that they possess flags of subbundles and quotient bundles, respectively. Then given a linear map $\mathcal{E} \to \mathcal{F}$, we can restrict the map to a given subbundle of $\mathcal{E}$ and project to a given quotient bundle of $\mathcal{F}$ and ask about its rank. Given a permutation σ, one can produce a rank function, and the corresponding Schubert degeneracy locus is the locus of points in X where the various ranks fall below the values given by the rank function. These loci have expected codimensions, and we are able to construct a MGMCM when this expectation is met. More details are given in Chapter 3. Besides its connection to the theory of cohomology tables, the MGMCM that we construct are related to a lot of rich combinatorics. In particular, the linear free resolutions are homological incarnations of double Schubert polynomials.

Finally, in Chapter 4 (which is joint work with Berkesch, Erman, and Kummini [BEKS3]), we address the question of classifying the possible ranks of minimal free resolutions over local rings $(R, \mathfrak{m})$ (when there is no longer any grading to keep track of). In this case, the Betti table is replaced by the sequence $\beta(M)_i = \dim \operatorname{Tor}_i^R(M, R/\mathfrak{m})$. We completely answer the question for regular local rings and show that the obvious inequalities on partial Euler characteristics that need to hold are the only ones. Furthermore, the convex cone spanned by the Betti sequences fails to be closed. One main difference between this case and the Eisenbud–Schreyer case is that the result is valid for regular local rings of mixed characteristic (i.e., do not contain a field). The question of hypersurface rings is also addressed, but not completely solved, and some conjectures about the complete picture are given. Beyond hypersurface rings, one probably cannot expect a meaningful list of all of the necessary inequalities that need to hold.

Each chapter begins with a self-contained introduction which provides a more detailed description about its contents.

Chapter 2

Poset structures in Boij–Söderberg theory

2.1 Introduction

Boij–Söderberg theory is the study of the cone of Betti diagrams over the standard graded polynomial ring $S = K[x_1, \ldots, x_n]$ and – dually – the cone of cohomology tables of coherent sheaves on $\mathbf{P}^{n-1}_K$, where K is a field. The extremal rays of these cones correspond to special modules and sheaves: Cohen–Macaulay modules with pure resolutions (Definition 2.2.1) and supernatural sheaves (Definition 2.5.1), respectively. Each set of extremal rays carries a partial order $\preceq$ (Definitions 2.2.2 and 2.5.2) that induces a simplicial decomposition of the corresponding cone.

Each partial order $\preceq$ is defined in terms of certain combinatorial data associated to these special modules and sheaves. For a module with a pure resolution, this data is a degree sequence, and for a supernatural sheaf, this data is a root sequence. Our main results reinterpret these partial orders $\preceq$ in terms of the existence of nonzero homomorphisms between Cohen–Macaulay modules with pure resolutions and between supernatural sheaves.

Theorem 2.1.1. *Let ρ_d and $\rho_{d'}$ be extremal rays of the cone of Betti diagrams for S corresponding to Cohen–Macaulay modules with pure resolutions of types d and d', respectively. Then $\rho_d \preceq \rho_{d'}$ if and only if there exist Cohen–Macaulay modules M and M' with pure resolutions of types d and d', respectively, with* $\mathrm{Hom}_S(M', M)_{\leq 0} \neq 0$.

Theorem 2.1.2. *Let ρ_f and $\rho_{f'}$ be extremal rays of the cone of cohomology tables for $\mathbf{P}^{n-1}$ corresponding to supernatural sheaves of types f and f', respectively. Then $\rho_f \preceq \rho_{f'}$ if and only if there exist supernatural sheaves $\mathcal{E}$ and $\mathcal{E}'$ of types f and f', respectively, with* $\mathrm{Hom}_{\mathbf{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$.

Though the statements of these two theorems are quite parallel, Theorem 2.1.1 is far more subtle than Theorem 2.1.2. Theorem 2.1.2 follows nearly directly from the Eisenbud–Schreyer pushforward construction of supernatural sheaves, but without modification, it is not clear how to compare the modules constructed in [ES2, §5].

We illustrate this via an example. Let $n = 3$, $d = (0, 2, 3, 5)$, $d' = (0, 3, 9, 10)$, and M and M' be finite length modules with pure resolutions of types d and d', as constructed in [ES2, §5]. We know of no method to produce a nonzero element of $\mathrm{Hom}(M, M')_{\leq 0}$, even in this specific case. The difficulty here stems from differences in the constructions of M and M': the module M is constructed by pushing forward a complex of projective dimension 5 along $\mathbf{P}^2 \times (\mathbf{P}^1)^2 \to \mathbf{P}^2$, whereas M' is constructed by pushing forward a complex of projective dimension 10 along $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^5 \to \mathbf{P}^2$. Thus, the construction of [ES2, §5] does not even suggest that Theorem 2.1.1 ought to be true.

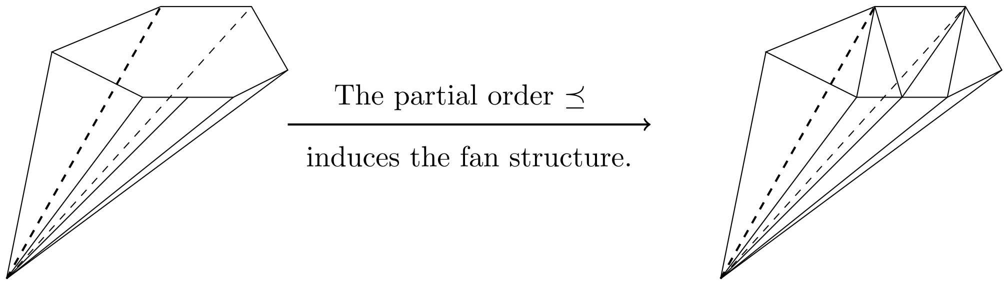


Figure 2-1: The partial order $\preceq$ on the extremal rays induces a simplicial decomposition of the cone of Betti diagrams, where the simplices correspond to chains of extremal rays with respect to the partial order. This simplicial decomposition is essential to many applications of Boij–Söderberg theory.

Our motivation for conjecturing the statement of Theorem 2.1.1 – and the first key idea behind its proof – is based on a flexible version of the Eisenbud–Schreyer construction of pure resolutions. This is Construction 2.3.3 below, and we show that the basic results of [ES2, §5] can be adapted to this construction. This extension enables us to use a single projection map to simultaneously produce modules N and N' with pure resolutions of types d and d'. In the case under consideration, we construct both N and N' by pushing forward complexes of projective dimension 10 along the projection map $\mathbf{P}^2 \times (\mathbf{P}^1)^7 \to \mathbf{P}^2$.[1]

We may then produce elements of $\mathrm{Hom}(N, N')_{\leq 0}$ by working with the complexes on the source $\mathbf{P}^2 \times (\mathbf{P}^1)^7$ of the projection map. However, finding such a nonzero element poses a second technical challenge in the proof of Theorem 2.1.1. This requires an explicit and somewhat delicate computation involving the pushforward of a morphism of complexes along the projection $\mathbf{P}^2 \times (\mathbf{P}^1)^7 \to \mathbf{P}^2$. This computation is carried out in the proof of Theorem 2.3.1, thus providing a new understanding of how certain modules with pure resolutions are related.

Besides providing greater insight into the structure of modules with pure resolutions and supernatural sheaves, our results have two further implications. First, the partial orders $\preceq$ are defined in terms of the combinatorial data of degree sequences and root sequences (see Sections 2.2 and 2.5), and depend on the total order of $\mathbf{Z}$; thus, they are only formally related to S and $\mathbf{P}^{n-1}$. However, our reinterpretations of $\preceq$ in terms of module- and sheaf-theoretic properties suggest the naturality not only of $\preceq$, but also of the induced simplicial decompositions of both cones. In other words, while there exist graded modules whose Betti diagrams can be written as a positive sum of pure tables in several ways, Theorem 2.1.1 suggests that the most natural of these decompositions is the Boij–Söderberg decomposition produced by [ES2, Decomposition Algorithm], and similarly for Theorem 2.1.2 and cohomology tables.

A second implication involves the extension of Boij–Söderberg theory to more complicated projective varieties or graded rings. For instance, the cone of free resolutions over a quadric hypersurface ring of $K[x, y]$ is described in [BBEG]. The extremal rays in this case correspond to pure resolutions of finite or infinite length. We could thus consider a partial order defined in parallel to Boij–Söderberg's original definition (based on the combinatorial data of a degree sequence), or, following our result, we could consider a partial order defined in terms of nonzero homomorphisms. These partial orders are different in this hypersurface case; only the second definition leads to a decomposition algorithm for Betti diagrams. See Example 2.8.1 below for details.

For more general graded rings there even exist extremal rays that do not correspond to pure

[1]We note that $M \neq N$ and $M' \neq N'$ in this example.

resolutions. (Similar statements hold for more general projective varieties.) There is thus no obvious extension of Boij–Söderberg's original partial order to these cases. By contrast, the reinterpretations of $\preceq$ provided by Theorems 2.1.1 and 2.1.2 are readily applicable to arbitrary projective varieties and graded rings. We discuss one such case in Example 2.8.2.

Theorems 2.1.1 and 2.1.2 hold over an arbitrary field K, and their proofs involve variants of the constructions in [ES2] for supernatural sheaves and modules with pure resolutions. When $\mathrm{char}(K) = 0$, there also exist equivariant constructions of supernatural vector bundles [ES2, Thm. 6.2] and of finite length modules with pure resolutions [EFW, Thm. 0.1]. For these we prove the most natural equivariant analogues of our main results.

Theorem 2.1.3. *Let V be an n-dimensional K-vector space with $\mathrm{char}(K) = 0$, and let ρ_d and $\rho_{d'}$ be the extremal rays of the cone of Betti diagrams for $S = \mathrm{Sym}(V)$ corresponding to finite length modules with pure resolutions of types d and d'. Then $\rho_d \preceq \rho_{d'}$ if and only if there exist finite length $\mathbf{GL}(V)$-equivariant modules M and M' with pure resolutions of types d and d', respectively, with $\mathrm{Hom}_{\mathbf{GL}(V)}(M', M)_{\leq 0} \neq 0$.*

Theorem 2.1.4. *Let V be an n-dimensional K-vector space with $\mathrm{char}(K) = 0$, and let ρ_f and $\rho_{f'}$ be the extremal rays of the cone of cohomology tables for $\mathbf{P}^{n-1} = \mathbf{P}(V)$ corresponding to supernatural vector bundles of types f and f'. Then $\rho_f \preceq \rho_{f'}$ if and only if there exist $\mathbf{GL}(V)$-equivariant supernatural vector bundles $\mathcal{E}$ and $\mathcal{E}'$ of types f and f', respectively, with $\mathrm{Hom}_{\mathbf{GL}(V)}(\mathcal{E}', \mathcal{E}) \neq 0$.*

The action of $\mathbf{GL}(V)$ has two orbits on the maximal ideals of S: one consisting of the maximal ideal $(x_1, \ldots, x_n)$ and the other consisting of its complement. An equivariant Cohen–Macaulay module therefore has only two options for its support, and hence either has finite length or must be a free module. Thus the finite length hypothesis in Theorem 2.1.3 is the natural equivariant analogue of the Cohen–Macaulay hypothesis in Theorem 2.1.1.

As above, the statement for pure resolutions is more subtle than the corresponding statement for supernatural vector bundles. The modules constructed in [EFW, §3] do not have nonzero equivariant homomorphisms between them, but the explicit combinatorics of the representation theory involved suggests a minor modification which does work. This also suggests how the maps should be defined in terms of the explicit presentation of the modules; the remaining nontrivial step is to show that these maps are in fact well-defined. The main obstacle is that such maps must be compatible with the actions of both the general linear group and the symmetric algebra, and the interplay between the two is delicate. This key issue in the proof of Theorem 2.1.3 is accomplished through a careful computation involving Pieri maps (combined with results from [SW]).

Outline

In Section 2.2, we prove the reverse implications of Theorems 2.1.1 and 2.1.3. We then construct nonzero morphisms between modules with pure resolutions. Sections 2.3 and 2.4, respectively, address the forward directions of Theorems 2.1.1 and 2.1.3. We next address the cone of cohomology tables for $\mathbf{P}^{n-1}$. In Section 2.5, we prove the reverse implications of Theorems 2.1.2 and 2.1.4. We then turn to the construction of nonzero morphisms between supernatural sheaves: Sections 2.6 and 2.7, respectively, address the forward directions of Theorems 2.1.2 and 2.1.4. Finally, we provide in Section 2.8 a brief discussion of how Theorem 2.1.1 has been applied in the study of Boij–Söderberg theory over other graded rings.

2.2 The poset of degree sequences

Let M be a finitely generated graded S-module. The (i,j)**th graded Betti number** of M, denoted $\beta_{i,j}(M)$, is $\dim_K \operatorname{Tor}_i^S(K,M)_j$. The **Betti diagram** of M is a table, with rows indexed by $\mathbf{Z}$ and columns by $0,\dots,n$, such that the entry in column i and row j is $\beta_{i,i+j}(M)$. A sequence $d=(d_0,\dots,d_n)\in(\mathbf{Z}\cup\{\infty\})^{n+1}$ is called a **degree sequence** for S if $d_i>d_{i-1}$ for all i (with the convention that $\infty>\infty$). The **length** of d, denoted $\ell(d)$, is the largest integer t such that d_t is finite.

Definition 2.2.1. A graded S-module M is said to have a **pure resolution of type** d if a minimal free resolution of M has the form

$$0\leftarrow M\leftarrow S(-d_0)^{\beta_{0,d_0}}\leftarrow S(-d_1)^{\beta_{1,d_1}}\leftarrow\cdots\leftarrow S(-d_{\ell(d)})^{\beta_{\ell(d),d_{\ell(d)}}}\leftarrow 0.$$

For every degree sequence d, there exists a Cohen–Macaulay module with a pure resolution of type d [ES2, Theorem 0.1] (see also [BS1, Conjecture 2.4], [EFW, Theorem 0.1]). The Betti diagram of any finitely generated S-module can be written as a positive rational combination of the Betti diagrams of Cohen–Macaulay modules with pure resolutions (see [ES2, Theorem 0.2] and [BS2, Theorem 2]). The **cone of Betti diagrams** for S is the convex cone inside $\bigoplus_{j\in\mathbf{Z}}\mathbf{Q}^{n+1}$ generated by the Betti diagrams of all finitely generated S-modules. Each degree sequence d corresponds to a unique extremal ray of this cone, which we denote by ρ_d, and every extremal ray is of the form ρ_d for some degree sequence d.

Definition 2.2.2. For two degree sequences d and d', we say that $d\preceq d'$ and that $\rho_d\preceq\rho_{d'}$ if $d_i\le d'_i$ for all i.

This partial order induces a simplicial fan structure on the cone of Betti diagrams, where simplices correspond to chains of degree sequences under the partial order $\preceq$. We now show that the existence of a nonzero homomorphism between two modules with pure resolutions implies the comparability of their corresponding degree sequences. This result provides the reverse implications for Theorems 2.1.1 and 2.1.3.

Proposition 2.2.3. *Let M and M' be graded Cohen–Macaulay S-modules with pure resolutions of types d and d', respectively. If $\operatorname{Hom}(M',M)_{\le 0}\neq 0$, then $d\preceq d'$.*

Proof. Write $\ell'=\ell(d')$ and $\ell=\ell(d)$. If $\ell'>\ell$, then $\operatorname{codim} M'>\operatorname{codim} M$, and, by [BH, Propositions 1.2.3, 1.2.1], $\operatorname{Hom}(M',M)=0$.

Therefore we may assume that $\ell'\le\ell$. By hypothesis, we may fix a nonzero homomorphism $\varphi\in\operatorname{Hom}(M',M)_t$ for some $t\le 0$. Let $F_\bullet$ and $F'_\bullet$ be minimal graded free resolutions of M and M', respectively, and let $\{\varphi_i\colon F'_i\to F_i\}_{i\ge 0}$ be the comparison maps in a lifting of φ. Suppose by way of contradiction that there is a j such that $d'_j<d_j$. Since $d'_j<d_j$, we see that $\varphi_j=0$. Hence, each φ_i such that $j\le i\le\ell'$ can be made zero by some homotopy equivalence. Write $(-)^\vee=\operatorname{Hom}_S(-,S(-n))$. Since M and M' are Cohen–Macaulay, we note that $(F_\bullet)^\vee$ and $(F'_\bullet)^\vee$ are minimal graded free resolutions of $\operatorname{Ext}^\ell_S(M,S(-n))$ and $\operatorname{Ext}^{\ell'}_S(M',S(-n))$. Further, the maps $\{\varphi_i^\vee\}_{i\ge 0}$ define an element of $\operatorname{Ext}_S^{\ell-\ell'}\Big(\operatorname{Ext}^\ell_S(M,S(-n)),\operatorname{Ext}^{\ell'}_S(M',S(-n))\Big)$. In fact, if we write $N=\operatorname{coker}\left((F_{\ell'-1})^\vee\longrightarrow(F_{\ell'})^\vee\right)$, then $(\varphi_{\ell'})^\vee\colon N\longrightarrow\operatorname{Ext}^{\ell'}_S(M',S(-n)))$ is the zero homomorphism. Hence $\varphi_i^\vee=0$ for all $0\le i\le\ell'$, and therefore $\varphi=0$. □

Proposition 2.2.3 is untrue if we do not assume that M' is Cohen–Macaulay. For example, consider $S=K[x,y]$, $M=S/\langle x^2\rangle$, and $M'=S\oplus K$. We used the hypothesis that M' is Cohen–Macaulay to have that $\operatorname{codim} M'=\ell(d')$ and that $\operatorname{Hom}_S(F'_\bullet,S(-n))$ is a resolution.

2.3 Construction of morphisms between modules with pure resolutions

In Theorem 2.1.1 we must, necessarily, consider more than $\mathrm{Hom}(M', M)_0$. For instance, if $n = 2, d = (0, 1, 2)$, and $d' = (1, 2, 3)$, then any M and M' with pure resolutions of types d and d' will be isomorphic to K^m and $K(-1)^{m'}$, respectively, for some integers m, m'. In this case, $\mathrm{Hom}(M', M)_0 = 0$, whereas $\mathrm{Hom}(M', M)_{-1} \neq 0$.

However, it is possible to reduce to the consideration of $\mathrm{Hom}(M', M)_0$. To do this, let $t := \min\{d'_i - d_i \mid d'_i \neq \infty\}$. By replacing d' by $d' - (t, \dots, t)$, the forward direction of Theorem 2.1.1 is an immediate corollary of the following result.

Theorem 2.3.1. *Let $d \preceq d'$ be degree sequences for S with $d_j = d'_j$ for some $0 \leq j \leq \ell(d')$. Then there exist finitely generated graded Cohen–Macaulay modules M and M' with pure resolutions of types d and d', respectively, with* $\mathrm{Hom}(M', M)_0 \neq 0$.

Remark 2.3.2. The homomorphism group in Theorems 2.1.1 and 2.3.1 is nonzero only for specific choices of the modules M and M'. For two degree sequences $d \preceq d'$, there exist many pairs of modules M, M' with pure resolutions of types d and d', respectively, such that $\mathrm{Hom}(M', M)_{\leq 0} = 0$. For example, take $d = d' = (0, 2, 4)$, $M = S/\langle x^2, y^2\rangle$, and $M' = S/\langle l_1^2, l_2^2\rangle$ for general linear forms l_1 and l_2. As another example, consider $d = (0, 3, 6) \prec d' = (0, 4, 8)$. When $M = S/\langle x^3, y^3\rangle$ and $M' = S/\langle f, g\rangle$ for general quartic forms f and g, we again have $\mathrm{Hom}(M', M)_{\leq 0} = 0$. □

The proof of Theorem 2.3.1 is given at the end of this section and involves two main steps.

1. Construct twisted Koszul complexes $\mathcal{K}_\bullet$ and $\mathcal{K}'_\bullet$ on a product $\mathbf{P}$ of projective spaces (including a copy of $\mathbf{P}^{n-1}$) and push them forward along the projection $\pi\colon \mathbf{P} \to \mathbf{P}^{n-1}$. This yields pure resolutions $F_\bullet$ and $F'_\bullet$ of types d and d' that respectively resolve modules M and M'.
2. Show that there exists a morphism $h_\bullet\colon \mathcal{K}'_\bullet \to \mathcal{K}_\bullet$ such that the induced map $\nu_\bullet\colon F'_\bullet \to F_\bullet$ is not null-homotopic. This yields a nonzero element $\psi \in \mathrm{Hom}_S(M', M)_0$.

We achieve (1) by modifying the construction of pure resolutions by Eisenbud and Schreyer [ES2, §5]. We replace their use of $\prod_i \mathbf{P}^{d_i - d_{i-1}}$ with a product of copies of $\mathbf{P}^1$. This enables us to simultaneously construct pure resolutions of types d and d' and a nonzero map between the modules they resolve. The details of (1) are contained in Construction 2.3.3. For (2), we apply Construction 2.3.3 so as to produce the morphism $h_\bullet$. Checking that the induced map $\nu_\bullet$ is not null-homotopic uses, in an essential way, the hypothesis that $d_j = d'_j$ for some $0 \leq j \leq \ell(d')$. Example 2.3.5 demonstrates these arguments. Write $\mathbf{P}^{1\times r}$ for the r-fold product of $\mathbf{P}^1$.

Construction 2.3.3 (Modification of the Eisenbud–Schreyer construction of pure resolutions)**.** *The objects involved in this construction of a pure resolution $F_\bullet$ of type d will be denoted by* $\mathrm{Kos}^d_\bullet$, *$\mathcal{K}_\bullet$, and $\mathcal{L}$. The corresponding objects for the pure resolution $F'_\bullet$ of type d' are* $\mathrm{Kos}^{d'}_\bullet$, *$\mathcal{K}'_\bullet$, and $\mathcal{L}'$. Let*

$$r := \max\{d_{\ell(d)} - d_0 - \ell(d), d'_{\ell(d')} - d_0 - \ell(d')\} \tag{2.3.4}$$

and $\mathbf{P} := \mathbf{P}^{n-1} \times \mathbf{P}^{1\times r}$. On $\mathbf{P}$, fix the coordinates

$$\Big([x_1 : x_2 : \cdots : x_n], [y_0^{(1)} : y_1^{(1)}], \dots, [y_0^{(r)} : y_1^{(r)}]\Big)$$

and consider the multilinear forms

$$f_p := \sum_{i_0 + \cdots + i_r = p} \left(x_{i_0} \cdot \prod_{j=1}^{r} y_{i_j}^{(j)} \right) \qquad \textit{for } p = 1, 2, \dots, n + r.$$

(Note that $i_0 \in \{1, \ldots, n\}$ and $i_j \in \{0,1\}$ for all $1 \leq j \leq r$.) We now define

$$\begin{aligned}
D &:= \{d_0, d_0+1, \ldots, d_0+\ell(d)+r\}, & D' &:= \{d_0, d_0+1, \ldots, d_0+\ell(d')+r\},\\
\delta &:= (\delta_1 < \cdots < \delta_r) = D \smallsetminus d, & \delta' &:= (\delta'_1 < \cdots < \delta'_r) = D' \smallsetminus d',\\
a &:= \delta - (d_0+1, \ldots, d_0+1), & a' &:= \delta' - (d_0+1, \ldots, d_0+1),\\
\mathcal{L} &:= \mathcal{O}_{\mathbf{P}}(-d_0, a), \qquad \text{and} & \mathcal{L}' &:= \mathcal{O}_{\mathbf{P}}(-d_0, a').
\end{aligned}$$

(We view δ and δ' as ordered sequences.) Let $\mathrm{Kos}^d_\bullet$ be the Koszul complex on $f_1, \ldots, f_{\ell(d)+r}$, which is an acyclic complex of sheaves on $\mathbf{P}$ of length $\ell(d)+r$ (see [ES2, Proposition 5.2]). Let $\mathcal{K}_\bullet := \mathrm{Kos}^d_\bullet \otimes \mathcal{L}$. Let $\pi\colon \mathbf{P} \to \mathbf{P}^{n-1}$ denote the projection onto the first factor. By repeated application of [ES2, Proposition 5.3], $\pi_*\mathcal{K}_\bullet$ is an acyclic complex of sheaves on $\mathbf{P}^{n-1}$ of length $\ell(d)$ such that each term is a direct sum of line bundles. Taking global sections of this complex in all twists yields the pure resolution $F_\bullet$ of a graded S-module (that is finitely generated and Cohen–Macaulay). We can write the free module F_i explicitly as follows. If $s = \max\{i \mid a_i - d_j + d_0 \leq -2\}$, then we have

$$F_j = S(-d_j)^{\binom{\ell(d)+r}{d_j-d_0}} \otimes \left(\bigotimes_{i=1}^{s} \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(a_i - d_j + d_0))\right) \otimes \left(\bigotimes_{i=s+1}^{r} \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(a_i - d_j + d_0))\right).$$

Let $\mathrm{Kos}^{d'}_\bullet$ be the Koszul complex on $f_1, \ldots, f_{\ell(d')+r}$ and $\mathcal{K}'_\bullet := \mathrm{Kos}^{d'}_\bullet \otimes \mathcal{L}'$, and define $F'_\bullet$ in a similar manner.

The value of r in (2.3.4) is the least integer such that we are able to fit both the twists $-d_0$ and $\min\{-d_{\ell(d)}, -d'_{\ell(d')}\}$ in the $\mathbf{P}^{n-1}$ coordinate of the bundles of the complexes $\mathcal{K}_\bullet$ and $\mathcal{K}'_\bullet$. The choices of a and a', which ensure that $F_\bullet$ and $F'_\bullet$ are pure of types d and d', are dictated by the homological degrees in $\mathcal{K}_\bullet$ and $\mathcal{K}'_\bullet$ that need to be eliminated in each projection away from a $\mathbf{P}^1$ component of $\mathbf{P}$. In Example 2.3.5, these homological degrees are those with an underlined -1 in Table 2.1. Observe that $a - a' \in \mathbf{N}^r$ since $d \preceq d'$. Thus there is a nonzero map $h_\bullet\colon \mathcal{K}'_\bullet \to \mathcal{K}_\bullet$ that is induced by a polynomial of multidegree $(0, a-a')$. In (2), we show that $\pi_* h_\bullet$ induces the desired nonzero map.

The following extended example contains all of the main ideas behind the proof of Theorem 2.3.1.

Example 2.3.5. Consider $d = (0,2,4,5,6)$ and $d' = (1,2,4,7) = (1,2,4,7,\infty)$. Note that $d_2 = d'_2 = 4$, so that d and d' satisfy the hypotheses of Theorem 2.3.1. Here $r = 4$ and $\mathbf{P} = \mathbf{P}^3 \times \mathbf{P}^{1\times 4}$. On $\mathbf{P}$, we have the Koszul complexes $\mathrm{Kos}^d_\bullet = \mathrm{Kos}_\bullet(\mathcal{O}_{\mathbf{P}}; f_1, \ldots, f_8)$ and $\mathrm{Kos}^{d'}_\bullet = \mathrm{Kos}_\bullet(\mathcal{O}_{\mathbf{P}}; f_1, \ldots, f_7)$. There is a natural map $\mathrm{Kos}^{d'}_\bullet \to \mathrm{Kos}^d_\bullet$ induced by the inclusion $\langle f_1, \ldots, f_7\rangle \subseteq \langle f_1, \ldots, f_8\rangle$. Here we have

$$\begin{aligned}
\delta &= (1,3,7,8), & \delta' &= (0,3,5,6),\\
a &= (0,2,6,7), & a' &= (-1,2,4,5),\\
\mathcal{K}_\bullet &= \mathrm{Kos}^d_\bullet \otimes \mathcal{O}_{\mathbf{P}}(0,a), \qquad \text{and} & \mathcal{K}'_\bullet &= \mathrm{Kos}^{d'}_\bullet \otimes \mathcal{O}_{\mathbf{P}}(0,a').
\end{aligned}$$

Table 2.1 shows the twists in each homological degree of these complexes.

Let h be a nonzero homogeneous polynomial on $\mathbf{P}$ of multidegree $(0, a-a') = (0,1,0,2,2)$. Then multiplication by h induces a nonzero map $h\colon \mathcal{K}'_0 \to \mathcal{K}_0$. To write h, we use matrix multi-index notation for the monomials in $K[y_0^{(1)}, y_1^{(1)}, \ldots, y_0^{(4)}, y_1^{(4)}]$, where the ith column represents the multi-index of the $y^{(i)}$-coordinates. With this convention, fix

$$h = \mathbf{y}^{\left(\begin{smallmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)} := y_0^{(1)} \cdot \left(y_0^{(3)}\right)^2 \cdot \left(y_0^{(4)}\right)^2.$$

$d = (0, 2, 4, 5, 6)$

i	Twist in $\mathcal{K}_i$
0	$(0, 0, 2, 6, 7)$
-1	$(-1, \underline{-1}, 1, 5, 6)$
-2	$(-2, -2, 0, 4, 5)$
-3	$(-3, -3, \underline{-1}, 3, 4)$
-4	$(-4, -4, -2, 2, 3)$
-5	$(-5, -5, -3, 1, 2)$
-6	$(-6, -6, -4, 0, 1)$
-7	$(-7, -7, -5, \underline{-1}, 0)$
-8	$(-8, -8, -6, -2, \underline{-1})$

$d' = (1, 2, 4, 7)$

i	Twist in $\mathcal{K}'_i$
0	$(0, \underline{-1}, 2, 4, 5)$
-1	$(-1, -2, 1, 3, 4)$
-2	$(-2, -3, 0, 2, 3)$
-3	$(-3, -4, \underline{-1}, 1, 2)$
-4	$(-4, -5, -2, 0, 1)$
-5	$(-5, -6, -3, \underline{-1}, 0)$
-6	$(-6, -7, -4, -2, \underline{-1})$
-7	$(-7, -8, -5, -3, -2)$

Table 2.1: Twists appearing in $\mathcal{K}_\bullet$ and $\mathcal{K}'_\bullet$ in Example 2.3.5.

Denote the induced map of complexes $\mathcal{K}'_\bullet \to \mathcal{K}_\bullet$ by $h_\bullet$. Taking the direct image of $h_\bullet$ along the natural projection $\pi\colon \mathbf{P} \to \mathbf{P}^3$ and its global sections in all twists induces a map $\nu_\bullet\colon F'_\bullet \to F_\bullet$.

We claim that $\nu_\bullet$ is not null-homotopic. This need not hold for an arbitrary pair $d \preceq d'$, however it does hold for a pair of degree sequences which satisfy the hypotheses of Theorem 2.3.1. We use the fact that $d_2 = d'_2 = 4$, as this implies that $\nu_2\colon F'_2 \to F_2$ is a matrix of scalars. Since $F'_\bullet$ and $F_\bullet$ are both minimal free resolutions, it then follows that the map ν_2 factors through a null-homotopy only if ν_2 is itself the zero map. Thus it is enough to show that $\nu_2 \neq 0$. For this, note that

$$\begin{aligned} F_2 &= S(-4)^{\binom{8}{4}} \otimes \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-4)) \otimes \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-2)) \otimes \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(3)) \\ \text{and} \quad F'_2 &= S(-4)^{\binom{7}{4}} \otimes \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-5)) \otimes \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-2)) \otimes \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(0)) \otimes \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(1)) \end{aligned}$$

and that F_2 and F'_2 have H^1 terms in precisely the same positions, and similarly for the H^0 terms. We may then use [BEKS2, Lemma 7.3] to compute the map $\nu_2\colon F'_2 \to F_2$ explicitly. Since the matrix is too large to be written down, we simply exhibit a basis element of F'_2 that is not mapped to zero.

For $I = \{i_1 < \cdots < i_4\}$ a subset of either $\{1, \ldots, 8\}$ or $\{1, \ldots, 7\}$, we use the notation $\epsilon_I := \epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_4}$ to write S-bases for $S(-4)^{\binom{8}{4}}$ and for $S(-4)^{\binom{7}{4}}$. Choose the natural monomial bases for the cohomology groups appearing in the tensor product expressions for F_2 and F'_2, and write these monomials in multi-index notation. Recalling the above definition of h, we then have that

$$\epsilon_{1,2,3,4} \otimes \mathbf{y}^{\left(\begin{smallmatrix} -4 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{smallmatrix}\right)}$$

is a basis element of F_2. We compute

$$\begin{aligned} \nu_2\left(\epsilon_{1,2,3,4} \otimes \mathbf{y}^{\left(\begin{smallmatrix} -4 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{smallmatrix}\right)}\right) &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\left(\begin{smallmatrix} -4 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{smallmatrix}\right)} \cdot h \\ &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\left(\begin{smallmatrix} -4 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)} \\ &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\left(\begin{smallmatrix} -3 & -1 & 2 & 3 \\ -1 & -1 & 0 & 0 \end{smallmatrix}\right)}. \end{aligned}$$

Since this yields a basis element of F'_2, it is clear that ν_2 is a nonzero map, so $\nu_\bullet$ is not null-homotopic. $\square$

Proof of Theorem 2.3.1. Construction 2.3.3 yields finitely generated graded Cohen–Macaulay modules M and M' that have pure resolutions $F_\bullet$ and $F'_\bullet$ of types d and d', respectively. To construct the desired nonzero map $\psi\colon M' \to M$, we fix a generic homogeneous form h on $\mathbf{P}$ of multidegree $(0, a-a')$, which exists because $a - a' = \delta - \delta' \in \mathbf{N}^r$. Multiplication by h induces a map $h_\bullet\colon \mathcal{K}'_\bullet \to \mathcal{K}_\bullet$. The functoriality of π_* induces a map $\pi_*\mathcal{K}'_\bullet \to \pi_*\mathcal{K}_\bullet$ that, upon taking global sections in all twists, yields a map $\nu_\bullet\colon F'_\bullet \to F_\bullet$. Let $\psi\colon M' \to M$ be the map induced by $\nu_\bullet$.

To show that ψ is nonzero, it suffices to show that $\nu_\bullet$ is not null-homotopic. Let j be the index such that $d_j = d'_j$. Then F_j and F'_j are generated entirely in the same degree. Since $F_\bullet$ and $F'_\bullet$ are minimal free resolutions, $\nu_j\colon F'_j \to F_j$ is given by a matrix of scalars. Thus it follows that $\nu_\bullet$ is null-homotopic only if ν_j is the zero map. We now use the description of ν_j given in [BEKS2, Lemma 7.3]. (The relevant homological degree in both $\mathcal{K}_\bullet$ and $\mathcal{K}'_\bullet$ is $d_j - d_0$.)

Let $s = \max\{i \mid a_i - d_j + d_0 \le -2\}$ and let $s' = \max\{i \mid a'_i - d'_j + d_0 \le -2\}$. Note that, since $d_j = d'_j$, the construction of a and a' implies that $s = s'$. We then have

$$F_j = S(-d_j)^{\binom{\ell(d)+r}{d_j-d_0}} \otimes \left(\bigotimes_{i=1}^{s} \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(a_i - d_j + d_0))\right) \otimes \left(\bigotimes_{i=s+1}^{r} \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(a_i - d_j + d_0))\right) \text{ and}$$
$$F'_j = S(-d_j)^{\binom{\ell(d')+r}{d_j-d_0}} \otimes \left(\bigotimes_{i=1}^{s} \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(a'_i - d_j + d_0))\right) \otimes \left(\bigotimes_{i=s+1}^{r} \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(a'_i - d_j + d_0))\right),$$

where both F_j and F'_j have the same number of factors involving H^0 (and therefore also the same number involving H^1). Hence we can repeatedly apply [BEKS2, Lemma 7.3] to conclude that ν_j is simply the map induced on cohomology by the map $h_{d_j-d_0}\colon \mathcal{K}'_{d_j-d_0} \to \mathcal{K}_{d_j-d_0}$.

We now fix a specific value of h and show that $\nu_j \neq 0$. Let $c := a - a' \in \mathbb{N}^r$ and write $c = (c_1, \dots, c_r)$. Let

$$h := \left(y_0^{(1)}\right)^{c_1} \cdot \left(y_0^{(2)}\right)^{c_2} \cdots \left(y_0^{(r)}\right)^{c_r} = \mathbf{y}^{\left(\begin{smallmatrix} c_1 & \dots & c_r \\ 0 & \dots & 0 \end{smallmatrix}\right)},$$

so that h is the unique monomial of multidegree $(0, c)$ that involves only the $y_0^{(i)}$-variables.

For $I = \{i_1 < \dots < i_{d_j-d_0}\}$ a subset of either $\{1, \dots, \ell(d)+r\}$ or $\{1, \dots, \ell(d')+r\}$, we use the notation $\epsilon_I := \epsilon_{i_1} \wedge \dots \wedge \epsilon_{d_j-d_0}$ to write S-bases for $S(-d_j)^{\binom{\ell(d)+r}{d_j-d_0}}$ and for $S(-d_j)^{\binom{\ell(d')+r}{d_j-d_0}}$. Choose the natural monomial bases for the cohomology groups appearing in the tensor product expression for F_j and F'_j, and write these monomials in matrix multi-index notation, as in Example 2.3.5. For each i corresponding to an H^1-term (i.e. $i \in \{1, \dots, s\}$), let $u_i := -(a_i - d_j + d_0) + 1$. For each i corresponding to an H^0 term (i.e. $i \in \{s+1, \dots, r\}$), let $w_i := -(a_i - d_j + d_0)$. Observe that

$$\epsilon_{\{1,\dots,d_j-d_0\}} \otimes \mathbf{y}^{\left(\begin{smallmatrix} u_1 & \dots & u_s & w_{s+1} & \dots & w_r \\ -1 & \dots & -1 & 0 & \dots & 0 \end{smallmatrix}\right)}$$

is a basis element of F_j. We then have that

$$\begin{aligned}
\nu_j\left(\epsilon_{\{1,\dots,d_j-d_0\}} \otimes \mathbf{y}^{\left(\begin{smallmatrix} u_1 & \dots & u_s & w_{s+1} & \dots & w_r \\ -1 & \dots & -1 & 0 & \dots & 0 \end{smallmatrix}\right)}\right) &= \epsilon_{\{1,\dots,d_j-d_0\}} \otimes \mathbf{y}^{\left(\begin{smallmatrix} u_1 & \dots & u_s & w_{s+1} & \dots & w_r \\ -1 & \dots & -1 & 0 & \dots & 0 \end{smallmatrix}\right)} \cdot h \\
&= \epsilon_{\{1,\dots,d_j-d_0\}} \otimes \mathbf{y}^{\left(\begin{smallmatrix} u_1 & \dots & u_s & w_{s+1} & \dots & w_r \\ -1 & \dots & -1 & 0 & \dots & 0 \end{smallmatrix}\right)} \cdot \mathbf{y}^{\left(\begin{smallmatrix} c_1 & \dots & c_r \\ 0 & \dots & 0 \end{smallmatrix}\right)} \\
&= \epsilon_{\{1,\dots,d_j-d_0\}} \otimes \mathbf{y}^{\left(\begin{smallmatrix} u_1+c_1 & \dots & u_s+c_s & w_{s+1}+c_{s+1} & \dots & w_r+c_r \\ -1 & \dots & -1 & 0 & \dots & 0 \end{smallmatrix}\right)}.
\end{aligned}$$

One may check that this is a basis element of F'_j, and hence the map ν_j is nonzero. Therefore $\nu_\bullet$ is not null-homotopic, as desired. ☐

2.4 Equivariant construction of morphisms between modules with pure resolutions

Throughout this section, we assume that K is a field of characteristic 0 and that all degree sequences have length n. Let V be an n-dimensional K-vector space, and let $S = \mathrm{Sym}(V)$. We use $\mathbf{S}_\lambda$ to denote a Schur functor, as in Section 2.7. As in Section 2.3, a shift of d' reduces the remaining direction of Theorem 2.1.3 to the following result.

Theorem 2.4.1. *Let $d \preceq d'$ be two degree sequences such that $d_k = d'_k$ for some k. Then there exist finite length $\mathbf{GL}(V)$-equivariant S-modules M and M' with pure resolutions of types d and d', respectively, with $\mathrm{Hom}_{\mathbf{GL}(V)}(M', M)_0 \neq 0$.*

Our proof of Theorem 2.4.1 relies on Lemma 2.4.2, which handles the special case when the degree sequences d and d' differ by 1 in a single position. This proof will repeatedly appeal to Pieri's rule for decomposing the tensor product of a Schur functor by a symmetric power. We refer the reader to [SW, §1.1 and Theorem 1.3] for a statement of this rule, as our main use of it will be through [SW, Lemma 1.6].

Given a degree sequence d, let $M(d)$ be the $\mathbf{GL}(V)$-equivariant graded S-module constructed in [EFW, §3] (see also [SW, §2.1]), and let $\mathbf{F}(d)_\bullet$ be its $\mathbf{GL}(V)$-equivariant free resolution. By construction, the generators for each S-module $\mathbf{F}(d)_j$ form an irreducible $\mathbf{GL}(V)$-module whose highest weight we call $\lambda(d)_j$. For instance, if $d = (0, 2, 5, 7, 8)$, then $\lambda(d)_0 = (3, 1, 0, 0)$ and $\lambda(d)_1 = (5, 1, 0, 0)$ [EFW, Example 3.3]. Note that $M(d) \otimes V$ is also an equivariant module with a pure resolution of type d.

Lemma 2.4.2. *Let $d = (d_0, \ldots, d_n) \in \mathbf{Z}^{n+1}$ be a degree sequence, and let d' be the degree sequence obtained from d by replacing d_i by $d_i + 1$ for some i. Then there exists an equivariant nonzero morphism $\varphi\colon M(d') \otimes V \to M(d)$.*

Further, if $F_\bullet$ and $F'_\bullet$ are the minimal free resolutions of $M(d)$ and $M(d') \otimes V$ respectively, then we may choose φ so that the induced map $F'_j \to F_j$ is surjective for all $j \neq i$.

Remark 2.4.3. Let d and d' be degree sequences as in the statement of Lemma 2.4.2. We observe that

(i) $\lambda(d')_i = \lambda(d)_i$.
(ii) If $j < i$, then $\lambda(d')_j$ is obtained from $\lambda(d)_j$ by removing a box from the ith part.
(iii) If $j > i$, then $\lambda(d')_j$ is obtained from $\lambda(d)_j$ by removing a box from the $(i + 1)$st part.

For instance, if $d = (0, 2, 4)$ and $d' = (0, 3, 4)$, then we have

$$\lambda(d)_j = \begin{cases} (1,0) & \text{if } j = 0 \\ (3,0) & \text{if } j = 1 \\ (3,2) & \text{if } j = 2 \end{cases} \quad \text{and} \quad \lambda(d')_j = \begin{cases} (0,0) & \text{if } j = 0 \\ (3,0) & \text{if } j = 1 \\ (3,1) & \text{if } j = 2. \end{cases} \qquad \square$$

Remark 2.4.4. In the proof of Lemma 2.4.2, we repeatedly use [SW, Lemma 1.6]. The statement of the lemma is for factorizations of Pieri maps into simple Pieri maps $\mathbf{S}_\nu V \to \mathbf{S}_\eta V \otimes V$, but we need to factor into simple Pieri maps as well as simple co-Pieri maps $\mathbf{S}_\eta V \otimes V \to \mathbf{S}_\nu V$. No modification of the proof is needed: we simply use the fact that the composition of a co-Pieri map and a Pieri map of the same type is an isomorphism and that in each case that we apply [SW, Lemma 1.6], the Pieri maps may be factored so that the simple Pieri maps and simple co-Pieri maps of the same type appear consecutively. $\square$

Proof of Lemma 2.4.2. Set $\lambda_\ell = \sum_{j=\ell}^{n-1}(d_{j+1} - d_j - 1)$ for $1 \leq \ell \leq n-1$, $\lambda_n = 0$, $\mu_1 = \lambda_1 + d_1 - d_0$, and $\mu_\ell = \lambda_\ell$ for $1 \leq \ell \leq n$. If $i = n$, we modify λ and μ by adding 1 to all of its parts (so in particular, $\lambda_n = \mu_n = 1$). As in [EFW, §3], define M to be the cokernel of the Pieri map

$$\psi_{\mu/\lambda} \colon S(-d_1) \otimes \mathbf{S}_\mu V \to S(-d_0) \otimes \mathbf{S}_\lambda V.$$

We will choose partitions λ' and μ' so that M' is the cokernel of the Pieri map

$$\psi_{\mu'/\lambda'} \colon S(-d_1') \otimes \mathbf{S}_{\mu'} V \to S(-d_0') \otimes \mathbf{S}_{\lambda'} V.$$

To do this, we separately consider the three cases $i = 0$, $i = 1$, and $i \geq 2$. In each case, we specify λ' and μ' (these descriptions are special cases of Remark 2.4.3) and construct a commutative diagram of equivariant degree 0 maps

$$\begin{array}{ccc} S(-d_1) \otimes \mathbf{S}_\mu V & \xrightarrow{\psi_{\mu/\lambda}} & S(-d_0) \otimes \mathbf{S}_\lambda V \\ \uparrow{\scriptstyle \varphi_\mu} & & \uparrow{\scriptstyle \varphi_\lambda} \\ S(-d_1') \otimes \mathbf{S}_{\mu'} V \otimes V & \xrightarrow{\psi_{\mu'/\lambda'} \otimes 1_V} & S(-d_0') \otimes \mathbf{S}_{\lambda'} V \otimes V \end{array} \tag{2.4.5}$$

that induces an equivariant degree 0 map of the cokernels $\varphi \colon M' \to M$. Since the Pieri maps are only well-defined up to a choice of nonzero scalar, we only prove that the square commutes up to a choice of nonzero scalar. One may scale appropriately to obtain strict commutativity.

Finally, after handling the three separate cases, we prove that the induced maps $F_j' \to F_j$ are surjective whenever $j \neq i$. Since $F_\bullet'$ is a minimal free resolution, this implies that the map $F_\bullet' \to F_\bullet$ is not null-homotopic, and hence $\varphi \colon M' \to M$ is nonzero.

Case $i = 1$. Set $\lambda_1' = \lambda_1 - 1$, $\lambda_j' = \lambda_j$ for $2 \leq j \leq n$, and $\mu' = \mu$. Also, let $d_0' = d_0$ and $d_1' = d_1 + 1$. Using the notation of (2.4.5), we define φ_μ by identifying $\mathbf{S}_{\mu'} V \otimes V$ with $\mathrm{Sym}^1 V \otimes \mathbf{S}_\mu V$ and then extending it to an S-linear map. Let φ_λ be the projection of $\mathbf{S}_{\lambda'} V \otimes V \to \mathbf{S}_\lambda V$ tensored with the identity of $S(-d_0)$. From the degree $d_1 + 1$ part of (2.4.5), we obtain

$$\begin{array}{ccc} \mathrm{Sym}^1 V \otimes \mathbf{S}_\mu V & \xrightarrow{\alpha} & \mathrm{Sym}^{d_1 - d_0 + 1} V \otimes \mathbf{S}_\lambda V \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} \\ \mathbf{S}_\mu V \otimes V & \xrightarrow{\delta} & \mathrm{Sym}^{d_1 - d_0 + 1} V \otimes \mathbf{S}_{\lambda'} V \otimes V. \end{array}$$

Note that α is the linear part of $\mathbf{F}_1 \to \mathbf{F}_0$ and is hence injective because $d_2 - d_1 > 1$. Since β is an isomorphism, $\alpha\beta$ is injective. Also we have $\lambda_1 > \lambda_2$ because $d_2 - d_1 > 1$, so by Pieri's rule, every summand of $\mathbf{S}_\mu V \otimes V$ is also a summand of $\mathrm{Sym}^{d_1 - d_0 + 1} V \otimes \mathbf{S}_\lambda V$. Using [SW, Lemma 1.6], one can show that $\gamma\delta$ is also injective. Since the tensor product $\mathrm{Sym}^{d_1 - d_0 + 1} V \otimes \mathbf{S}_\lambda V$ is multiplicity-free by the Pieri rule, this implies that these maps are equal after rescaling the image of each direct summand of $\mathbf{S}_\mu V \otimes V$ by some nonzero scalar. Hence this diagram is commutative, and the same is true for (2.4.5).

Case $i \geq 2$. Set $\lambda_i' = \lambda_i - 1$ and $\lambda_j = \lambda_j$ for $j \neq i$. Similarly, set $\mu_i' = \mu_i - 1$ and $\mu_j' = \mu_j$ for $j \neq i$. Using the notation of (2.4.5), let φ_μ be a nonzero projection of $\mathbf{S}_{\mu'} V \otimes V$ onto $\mathbf{S}_\mu V$ tensored with the identity on $S(-d_1)$. Similar to the previous case, choose a nonzero projection $\mathbf{S}_{\lambda'} V \otimes V \to \mathbf{S}_\lambda V$ and tensor it with the identity map on $S(-d_0)$ to get φ_λ. From the degree d_1 part of (2.4.5), we obtain

$$\begin{array}{ccc}
\mathbf{S}_\mu V & \xrightarrow{\alpha} & \mathrm{Sym}^{d_1-d_0} V \otimes \mathbf{S}_\lambda V \\
\beta \uparrow & & \uparrow \gamma \\
\mathbf{S}_{\mu'} V \otimes V & \xrightarrow{\delta} & \mathrm{Sym}^{d_1-d_0} V \otimes \mathbf{S}_{\lambda'} V \otimes V.
\end{array}$$

Let $\mathbf{S}_\nu V$ be a direct summand of $\mathbf{S}_{\mu'} V \otimes V$. If $\nu \neq \mu$, then $\mathbf{S}_\nu V$ is not a summand of $\mathrm{Sym}^{d_1-d_0} V \otimes \mathbf{S}_\lambda V$, as otherwise we would have $\nu_i = \lambda_i - 1$, and both of the compositions $\alpha\beta$ and $\gamma\delta$ would therefore be 0 on such a summand. If $\nu = \mu$, then the composition $\alpha\beta$ is nonzero, so it is enough to check that the same is true for $\gamma\delta$; this holds by [SW, Lemma 1.6], and hence this diagram and (2.4.5) are commutative.

Case $i = 0$. Set $d^\vee := (-d_n, -d_{n-1}, \dots, -d_0)$ and $d'^\vee := (-d'_n, -d'_{n-1}, \dots, -d'_0)$. Since $d_j = d'_j$ for all $j \neq i = 0$, we see that $d^\vee$ and $d'^\vee$ only differ in position n. Hence, by the case $i \geq 2$ above (we assume that $n \geq 2$ since the $n = 1$ case is easily done directly), we have finite length modules $M(d^\vee)$ and $M(d'^\vee)$ with pure resolutions of types $d^\vee$ and $d'^\vee$, respectively, along with a nonzero morphism $\psi\colon M(d^\vee) \otimes V \to M(d'^\vee)$. If we define $N^\vee := \mathrm{Ext}^n(N, S)$, then $M(d'^\vee)^\vee \cong M(d')$ and $(M(d^\vee) \otimes V)^\vee \cong M(d) \otimes V^*$ (both isomorphisms are up to some power of $\bigwedge^n V$ which we cancel off). In addition, since $\mathrm{Ext}^n(-, S)$ is a duality functor on the space of finite length S-modules, we obtain a nonzero map

$$\psi^\vee\colon M(d') \to M(d) \otimes V^*.$$

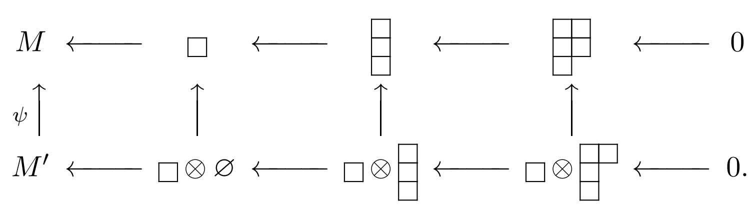
By adjunction, we then obtain a nonzero map $M(d') \otimes V \to M(d)$.

Fixing some $j \neq i$, we now prove the surjectivity of the maps $F'_j \to F_j$, which implies that φ is a nonzero morphism, as observed above. The key observation is that, in each of the above three cases, F_j is an irreducible Schur module. Since $d_j = d'_j$, the map

$$F'_j = S(-d'_j) \otimes \mathbf{S}_{\lambda(d')_j} V \otimes V \to F_j = S(-d_j) \otimes \mathbf{S}_{\lambda(d)_j} V$$

is induced by a nonzero equivariant map $\mathbf{S}_{\lambda(d')_j} V \otimes V \to \mathbf{S}_{\lambda(d)_j} V$. Since the target is an irreducible representation, this morphism, and hence the map $F'_j \to F_j$, is surjective. More specifically, the map $\mathbf{S}_{\lambda(d')_j} V \otimes V \to \mathbf{S}_{\lambda(d)_j} V$ is a projection onto one of the factors in the Pieri rule decomposition of $\mathbf{S}_{\lambda(d')_j} V \otimes V$. □

Example 2.4.6. This example illustrates the construction of Lemma 2.4.2 when $d = (0, 2, 4)$ and $d' = (0, 3, 4)$. When writing the free resolutions, we simply write the Young diagram of λ in place of the corresponding graded equivariant free module. Also, we follow the conventions in [EFW] and [SW] and draw the Young diagram of λ by placing λ_i boxes in the ith *column*, rather than the usual convention of using rows. The morphism from Lemma 2.4.2 yields a map of complexes, which we write as



Observe that $d_2 = 4 = d'_2$ and that the vertical arrow in homological position 2 is surjective, as it corresponds to a Pieri rule projection. A similar statement holds in position 0. □

Proof of Theorem 2.4.1. Set $r := \sum_{j=0}^{n} d'_j - d_j$. We may construct a sequence of degree sequences $d =: d^0 < d^1 < \cdots < d^r := d'$ such that d^j and d^{j+1} satisfy the hypotheses of Lemma 2.4.2 for any j. Lemma 2.4.2 yields a nonzero morphism

$$\varphi^{(j+1)} \colon M(d^{j+1}) \otimes V \to M(d^j)$$

for any $j = 1, \ldots, r$. If we set $M^{(j)} := M(d^j) \otimes V^{\otimes j}$, and we set $\psi^{(j+1)}$ to be the natural map

$$\psi^{(j+1)} \colon M^{(j+1)} \to M^{(j)}$$

given by $\varphi^{(j)} \otimes \mathrm{id}_V^{\otimes j}$, then we may compose the map $\psi^{(j+1)}$ with the map $\psi^{(j)}$.

Let $M := M^{(0)} = M(d)$, and let $M' := M^{(r)} = M(d') \otimes V^{\otimes r}$. We then have an equivariant map $\psi := \psi^{(1)} \circ \cdots \circ \psi^{(r)} \colon M' \to M$, and we must finally show that ψ is nonzero. Let $F_\bullet^{(j)}$ be the minimal free resolution of $M^{(j)}$. Since $d_k = d'_k$, it follows that $d_k^{(j)} = d_k^{(j+1)}$ for all j. Lemma 2.4.2 then implies that we can choose each $\varphi^{(j+1)}$ such that the map $\psi^{(j+1)}$ induces a surjection $F_k^{(j+1)} \to F_k^{(j)}$. Since the composition of surjective maps is surjective, it follows that the map $F_k^{(r)} \to F_k^{(0)}$ induced by ψ is surjective. Since $F_\bullet^{(0)}$ is a minimal free resolution, we conclude that the map of complexes $F_\bullet^{(r)} \to F_\bullet^{(0)}$ is not null-homotopic, and hence $\psi \colon M' \to M$ is a nonzero morphism. □

Remark 2.4.7. By introducing a variant of Lemma 2.4.2, we may simplify the construction used in the proof of Theorem 2.4.1. Let d and d' be two degree sequences such that $d'_i = d_i + N$, and $d'_j = d_j$ for all $j \neq i$. Iteratively applying Lemma 2.4.2 yields a morphism $\varphi \colon M(d') \otimes V^{\otimes N} \to M(d)$. Since $\mathrm{char}(K) = 0$, we have an inclusion $\iota \colon \mathrm{Sym}^N V \to V^{\otimes N}$, and we let ψ be the morphism induced by composing φ and $\mathrm{id}_{M(d')} \otimes \iota$. Let $F'_\bullet$ and $F_\bullet$ be the minimal free resolutions of $M(d') \otimes \mathrm{Sym}^N V$ and $M(d)$ respectively. The map $F'_j \to F_j$ induced by ψ is induced by the equivariant map of vector spaces

$$\mathbf{S}_{\lambda(d')} V \otimes \mathrm{Sym}^N V \to \mathbf{S}_{\lambda(d)} V.$$

This map is surjective because it is a projection onto one of the factors in the Pieri rule decomposition of $\mathbf{S}_{\lambda(d')} V \otimes \mathrm{Sym}^N V$.

This simplifies the proof of Theorem 2.4.1 as follows. Let $i_1 > \cdots > i_\ell$ be the indices for which d and d' differ. By iteratively applying the construction outlined in this remark, we may construct the desired modules and nonzero morphism in ℓ steps. Since ℓ can be far smaller than $r := \sum_{j=0}^{n} d'_j - d_j$, this variant is useful for computing examples such as Example 2.4.8. □

Example 2.4.8. We illustrate Theorem 2.4.1 with $n = 4$, $d = (0, 2, 3, 6, 7)$, and $d' = (1, 2, 5, 6, 10)$. Using the notation of Remark 2.4.7, $d^{(1)} = (0, 2, 3, 6, 10)$, $d^{(2)} = (0, 2, 5, 6, 10)$. Following the same conventions as in Example 2.4.6, the corresponding resolutions are given in Figure 2-2. Notice that $d_3 = 6 = d'_3$. Focusing on the third terms of the resolutions, we see that the maps are simply projections from Pieri's rule. In particular, these maps are surjective and therefore nonzero. □

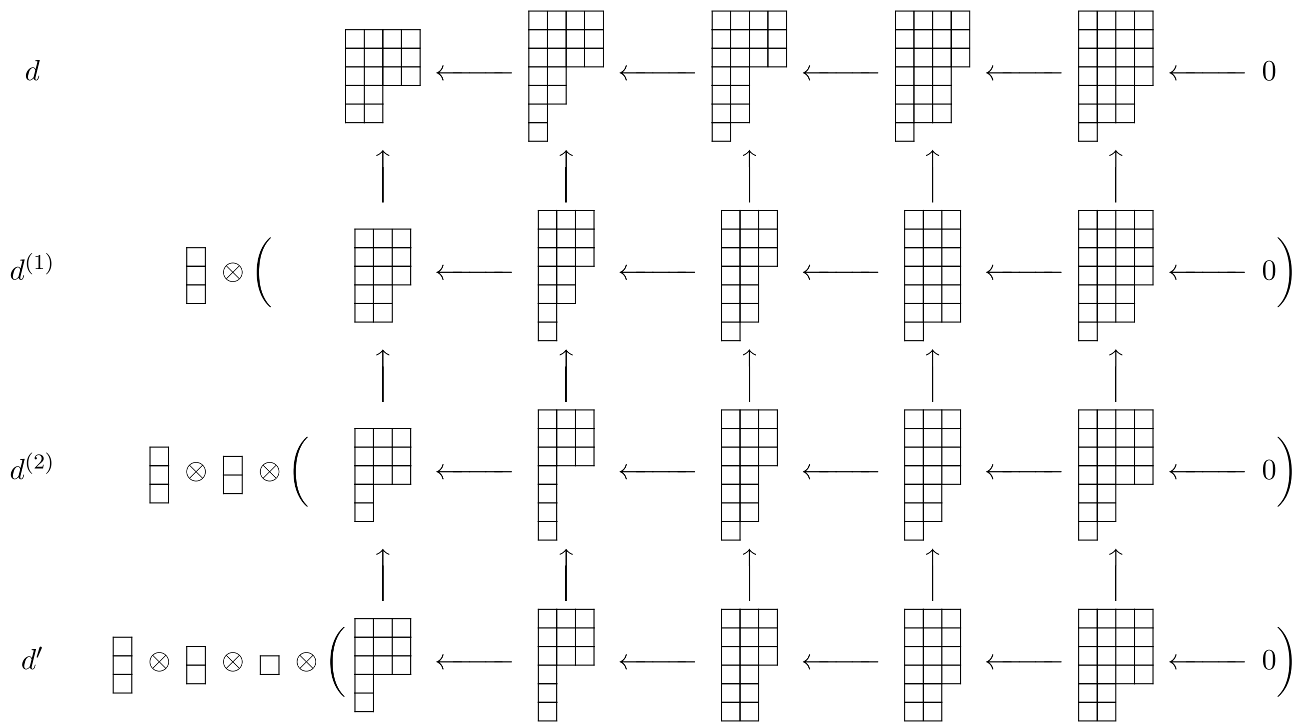


Figure 2-2: The Young diagram depictions of the resolutions in Example 2.4.8.

2.5 The poset of root sequences

Let $\mathcal{E}$ be a coherent sheaf on $\mathbf{P}^{n-1}$. The **cohomology table** of $\mathcal{E}$ is a table with rows indexed by $\{0, \dots, n-1\}$ and columns indexed by $\mathbf{Z}$, such that the entry in row i and column j is $\dim_K \mathrm{H}^i(\mathbf{P}^{n-1}, \mathcal{E}(j-i))$. A sequence $f = (f_1, \dots, f_{n-1}) \in (\mathbf{Z} \cup \{-\infty\})^{n-1}$ is called a **root sequence** for $\mathbf{P}^{n-1}$ if $f_i < f_{i-1}$ for all i (with the convention that $-\infty < -\infty$). The **length** of f, denoted $\ell(f)$, is the largest integer t such that f_t is finite.

Definition 2.5.1. Let f be a root sequence for $\mathbf{P}^{n-1}$. A sheaf $\mathcal{E}$ on $\mathbf{P}^{n-1}$ is **supernatural of type** $f = (f_1, \dots, f_{n-1})$ if the following are satisfied:

1. The dimension of $\operatorname{Supp} \mathcal{E}$ is $\ell(f)$.
2. For all $j \in \mathbb{Z}$, there exists at most one i such that $\dim_K \mathrm{H}^i(\mathbf{P}^{n-1}, \mathcal{E}(j)) \neq 0$.
3. The Hilbert polynomial of $\mathcal{E}$ has roots $f_1, \dots, f_{\ell(f)}$.

Dropping the reference to its root sequence, we also say that $\mathcal{E}$ is a **supernatural sheaf** (or a **supernatural vector bundle** if it is locally free).

For every root sequence f, there exists a supernatural sheaf of type f [ES2, Theorem 0.4]. Moreover, the cohomology table of any coherent sheaf can be written as a positive real combination of cohomology tables of supernatural sheaves [ES3, Theorem 0.1]. The **cone of cohomology tables** for $\mathbf{P}^{n-1}$ is the convex cone inside $\prod_{j \in \mathbf{Z}} \mathbf{R}^n$ generated by cohomology tables of coherent sheaves on $\mathbf{P}^{n-1}$. Each root sequence f corresponds to a unique extremal ray of this cone, which we denote by ρ_f, and every extremal ray is of the form ρ_f for some root sequence f.

Definition 2.5.2. For two root sequences f and f', we say that $f \preceq f'$ and that $\rho_f \preceq \rho_{f'}$ if $f_i \leq f'_i$ for all i.

This partial order induces a simplicial fan structure on the cone of cohomology tables, where simplices correspond to chains of root sequences under the partial order $\preceq$. We now show that the existence of a nonzero homomorphism between two supernatural sheaves implies the comparability of their corresponding root sequences, which provides the reverse implications for Theorems 2.1.2 and 2.1.4.

Proposition 2.5.3. *Let $\mathcal{E}$ and $\mathcal{E}'$ be supernatural sheaves of types f and f' respectively. If $\operatorname{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$, then $f \preceq f'$.*

Proof. Let $\mathrm{T}(\mathcal{E})$ and $\mathrm{T}(\mathcal{E}')$ denote the Tate resolutions of $\mathcal{E}$ and $\mathcal{E}'$ [EFS, §4]. These are doubly infinite acyclic complexes over the exterior algebra Λ, which is Koszul dual to S and has generators in degree -1. Since $\operatorname{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$, there is a map $\varphi \colon \mathrm{T}(\mathcal{E}') \to \mathrm{T}(\mathcal{E})$ that is not null-homotopic. Observe that for every cohomological degree j, $\varphi^j \colon \mathrm{T}(\mathcal{E}')^j \to \mathrm{T}(\mathcal{E})^j$ is nonzero. First, if $\varphi^j = 0$ for some j, then, we may take $\varphi^k = 0$ for all $k < j$. Secondly, if $k > j$, then after applying $\operatorname{Hom}_\Lambda(-, \Lambda)$ (which is exact because Λ is self-injective), we can take φ^k to be zero.

By [ES2, Theorem 6.4], we see that all the minimal generators of $T(\mathcal{E})^j$ (respectively, $T(\mathcal{E}')^j$) are of a single degree i (respectively, i'). (This is equivalent to stating that every column of the cohomology table of $\mathcal{E}$ and $\mathcal{E}'$ contains precisely one nonzero entry.) Since φ^j is nonzero and Λ is generated in elements of degree -1, we see that $i' \leq i$. Now, again by [ES2, Theorem 6.4], $f \preceq f'$. □

2.6 Construction of morphisms between supernatural sheaves

The goal of this section is to prove Theorem 2.6.1, which provides the forward direction of Theorem 2.1.2.

Theorem 2.6.1. *Let $f \preceq f'$ be two root sequences. Then there exist supernatural sheaves $\mathcal{E}$ and $\mathcal{E}'$ of types f and f', respectively, with* $\operatorname{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.

For the purposes of exposition, we separate the proof of Theorem 2.6.1 into two cases (with $\ell(f) = \ell(f')$ and with $\ell(f) < \ell(f')$), and handle these cases in Propositions 2.6.8 and 2.6.12 respectively. Examples 2.6.4 and 2.6.9 illustrate the essential ideas behind the proof in each case.

If $\ell(f) < n-1$, then we call $(f_1, \dots, f_{\ell(f)})$ the **truncation** of f, and write $\tau(f)$. Let $f = (f_1, \dots, f_{n-1})$ be a root sequence with $\ell(f) = s$. Denote the s-fold product of $\mathbf{P}^1$ by $\mathbf{P}^{1\times s}$. Fix homogeneous coordinates

$$\left([y_0^{(1)} : y_1^{(1)}], \dots, [y_0^{(s)} : y_1^{(s)}]\right) \quad \text{on} \quad \mathbf{P}^{1\times s}. \tag{2.6.2}$$

In order to produce a supernatural sheaf of type f on $\mathbf{P}^{n-1}$, we first construct a supernatural vector bundle of type $\tau(f)$ on $\mathbf{P}^s$. Its image under an embedding of $\mathbf{P}^s$ as a linear subvariety $\mathbf{P}^{n-1}$ will give the desired supernatural sheaf.

We now outline our approach to construct a nonzero map between supernatural sheaves on $\mathbf{P}^s$ of types $f \preceq f'$ in the case that $\ell(f) = \ell(f') = s$. This uses the proof of [ES2, Theorem 6.1].

1. Construct a finite map $\pi \colon \mathbf{P}^{1\times s} \to \mathbf{P}^s$.
2. Choose appropriate line bundles $\mathcal{L}$ and $\mathcal{L}'$ on $\mathbf{P}^{1\times s}$ so that $\pi_*\mathcal{L}$ and $\pi_*\mathcal{L}'$ are supernatural vector bundles of the desired types.
3. When $\ell(f) = \ell(f') = s$, construct a morphism $\mathcal{L}' \xrightarrow{\varphi} \mathcal{L}$ such that $\pi_*\varphi$ is nonzero.

For (1), we use the multilinear $(1, \dots, 1)$-forms

$$g_p := \sum_{i_1 + \cdots + i_s = p} \left(\prod_{j=1}^{s} y_{i_j}^{(j)} \right) \qquad \text{for} \quad p = 0, \dots, s \tag{2.6.3}$$

on $\mathbf{P}^{1\times s}$ to define the map $\pi \colon \mathbf{P}^{1\times s} \to \mathbf{P}^s$ via $[g_0 : \cdots : g_s]$. For (2), with $\mathbf{1} := (1, \dots, 1) \in \mathbf{Z}^s$,

$$\mathcal{E}_f := \pi_* \left(\mathcal{O}_{\mathbf{P}^{1\times s}}(-f - \mathbf{1}) \right)$$

is a supernatural vector bundle of type $\tau(f)$ on $\mathbf{P}^s$ of rank $s!$ (the degree of π). The next example illustrates (3).

Example 2.6.4. Here we find a nonzero morphism $\mathcal{E}_{f'} \to \mathcal{E}_f$ that is the direct image of a morphism of line bundles on $\mathbf{P}^{1\times(n-1)}$. Let $n = 5$ and $f := (-2, -3, -4, -5) \preceq f' := (-1, -2, -3, -4)$. The map $\pi \colon \mathbf{P}^{1\times 4} \to \mathbf{P}^4$ is finite of degree $4! = 24$. Following steps (1) and (2) as outlined above, we set $\mathcal{E} := \mathcal{E}_f = \pi_*\mathcal{O}_{\mathbf{P}^{1\times 4}}(1,2,3,4)$ and $\mathcal{E}' := \mathcal{E}_{f'} = \pi_*\mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3)$. There is a natural inclusion

$$\pi_*\mathcal{H}om_{\mathbf{P}^{1\times 4}} \left(\mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3), \mathcal{O}_{\mathbf{P}^{1\times 4}}(1,2,3,4) \right) \subseteq \mathcal{H}om_{\mathbf{P}^4} \left(\mathcal{E}', \mathcal{E} \right), \tag{2.6.5}$$

which induces an inclusion of global sections (see Remark 2.6.6). Therefore

$$\begin{aligned} \operatorname{Hom}(\mathcal{E}', \mathcal{E}) &\supseteq \mathrm{H}^0 \left(\mathbf{P}^4, \pi_*\mathcal{H}om_{\mathbf{P}^{1\times 4}} \left(\mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3), \mathcal{O}_{\mathbf{P}^{1\times 4}}(1,2,3,4) \right) \right) \\ &= \mathrm{H}^0(\mathbf{P}^{1\times 4}, \mathcal{O}_{\mathbf{P}^{1\times 4}}(1,1,1,1)) \\ &\simeq K^{16}. \end{aligned}$$

We thus conclude that $\operatorname{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.

The inclusion (2.6.5) is strict. Note that, by definition, neither $\mathcal{E}'$ nor $\mathcal{E}$ has intermediate cohomology, and hence, by Horrocks' Splitting Criterion, both $\mathcal{E}$ and $\mathcal{E}'$ must split as the sum of line bundles. Thus $\mathcal{E}' = \mathcal{O}_{\mathbf{P}^4}^{24}$ and $\mathcal{E} = \mathcal{O}_{\mathbf{P}^4}(1)^{24}$, and it follows that $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) = \mathrm{H}^0(\mathbf{P}^4, \mathcal{O}(1)^{576}) \simeq K^{2880}$. $\square$

Remark 2.6.6. Let $\pi\colon \mathbf{P}^{1\times s} \to \mathbf{P}^s$ be as in (1). For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbf{P}^{1\times s}$, we have

$$\pi_* \mathcal{H}om_{\mathcal{O}_{\mathbf{P}^{1\times s}}}(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{H}om_{\mathcal{O}_{\mathbf{P}^s}}(\pi_*\mathcal{F}, \pi_*\mathcal{G}).$$

Indeed, this can be checked locally. Let $U \subseteq \mathbf{P}^s$ be an affine open subset, and write $A = \mathrm{H}^0(U, \mathcal{O}_{\mathbf{P}^s})$ and $B = \mathrm{H}^0(U, \pi_*\mathcal{O}_{\mathbf{P}^{1\times s}})$. For all B-modules M and N, every nonzero B-module homomorphism is also a nonzero A-module homomorphism via the map $A \to B$. Injectivity is immediate. $\square$

Remark 2.6.7. Suppose that $\beta\colon \mathbf{P}^s \to \mathbf{P}^{n-1}$ is a closed immersion as a linear subvariety. Let $\mathcal{E}$ be a coherent sheaf on $\mathbf{P}^s$. It follows from the projection formula and from the finiteness of β that $\mathcal{E}$ is a supernatural sheaf on $\mathbf{P}^s$ of type $(f_1, \dots, f_s)$ if and only if $\beta_*\mathcal{E}$ is a supernatural sheaf on $\mathbf{P}^{n-1}$ of type $(f_1, \dots, f_s, -\infty, \dots, -\infty)$. $\square$

Proposition 2.6.8. *If $\ell(f) = \ell(f')$, then Theorem 2.6.1 holds.*

Proof. We first reduce to the case $\ell(f') = n-1$. Let $\beta\colon \mathbf{P}^{\ell(f')} \to \mathbf{P}^{n-1}$ be a closed immersion as a linear subvariety. Let $\ell(f') = s$ and write $f = (f_1, \dots, f_s, -\infty, \dots, -\infty)$ and $f' = (f'_1, \dots, f'_s, -\infty, \dots, -\infty)$. Assume that $\mathcal{E}$ and $\mathcal{E}'$ are supernatural sheaves of type $(f_1, \dots, f_s)$ and $(f'_1, \dots, f'_s)$ on $\mathbf{P}^s$ and that $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$. Then, by Remark 2.6.7, $\beta_*\mathcal{E}$ and $\beta_*\mathcal{E}'$ are supernatural sheaves of types f and f', and $\mathrm{Hom}(\beta_*\mathcal{E}', \beta_*\mathcal{E}) \neq 0$.

We may thus assume that $\ell(f') = n-1$. Let $\mathbf{1} := (1, \dots, 1) \in \mathbf{Z}^{n-1}$. Let $\pi\colon \mathbf{P}^{1\times(n-1)} \to \mathbf{P}^{n-1}$ be the morphism given by the forms g_p defined in (2.6.3) (with $s = n-1$). Let $\mathcal{E} := \mathcal{E}_f = \pi_*\mathcal{O}(-f-\mathbf{1})$ and $\mathcal{E}' := \mathcal{E}_{f'} = \pi_*\mathcal{O}(-f'-\mathbf{1})$. Remark 2.6.6 shows that

$$\mathrm{H}^0\left(\mathbf{P}^{n-1}, \pi_*\mathcal{H}om_{\mathbf{P}^{1\times(n-1)}}\left(\mathcal{O}(-f'-\mathbf{1}), \mathcal{O}(-f-\mathbf{1})\right)\right) \subseteq \mathrm{Hom}_{\mathbf{P}^{n-1}}(\mathcal{E}', \mathcal{E}).$$

Note that $\mathcal{H}om_{\mathbf{P}^{1\times(n-1)}}(\mathcal{O}(-f'-\mathbf{1}), \mathcal{O}(-f-\mathbf{1})) = \mathcal{O}(f'-f)$. Since $f \preceq f'$, we have that $\mathrm{H}^0(\mathbf{P}^{1\times(n-1)}, \mathcal{O}(f'-f)) \neq 0$, and thus $\mathrm{Hom}_{\mathbf{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$. $\square$

When $\ell(f) < \ell(f')$, the supernatural sheaves constructed using (1) and (2) above have supports of different dimensions. Before addressing this general case, we provide an example.

Example 2.6.9. Let $n = 5$ and $f = (-2, -3, -4, -\infty) \preceq f' = (-1, -2, -3, -4)$, so that $\ell(f) = 3 < \ell(f') = 4 = n-1$. We proceed by modifying steps (1)-(3) above.

i′. We extend the construction of (1) to the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^{1\times 3} & \xrightarrow{\alpha} & \mathbf{P}^{1\times 4} \\ \downarrow{\scriptstyle \pi^{(3)}} & & \downarrow{\scriptstyle \pi^{(4)}} \\ \mathbf{P}^3 & \xrightarrow{\beta} & \mathbf{P}^4. \end{array}$$

ii′. Choose appropriate line bundles $\mathcal{L}$ on $\mathbf{P}^{1\times 3}$ and $\mathcal{L}'$ on $\mathbf{P}^{1\times 4}$, so that $\pi^{(3)}_*\mathcal{L}$ and $\pi^{(4)}_*\mathcal{L}'$ are supernatural sheaves of the desired types.

iii′. Construct a morphism $\mathcal{L}' \xrightarrow{\varphi} \alpha_*\mathcal{L}$ such that $\pi^{(4)}_*\varphi$ is nonzero.

For (i′), we use the homogeneous coordinates from (2.6.2). The maps $\pi^{(3)}$ and $\pi^{(4)}$ are instances of the map π from (1) for $\mathbf{P}^{1\times 3}$ and $\mathbf{P}^{1\times 4}$, respectively. Define a closed immersion $\alpha\colon \mathbf{P}^{1\times 3} \to \mathbf{P}^{1\times 4}$ by the vanishing of the coordinate $y_1^{(4)}$. Fix coordinates $x_0, \dots, x_4$ for $\mathbf{P}^4$, and let $\beta\colon \mathbf{P}^3 \to \mathbf{P}^4$ be the closed immersion given by the vanishing of x_4. We now have that the diagram in (i$'$) is indeed commutative.

In (ii$'$), we take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{1\times 3}}(1,2,3)$ and $\mathcal{L}' = \mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3)$ and set $\mathcal{E}_f = \pi^{(3)}_* \mathcal{L}$ and $\mathcal{E}_{f'} = \pi^{(4)}_* \mathcal{L}'$. Set $\mathcal{E} := \beta_* \mathcal{E}_f$ and $\mathcal{E}' := \mathcal{E}_{f'}$. Then $\mathcal{E}$ is a supernatural sheaf on $\mathbf{P}^4$ (see Remark 2.6.7), and

$$\operatorname{Hom}_{\mathbf{P}^4}(\mathcal{E}', \mathcal{E}) = \mathrm{H}^0\left(\mathbf{P}^4, \mathcal{H}om\left(\pi^{(4)}_*\left(\mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3)\right), \pi^{(4)}_*\left(\alpha_* \mathcal{O}_{\mathbf{P}^{1\times 3}}(1,2,3)\right)\right)\right).$$

By Remarks 2.6.6 and 2.6.10, we obtain the containment

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{P}^4}(\mathcal{E}', \mathcal{E}) &\supseteq \mathrm{H}^0\left(\mathbf{P}^4, \pi^{(4)}_* \mathcal{H}om\left(\mathcal{O}_{\mathbf{P}^{1\times 4}}(0,1,2,3), \alpha_* \mathcal{O}_{\mathbb{P}^{1\times 3}}(1,2,3)\right)\right) \\
&\cong \mathrm{H}^0\left(\mathbf{P}^{1\times 4}, \mathcal{H}om\left(\mathcal{O}_{\mathbb{P}^{1\times 4}}(0,1,2,3), \alpha_* \mathcal{O}_{\mathbb{P}^{1\times 3}}(1,2,3)\right)\right) \\
&\cong \mathrm{H}^0\left(\mathbf{P}^{1\times 4}, \left(\alpha_* \mathcal{O}_{\mathbb{P}^{1\times 3}}(1,1,1)\right)(0,0,0,-3)\right) \\
&\cong \mathrm{H}^0\left(\mathbf{P}^{1\times 4}, \alpha_* \mathcal{O}_{\mathbb{P}^{1\times 3}}(1,1,1)\right) \cong K^8.
\end{aligned}$$

In particular, $\operatorname{Hom}_{\mathbf{P}^4}(\mathcal{E}', \mathcal{E}) \neq 0$, as desired. □

Remark 2.6.10. Let $1 \le s < t$, and let $\alpha\colon \mathbf{P}^{1\times s} \to \mathbf{P}^{1\times t}$ be the embedding given by the vanishing of $y_1^{(s+1)}, \dots, y_1^{(t)}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^{1\times s}$ and $b \in \mathbf{Z}^{t-s}$. Write $\mathbf{0}_s$ for the 0-vector in $\mathbf{Z}^s$. Then

$$\mathrm{H}^i\left(\mathbf{P}^{1\times t}, (\alpha_*\mathcal{F})(\mathbf{0}_s, b)\right) \cong \mathrm{H}^i\left(\mathbf{P}^{1\times t}, \alpha_*\mathcal{F}\right) \cong \mathrm{H}^i\left(\mathbf{P}^{1\times s}, \mathcal{F}\right) \tag{2.6.11}$$

The first isomorphism follows from the projection formula, taken along with the fact that, by the definition of α, the line bundle $\mathcal{O}_{\mathbf{P}^{1\times t}}(\mathbf{0}_s, b)$ is trivial when restricted to the support of $\alpha_*\mathcal{F}$ (which is contained in $\mathbf{P}^{1\times s}$). The second isomorphism holds because α is a finite morphism. □

Proposition 2.6.12. *If $\ell(f) < \ell(f')$, then Theorem 2.6.1 holds.*

Proof. We may reduce to the case $\ell(f') = n-1$ by the same argument as in the beginning of the proof of Proposition 2.6.8.

Let $s = \ell(f)$ and consider the line bundles $\mathcal{L} = \mathcal{O}_{\mathbf{P}^{1\times s}}(-\tau(f) - \mathbf{1})$ on $\mathbf{P}^{1\times s}$ and $\mathcal{L}' = \mathcal{O}_{\mathbf{P}^{1\times(n-1)}}(-f' - \mathbf{1})$ on $\mathbf{P}^{1\times(n-1)}$. Let $\pi\colon \mathbf{P}^{1\times s} \to \mathbf{P}^s$ and $\pi'\colon \mathbf{P}^{1\times(n-1)} \to \mathbf{P}^{n-1}$ be the maps defined by the forms in (2.6.3). Let $\mathcal{E}_f = \pi_*\mathcal{L}$ and $\mathcal{E}_{f'} = (\pi')_*\mathcal{L}'$, and define the closed immersion $\alpha\colon \mathbf{P}^{1\times s} \to \mathbf{P}^{1\times(n-1)}$ by the vanishing of the coordinates $y_1^{(s+1)}, \dots, y_1^{(n-1)}$. Fix coordinates $x_0, \dots, x_{n-1}$ for $\mathbf{P}^{n-1}$, and let $\beta\colon \mathbf{P}^s \to \mathbf{P}^{n-1}$ be the closed immersion given by the vanishing of $x_{s+1}, \dots, x_{n-1}$. This yields the commutative diagram

$$\begin{array}{ccc}
\mathbf{P}^{1\times s} & \xrightarrow{\alpha} & \mathbf{P}^{1\times(n-1)} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
\mathbf{P}^s & \xrightarrow{\beta} & \mathbf{P}^{n-1}.
\end{array}$$

By Remark 2.6.7, $\mathcal{E} := \beta_*\mathcal{E}_f$ is a supernatural sheaf of type f. Also, $\mathcal{E}' := \mathcal{E}_{f'}$ is a supernatural sheaf of type f'.

We must show that $\operatorname{Hom}_{\mathbf{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$. It suffices to show that $\operatorname{Hom}_{\mathbf{P}^{1\times(n-1)}}(\mathcal{L}', \alpha_*\mathcal{L}) \neq 0$ by Remark 2.6.6. To see this, let $c := (f'_1, \dots, f'_s)$ and $b := (-f'_{s+1} - 1, \dots, -f'_{s'} - 1)$, and note that

$$\begin{aligned}
\mathcal{H}om(\mathcal{L}', \alpha_*\mathcal{L}) &= \mathcal{H}om(\mathcal{O}_{\mathbf{P}^{1\times(n-1)}}(-f' - \mathbf{1}), \alpha_*\mathcal{O}_{\mathbf{P}^{1\times s}}(-\tau(f) - \mathbf{1})) \\
&\cong (\alpha_*\mathcal{O}_{\mathbf{P}^{1\times s}}(c - \tau(f)))(\mathbf{0}_s, -b).
\end{aligned}$$

By Remark 2.6.10, $\mathrm{Hom}(\mathcal{L}', \alpha_* \mathcal{L}) = \mathrm{H}^0(\mathbf{P}^{1\times s}, \mathcal{O}(c - \tau(f)))$, which is nonzero as $\tau(f) \preceq c$. □

2.7 Equivariant construction of morphisms between supernatural sheaves

Throughout this section, we assume that K is a field of characteristic 0 and that all root sequences have length $n-1$. Let V be an n-dimensional K-vector space, identify $\mathbf{P}^{n-1}$ with $\mathbf{P}(V)$, and let $\mathcal{Q}$ denote the tautological quotient bundle of rank $n-1$ on $\mathbf{P}(V)$. We have a short exact sequence

$$0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{Q} \to 0.$$

We will use the fact that $\det \mathcal{Q} \cong \mathcal{O}(1) \otimes \bigwedge^n V$ is a $\mathbf{GL}(V)$-equivariant isomorphism. For a weakly decreasing sequence λ of non-negative integers, we let $\mathbf{S}_\lambda$ denote the corresponding Schur functor. See [Wey, Chapter 2] for more details (since we are working in characteristic 0, the functors $\mathbf{K}_\lambda$ and $\mathbf{L}_{\lambda^t}$ are isomorphic, where λ^t is the transpose partition of λ, and we call this $\mathbf{S}_\lambda$). We extend this definition to weakly decreasing sequences λ with possibly negative entries as follows. Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{n-1}$ and define $\mathbf{S}_\lambda \mathcal{Q} := \mathbf{S}_{\lambda - \lambda_{n-1}\mathbf{1}} \mathcal{Q} \otimes (\det \mathcal{Q})^{\lambda_{n-1}}$.

Proof of Theorem 2.1.4. The reverse implication has been shown in Proposition 2.5.3. For the forward implication, we proceed in two steps. First, we construct equivariant supernatural bundles $\mathcal{E}'$ and $\mathcal{E}$ with $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$ using the construction in the proof of [ES2, Theorem 6.2]. Second, we use this fact to construct a new supernatural bundle $\mathcal{E}''$ of type f' such that $\mathrm{Hom}_{\mathbf{GL}(V)}(\mathcal{E}'', \mathcal{E}) \neq 0$. Thus we will ignore powers of the trivial bundle $\bigwedge^n V$ that appear in the first step.

Write $N_i = f'_i - f_i$ and let $\lambda \in \mathbb{Z}^{n-1}$ be the partition defined by

$$\lambda_i := f_1 - f_{n-i} - n + 1 + i \quad \text{for } 1 \le i \le n-1.$$

Let λ' be the sequence of weakly decreasing integers defined by $\lambda'_{n-i} := \lambda_{n-i} - N_i$ and set

$$\mathcal{E} := \mathbf{S}_\lambda \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1) \quad \text{and} \quad \mathcal{E}' := \mathbf{S}_{\lambda'} \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1).$$

Observe that $\mathbf{S}_{\lambda'} \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1) \cong \mathbf{S}_{\lambda' + N_1 \cdot \mathbf{1}} \mathcal{Q} \otimes \mathcal{O}(-f'_1 - 1)$. Hence by the Borel–Weil–Bott theorem [Wey, Corollary 4.1.9], $\mathcal{E}$ and $\mathcal{E}'$ are supernatural vector bundles of types f and f', respectively.

To compute $\mathrm{Hom}(\mathcal{E}', \mathcal{E})$, let $\lambda'' := \lambda' + N_1 \cdot \mathbf{1}$. Define λ^c to be the complement of λ inside of the $(n-1) \times \lambda_1$ rectangle, so $\lambda^c_j = \lambda_1 - \lambda_{n-j}$ for $1 \le j \le n-1$. Then $\mathbf{S}_\lambda \mathcal{Q} \cong \mathbf{S}_{\lambda^c} \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1)$ by [Wey, Exercise 2.18]. We then obtain

$$\mathcal{H}om(\mathcal{E}', \mathcal{E}) \cong \mathbf{S}_{\lambda'} \mathcal{Q}^* \otimes \mathbf{S}_\lambda \mathcal{Q} \cong \mathbf{S}_{\lambda''} \mathcal{Q}^* \otimes \mathbf{S}_{\lambda^c} \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1 + N_1)$$

and seek to show that this bundle has a nonzero global section.

Fix μ so that $\mathbf{S}_\mu \mathcal{Q}^*$ is a direct summand of $\mathbf{S}_{\lambda''} \mathcal{Q}^* \otimes \mathbf{S}_{\lambda^c} \mathcal{Q}^*$. The Borel–Weil–Bott Theorem [Wey, Corollary 4.1.9] shows that $\mathbf{S}_\mu \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1 + N_1)$ has nonzero sections if and only if $\lambda_1 + N_1 \ge \mu_1$. This is equivalent to μ being inside of a $(n-1) \times (\lambda_1 + N_1)$ rectangle. By [F2, §9.4], the existence of such a μ is equivalent to the condition

$$\lambda''_i + \lambda^c_{n-i} \le \lambda_1 + N_1 \quad \text{for } i = 1, \dots, n-1. \tag{2.7.1}$$

Since $\lambda''_i + \lambda^c_{n-i} = \lambda_1 + N_1 - N_{n-i}$, we see that (2.7.1) holds for all i, and thus $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.

For the second step, replace $\mathcal{E}'$ by $\mathcal{E}'' := \mathcal{E}' \otimes \operatorname{Hom}(\mathcal{E}', \mathcal{E})$, where we view $\operatorname{Hom}(\mathcal{E}', \mathcal{E})$ as a trivial bundle over $\mathbf{P}(V)$. Note that

$$\mathrm{H}^i(\mathbf{P}(V), \mathcal{E}''(j)) \cong \mathrm{H}^i(\mathbf{P}(V), \mathcal{E}'(j)) \otimes \operatorname{Hom}(\mathcal{E}', \mathcal{E})$$

for all i, j, and hence $\mathcal{E}''$ is also supernatural of type f'. The space of sections $\operatorname{Hom}(\mathcal{E}'', \mathcal{E})$ is $\operatorname{Hom}(\mathcal{E}', \mathcal{E})^* \otimes \operatorname{Hom}(\mathcal{E}', \mathcal{E})$, which contains the $\mathbf{GL}(V)$-invariant section corresponding to the evaluation map. This gives a nonzero $\mathbf{GL}(V)$-equivariant map $\mathcal{E}'' \to \mathcal{E}$. □

Example 2.7.2. We reconsider Example 2.6.4 in the equivariant context. Here we will not ignore powers of $\bigwedge^n V$. Let $n = 4$ and $f = (-2, -3, -4, -5) \preceq f' = (-1, -2, -3, -4)$. With notation as in the proof of Theorem 2.1.4, we have $N = (1, 1, 1, 1)$, $\lambda = (0, 0, 0, 0)$, $\lambda' = (-1, -1, -1, -1)$,

$$\begin{aligned} \mathcal{E} &= \mathbf{S}_{(0,0,0,0)}\mathcal{Q} \otimes \mathcal{O}(2-1) = \mathcal{O}(1), \quad \text{and} \\ \mathcal{E}' &= \mathbf{S}_{(-1,-1,-1,-1)}\mathcal{Q} \otimes \mathcal{O}(2-1) = \left(\mathcal{O}(-1) \otimes \left(\bigwedge^n V\right)^{-1}\right) \otimes \mathcal{O}(1) = \left(\bigwedge^n V\right)^{-1} \otimes \mathcal{O}. \end{aligned}$$

Since $\lambda^c = (0, 0, 0, 0) = \lambda''$, we see that

$$\mathcal{H}om(\mathcal{E}', \mathcal{E}) \cong \mathcal{O}(1) \otimes \bigwedge^n V,$$

which certainly has nonzero global sections. In fact, $\operatorname{Hom}(\mathcal{E}', \mathcal{E}) \cong V \otimes \bigwedge^n V$. Note, however, that this implies that there is no nonzero equivariant morphism from $\mathcal{E}'$ to $\mathcal{E}$. We thus set $\mathcal{E}'' := \mathcal{E} \otimes \operatorname{Hom}(\mathcal{E}', \mathcal{E})$. Then $\operatorname{Hom}(\mathcal{E}'', \mathcal{E}) \cong V^* \otimes V$, and our desired nonzero equivariant morphism is given by the trace element. □

2.8 Remarks on other graded rings

Given any graded ring R, one could try to use an analog of Theorem 2.1.1 to induce a partial order on the extremal rays of the cone of Betti diagrams over R. This application has already proven useful in a couple of the other cases where Boij–Söderberg has been studied. In this section, we provide a sketch of some of these applications.

Example 2.8.1. We first consider an example involving hypersurface rings over $K[x, y]$. Let $f \in K[x, y]$ be a quadric polynomial, and set $R := K[x, y]/\langle f \rangle$. The cone of Betti diagrams over R is described in detail in [BBEG]. The extremal rays still correspond to Cohen–Macaulay modules with pure resolutions, though some of the degrees are infinite in length.

(i) *Finite pure resolutions.* For example, if h is a degree 7 polynomial that is not divisible by f, then the free resolution of $R/\langle h \rangle$ is

$$R \leftarrow R(-7) \leftarrow 0.$$

Following the notation of Section 2.2, we denote such a resolution by its corresponding degree sequence, i.e., $(0, 7, \infty, \infty, \dots)$.

(ii) *Infinite pure resolutions.* For example, the free resolution of the R-module $R/\langle x, y \rangle$ is

$$R \leftarrow R^2(-1) \leftarrow R^2(-2) \leftarrow R^2(-3) \leftarrow \cdots.$$

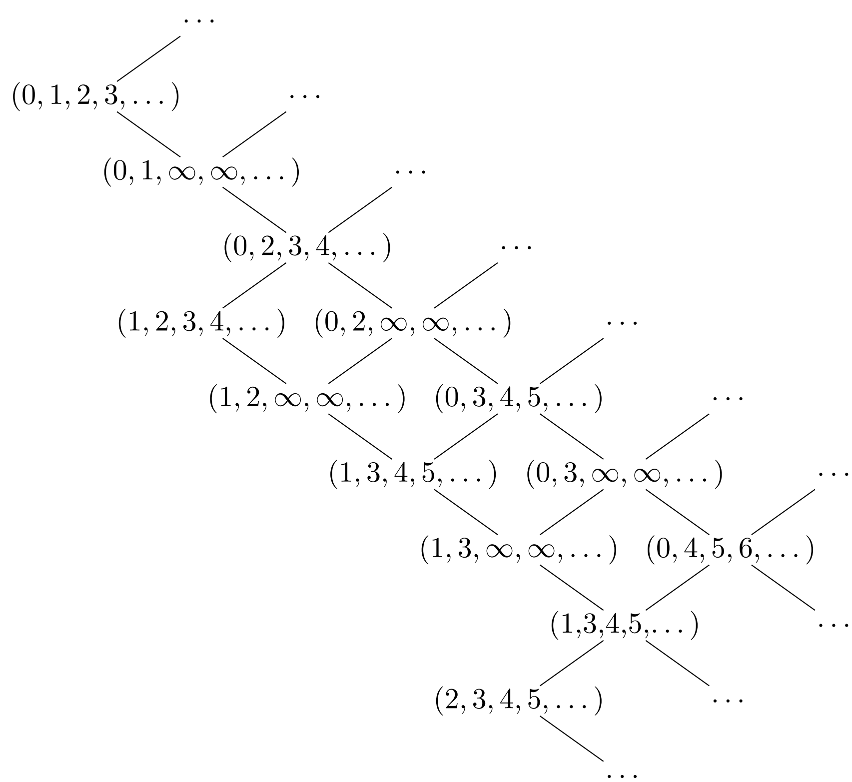


Figure 2-3: For the hypersurface ring R, this partial order provides a simplicial fan structure, as illustrated in [BBEG] and discussed in Example 2.8.1. The partial order is determined by an analog of Theorem 2.1.1.

We denote this by its corresponding degree sequence, i.e., $(0, 1, 2, 3, \dots)$.

There are two possible partial orders for these extremal rays:

- $\rho_d \preceq \rho_{d'}$ if $d_i \leq d'_i$ for all i.
- $\rho_d \preceq \rho_{d'}$ if there exist Cohen–Macaulay R-modules M and M' with pure resolutions of types d and d', respectively, with $\mathrm{Hom}_R(M', M)_{\leq 0} \neq 0$.

In contrast with the case of the polynomial ring, these partial orders are genuinely different. Only the second partial order leads to a greedy algorithm for decomposing Betti diagrams over R, in parallel to [ES2, Decomposition Algorithm]. This also provides an analog of the Multiplicity Conjecture for R. □

Example 2.8.2. We now consider $S = K[x, y]$ with the $\mathbf{Z}^2$-grading $\deg(x) := (1, 0)$ and $\deg(y) := (0, 1)$. In general, the cone of bigraded Betti diagrams over S remains poorly understood. However, portions of this cone have been worked out by the first three authors, and we now provide a brief sketch of these unpublished results.

We restrict attention to the cone of Betti diagrams of finite length S-modules M, where all of the Betti numbers of M are concentrated in bidegrees (a, b) with $0 \leq a, b \leq 2$. The extremal rays of this cone may be realized by quotients of monomial ideals of the form m_1/m_2, where each m_i is a monomial ideal generated by monomials of the form $x^\ell y^k$ with $0 \leq \ell, k \leq 2$. The natural analog of Theorem 2.1.1 induces a partial order on these rays, which also induces a simplicial structure on this cone of bigraded Betti diagrams. □

Chapter 3

Schubert complexes and degeneracy loci

3.1 Introduction.

Let X be an equidimensional Cohen–Macaulay (e.g., nonsingular) variety, and let $\varphi\colon E \to F$ be a map of vector bundles over X, with ranks e and f respectively. Given a number $k \le \min(e, f)$, let $D_k(\varphi)$ be the degeneracy locus of points x where the rank of φ restricted to the fiber of x is at most k. Then $\operatorname{codim} D_k(\varphi) \le (e-k)(f-k)$, and in the case of equality, the Thom–Porteous formula expresses the homology class of $D_k(\varphi)$ as an evaluation of a multi-Schur function at the Chern classes of E and F (see [Man, §3.5.4]). Also in the case of equality, the Schur complex associated with the rectangular partition $(f-k) \times (e-k)$ (see [ABW] or [Wey, §2.4] for more about Schur complexes) of φ is a linear locally free resolution for a Cohen–Macaulay coherent sheaf whose support is $D_k(\varphi)$. This resolution gives a formula in the K-theory of X. In the case that X is smooth, there is an isomorphism from an associated graded of the K-theory of X to the Chow ring of X (see Section 3.4.1 for more details). Then the image of this complex recovers the Thom–Porteous formula, and the complex provides a "linear approximation" of the syzygies of $D_k(\varphi)$.

The situation was generalized by Fulton [F1] as follows. We provide the additional data of a flag of subbundles $E_\bullet$ for E and a flag of quotient bundles $F_\bullet$ for F, and we can define degeneracy loci for an array of numbers which specifies the ranks of the restriction maps $E_p \to F_q$. The rank functions that give rise to irreducible degeneracy loci are indexed by permutations in a natural way. Under the right codimension assumptions, one can express the homology class of a given degeneracy locus as a substitution of a double Schubert polynomial with the Chern classes of the quotients E_i/E_{i-1} and the kernels $\ker(F_j \to F_{j-1})$. The motivation for this work was to complete the analogy of this situation with the previous one by constructing "Schubert complexes" which would be acyclic whenever the degeneracy loci has the right codimension.

Building on the constructions for Schubert functors by Kraśkiewicz and Pragacz of [KP], we construct these complexes over an arbitrary (commutative) ring R from the data of two free R-modules M_0, M_1, with given flags of submodules, respectively, quotient modules, and a map $\partial\colon M_0 \to M_1$. We can also extend the construction to an arbitrary scheme. We show that they are acyclic when a certain ideal defined in terms of minors of ∂ has the right depth, i.e., they are "depth-sensitive." Our main result is that in the situation of Fulton's theorem, the complex is acyclic and the Euler characteristic provides the formula in the same sense as above. Our proof uses techniques from commutative algebra, algebraic geometry, and combinatorics. Again, the complexes are linear and provide a "linear approximation" to the syzygies of Fulton's degeneracy loci. As a special case of Fulton's degeneracy loci, one gets Schubert varieties inside of (type A) partial flag varieties.

Using the work of Fomin, Greene, Reiner, and Shimozono [FGRS], we construct explicit bases for the Schubert complex in the case that M_0 and M_1 are free. This basis naturally extends their notion of balanced labelings and the generating function of the basis elements gives what seems to be a new combinatorial expression for double Schubert polynomials. Furthermore, the complex naturally affords a representation of the Lie superalgebra of upper triangular matrices (with respect to the given flags) in $\mathrm{Hom}(M_0, M_1)$, and its graded character is the double Schubert polynomial.

The article is structured as follows. In Section 3.2 we recall some facts about double Schubert polynomials and balanced labelings. We introduce balanced super labelings (BSLs) and prove some of their properties. In Section 3.3 we extend the construction for Schubert functors to the $\mathbf{Z}/2$-graded setting and show that they have a basis naturally indexed by the BSLs. In Section 3.4 we construct the Schubert complex from this $\mathbf{Z}/2$-graded Schubert functor. Using some facts about the geometry of flag varieties, we show that the acyclicity of these complexes is controlled by the depth of a Schubert determinantal ideal. In the case of acyclicity and when the coefficient ring is Cohen–Macaulay, we show that the cokernel of the complex is a Cohen–Macaulay module which is generically a line bundle on its support. We also give some examples of Schubert complexes. Finally, in Section 3.5, we relate the acyclicity of the Schubert complexes to a degeneracy locus formula of Fulton. We finish with some remarks and possible future directions.

Conventions.

The letter K is reserved for a field of arbitrary characteristic. If X is a scheme, then $\mathcal{O}_X$ denotes the structure sheaf of X. Throughout, all schemes are assumed to be separated. A variety means a reduced scheme which is of finite type over K. We treat the notions of locally free sheaves and vector bundles as the same, and points will always refer to closed points. The fiber of a vector bundle E at a point $x \in X$ is denoted $E(x)$ and refers to the stalk E_x tensored with the residue field $k(x)$. Given a line bundle L on X, $c_1(L)$ denotes the first Chern class of L, which we think of as a degree -1 endomorphism of the Chow groups $\mathrm{A}_*(X)$. For an element $\alpha \in \mathrm{A}_*(X)$, and an endomorphism c of $\mathrm{A}_*(X)$, we will use the notation $c \cap \alpha$ to denote c applied to α.

3.2 Double Schubert polynomials.

3.2.1 Preliminaries.

Let Σ_n be the permutation group on the set $\{1, \dots, n\}$. Since we are thinking of Σ_n as a group of functions, we will multiply them as functions, e.g., if s_1 and s_2 are the transpositions that switch 1 and 2, and 2 and 3, respectively, then $s_1 s_2$ is the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. We will use inline notation for permutations, so that w is written as $w(1)w(2)\cdots w(n)$. Proofs for the following statements about Σ_n can be found in [Man, §2.1]. Let s_i denote the transposition which switches i and $i+1$. Then Σ_n is generated by $\{s_1, \dots, s_{n-1}\}$, and for $w \in \Sigma_n$, we define the **length** of w to be the least number $\ell(w)$ such that $w = s_{i_1} \cdots s_{i_{\ell(w)}}$. Such a minimal expression is a **reduced decomposition** for w. All reduced expressions can be obtained from one another using only the **braid relations**: $s_i s_j = s_j s_i$ for $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. We can also write $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$. The **long word** w_0 is the unique word with maximal length, and is defined by $w_0(i) = n + 1 - i$.

We will use two partial orders on Σ_n. The **(left) weak Bruhat order**, denoted by $u \le_W w$, holds if some reduced decomposition of u is the suffix of some reduced decomposition of w.[1] We denote the **strong Bruhat order** by $u \le w$, which holds if some reduced decomposition of w contains a subword that is a reduced decomposition of u. It follows from the definition that $u \le w$ if and only if $u^{-1} \le w^{-1}$. For a permutation w, let $r_w(p,q) = \#\{i \le p \mid w(i) \le q\}$ be its **rank function**. Then $u \le w$ if and only if $r_u(p,q) \ge r_w(p,q)$ for all p and q (the inequality on rank functions is reversed).

Given a polynomial (with arbitrary coefficient ring) in the variables $\{x_i\}_{i\ge1}$, let ∂_i be the **divided difference operator**

$$(\partial_i P)(x_1, x_2, \dots) = \frac{P(\dots, x_{i-1}, x_i, x_{i+1}, \dots) - P(\dots, x_{i-1}, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}. \tag{3.2.1}$$

The operators ∂_i satisfy the braid relations: $\partial_i\partial_j = \partial_j\partial_i$ when $|i-j| > 1$ and $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.

For the long word $w_0 \in \Sigma_n$, set $\mathfrak{S}_{w_0}(x,y) = \prod_{i+j\le n}(x_i - y_j)$. In general, if $\ell(ws_i) = \ell(w) - 1$, we set $\mathfrak{S}_{ws_i}(x,y) = \partial_i \mathfrak{S}_w(x,y)$, where we interpret $\mathfrak{S}_w(x,y)$ as a polynomial in the variables $\{x_i\}_{i\ge1}$ with coefficients in the ring $\mathbf{Z}[y_1, y_2, \dots]$. These polynomials are the **double Schubert polynomials**, and are well-defined since the ∂_i satisfy the braid relations and the braid relations connect all reduced decompositions of a permutation. The definition of these polynomials is due to Lascoux and Schützenberger [LS]. They enjoy the following stability property: if we embed Σ_n into Σ_{n+m} by identifying permutations of Σ_n with permutations of Σ_{n+m} which pointwise fix $\{n+1, n+2, \dots, n+m\}$, then the polynomial $\mathfrak{S}_w(x,y)$ is the same whether we regard w as an element of Σ_n or Σ_{n+m} [Man, Corollary 2.4.5].

Define the **single Schubert polynomials** by $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$. We will use the identity [Man, Proposition 2.4.7]

$$\mathfrak{S}_w(x,y) = \sum_{u \le_W w} \mathfrak{S}_u(x)\mathfrak{S}_{uw^{-1}}(-y). \tag{3.2.2}$$

3.2.2 Balanced super labelings.

For the rest of this article, we fix a totally ordered alphabet $\cdots < 3' < 2' < 1' < 1 < 2 < 3 < \cdots$. The elements i' will be referred to as **marked** and the elements i will be referred to as **unmarked**.

For a permutation w, define its **diagram** $D(w) = \{(i, w(j)) \mid i < j,\ w(i) > w(j)\}$. Note that $\#D(w) = \ell(w)$. Our convention is that the box (i,j) means row number i going from top to bottom, column number j going from left to right, just as with matrix indexing. An alternative way to get the diagram of $D(w)$ is as follows: for each i, remove all boxes to the right of $(i, w(i))$ in the same row and all boxes below $(i, w(i))$ in the same column including $(i, w(i))$. The complement is $D(w)$. See Figure 3-1 for an example with $w = 35142$. Here the boxes $(i, w(i))$ are marked with $\bullet$ and the other removed boxes are marked with $\times$.

Let T be a labeling of $D(w)$. The **hook** of a box $b \in D(w)$ is the set of boxes in the same column below it, and the set of boxes in the same row to the right of it (including itself). A hook is **balanced** (with respect to T) if it satisfies the following property: when the entries are rearranged so that they are weakly increasing going from the top right end to the bottom left end, the label in the corner stays the same. A labeling is **balanced** if all of the hooks are balanced. Call a labeling T of $D(w)$ with entries in our alphabet a **balanced super labeling (BSL)** if it is balanced, column-strict (no repetitions in any column) with respect to the unmarked alphabet, row-strict

[1] In [Man], the weak Bruhat order is defined in terms of prefixes. We point out that these two definitions are distinct, but this will not cause any problems.

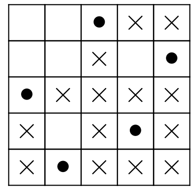


Figure 3-1: $D(35142)$

with respect to the marked alphabet, and satisfies $j' \le T(i,j) \le i$ for all i and j (this last condition will be referred to as the **flag conditions**). To be consistent with the identity permutation, we say that an empty diagram has exactly one labeling.

Example 3.2.3. We list the BSL for some long words.

$n = 3$, $\mathfrak{S}_{321}(x,y) = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$

$$\begin{matrix}1&1\\2\end{matrix}\quad \begin{matrix}1&1\\1'\end{matrix}\quad \begin{matrix}1&2'\\2\end{matrix}\quad \begin{matrix}1'&2'\\1\end{matrix}\quad \begin{matrix}1&1'\\2\end{matrix}\quad \begin{matrix}1'&1\\1'\end{matrix}\quad \begin{matrix}1'&2'\\2\end{matrix}\quad \begin{matrix}1'&2'\\1'\end{matrix}$$

$n = 4$, $\mathfrak{S}_{4321}(x,y) = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_1 - y_3)(x_2 - y_2)(x_3 - y_1)$.

$$\begin{matrix}1&1&1\\2&2\\3\end{matrix}\quad \begin{matrix}1&1&1\\2&2\\1'\end{matrix}\quad \begin{matrix}1&1&1\\2&2'\\3\end{matrix}\quad \begin{matrix}1&1&1\\1'&2'\\2\end{matrix}\quad \begin{matrix}1&1&3'\\2&2\\3\end{matrix}\quad \begin{matrix}1&1&3'\\2&2\\1'\end{matrix}\quad \begin{matrix}1&2'&3'\\2&1\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&1\\2\end{matrix}$$

$$\begin{matrix}1&1&1\\2&1'\\3\end{matrix}\quad \begin{matrix}1&1&1\\1'&2\\1'\end{matrix}\quad \begin{matrix}1&1&1\\1'&2'\\3\end{matrix}\quad \begin{matrix}1&1&1\\1'&2'\\1'\end{matrix}\quad \begin{matrix}1&1'&3'\\2&1\\3\end{matrix}\quad \begin{matrix}1'&1&3'\\1&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&1\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&1\\1'\end{matrix}$$

$$\begin{matrix}1&1&2'\\2&2\\3\end{matrix}\quad \begin{matrix}1&1&2'\\2&2\\1'\end{matrix}\quad \begin{matrix}1&2'&1\\2&2'\\3\end{matrix}\quad \begin{matrix}1&2'&1\\1'&2'\\2\end{matrix}\quad \begin{matrix}1&2'&3'\\2&2\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&2\\1\end{matrix}\quad \begin{matrix}1&2'&3'\\2&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&2'\\1\end{matrix}$$

$$\begin{matrix}1&1'&2'\\2&1\\3\end{matrix}\quad \begin{matrix}1&1'&1\\1'&2'\\2\end{matrix}\quad \begin{matrix}1&2'&1\\1'&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&1\\1'&2'\\1\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&1\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&2'\\1\end{matrix}$$

$$\begin{matrix}1&1&1'\\2&2\\3\end{matrix}\quad \begin{matrix}1&1&1'\\2&2\\1'\end{matrix}\quad \begin{matrix}1&1'&1\\2&2'\\3\end{matrix}\quad \begin{matrix}1'&1&2'\\1&2\\1'\end{matrix}\quad \begin{matrix}1&1'&3'\\2&2\\3\end{matrix}\quad \begin{matrix}1'&1&3'\\2&2\\1'\end{matrix}\quad \begin{matrix}1&2'&3'\\2&1'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&1'\\2\end{matrix}$$

$$\begin{matrix}1&1'&1\\2&1'\\3\end{matrix}\quad \begin{matrix}1'&1&1\\1'&2\\1'\end{matrix}\quad \begin{matrix}1&1'&1\\1'&2'\\3\end{matrix}\quad \begin{matrix}1'&1&1\\1'&2'\\1'\end{matrix}\quad \begin{matrix}1&1'&3'\\2&1'\\3\end{matrix}\quad \begin{matrix}1'&1&3'\\1'&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\1&1'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&1\\1'\end{matrix}$$

$$\begin{matrix}1&1'&2'\\2&2\\3\end{matrix}\quad \begin{matrix}1'&1&2'\\2&2\\1'\end{matrix}\quad \begin{matrix}1&1'&2'\\2&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&1\\1'&2'\\2\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&2\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&2'\\2\end{matrix}$$

$$\begin{matrix}1&1'&2'\\2&1'\\3\end{matrix}\quad \begin{matrix}1'&1&2'\\1'&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&1\\1'&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&1\\1'&2'\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\2&1'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&2\\1'\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&2'\\3\end{matrix}\quad \begin{matrix}1'&2'&3'\\1'&2'\\1'\end{matrix}$$

□

Let $A = (a_{i,j})$ be an $n \times n$ array. We define left and right actions of Σ_n on A as follows. For $w \in \Sigma_n$, set $(wA)_{i,j} = A_{i,w(j)}$, and $(Aw)_{i,j} = A_{w^{-1}(i),j}$. Equivalently, $Aw = (w^{-1}A^t)^t$ where t denotes transpose. In particular, if $A = D(w)$ is the diagram of a permutation, and $\ell(wu) = \ell(w) + \ell(u)$, then $D(w)u \subseteq D(wu)$. It is enough to check this when $u = s_i$ is a transposition. In this case, the condition $\ell(ws_i) = \ell(w) + 1$ means that $w(i) < w(i+1)$, and then $D(ws_i) = D(w)s_i \cup \{(i, w(i))\}$. Similarly, $wD(u) \subseteq D(wu)$.

If w is a permutation, then $(i,j) \in D(w)$ is a **border cell** if $w(i+1) = j$. In particular, if (i,j) is a border cell, then $w(i) > w(i+1)$, so $(D(w) \setminus (i,j))s_i = D(ws_i)$.

Lemma 3.2.4. *Let T be a labeling of $D(w)$ with largest label M.*
(a) *Suppose (i,j) is a border cell which contains M. Then T is balanced if and only if $(T \setminus (i,j))s_i$ is balanced.*
(b) *Suppose T is a BSL and M is unmarked. Then every row which contains M must contain an M in a border cell.*

Proof. See [FGRS, Theorem 4.8] for (a). Part (b) follows from [FGRS, Lemma 4.7] □

By convention, a BSL of $D(w)$ is an $n \times n$ array which is 0 outside of $D(w)$ and takes values in our alphabet otherwise. We use the convention that $0 + i = i + 0 = i$ and $0 + i' = i' + 0 = i'$ whenever i, i' is in our alphabet, and also that $1' < 0 < 1$.

Lemma 3.2.5. *Let u and v be two permutations such that $\ell(uv) = \ell(u) + \ell(v)$. Let T_u be a BSL of $D(u)$ using only marked letters, and let T_v be a BSL of $D(v)$ using only unmarked letters. Then $T = T_u v + uT_v$ is a BSL for $D(uv)$, and all BSLs of $w = uv$ come from such a "factorization" in a unique way.*

Proof. The condition $j' \leq T(i,j) \leq i$ is automatic since we assumed that T_u contains only marked letters and T_v contains only unmarked letters. Similarly, the respective column-strict and row-strict conditions are automatic. So it is enough to check that T is balanced.

By Lemma 3.2.4, we can factor $v = s_{i_1} s_{i_2} \cdots s_{i_{\ell(v)}}$ into simple transpositions such that if we write $v_j = s_{i_1} \cdots s_{i_{j-1}} s_{i_j}$, then $T_{v_{\ell(v)}} = T_v$, and for $j < \ell(v)$, T_{v_j} is the result of removing a border cell with the largest label L_j from $T_{v_{j+1}}$ and hence is a balanced labeling. In particular, $L_1 \leq L_2 \leq \cdots \leq L_{\ell(v)}$. Set $T_0 = T_u$ and $T_j = T_u v_j + uT_{v_j}$ for $1 \leq j \leq \ell(v)$. Then for $1 \leq j \leq \ell(v)$, T_j is the result of switching rows i_j and i_{j+1} in T_{j-1} and replacing the newly made 0 with L_j. Since all letters in T_0 are marked, and $L_1 \leq L_2 \leq \cdots \leq L_{\ell(v)}$, we conclude from Lemma 3.2.4(a) that each T_j is balanced, and hence $T = T_{\ell(v)}$ is balanced.

The last statement also follows from Lemma 3.2.4: given a BSL of $D(w)$, we can successively remove border cells containing the largest labels (which are unmarked), and the result will be a BSL of a diagram $D(u)$ for some permutation u which contains only unmarked letters. The removals give the desired permutation $v = u^{-1}w$.

For uniqueness, note that if at any point we have two choices of border cells to remove in rows i and j, then $|i - j| > 1$. Otherwise, if $j = i + 1$, for example, then by the balanced condition at the hook of box $(i, w(i+2))$, $T(i, w(i+1)) = T(i, w(i+2)) = T(i+1, w(i+2))$, which contradicts our strictness conditions. Since s_i and s_j commute for $|i - j| > 1$, it does not matter which one we do first. □

Given a BSL T of $D(w)$, let $f_T(i)$, respectively $f_T(i')$, be the number of occurrences of i, respectively i'. Define a monomial

$$m(T) = x_1^{f_T(1)} \cdots x_{n-1}^{f_T(n-1)} (-y_1)^{f_T(1')} \cdots (-y_{n-1})^{f_T((n-1)')}. \tag{3.2.6}$$

One more bit of notation: given a labeling T of $D(w)$, let T^* denote the labeling of $D(w^{-1})$ obtained by transposing T and performing the swap $i \leftrightarrow i'$.

Theorem 3.2.7. *For every permutation w,*

$$\mathfrak{S}_w(x, y) = \sum_T m(T),$$

where the sum is over all BSL T of $D(w)$.

Proof. Suppose we are given a BSL T of $D(w)$. By Lemma 3.2.5, there exists a unique pair of permutations v^{-1} and u such that $v^{-1}u = w$, $\ell(w) = \ell(v^{-1}) + \ell(u)$, a BSL $T_{v^{-1}}$ of $D(v^{-1})$ which only uses marked letters, and a BSL T_u of $D(u)$ which only uses unmarked letters, such that $T = T_u v^{-1} + u T_{v^{-1}}$. The labeling $T_v = T^*_{v^{-1}}$ gives a BSL of $D(v)$ which only uses unmarked letters.

Finally, using (3.2.2) coupled with the fact that $\mathfrak{S}_u(x) = \sum_T m(T)$, where the sum is over all BSL of $D(u)$ using only unmarked letters [FGRS, Theorem 6.2], we get the desired result. □

Remark 3.2.8. The operation $T \mapsto T^*$ gives a concrete realization of the symmetry $\mathfrak{S}_w(-y, -x) = \mathfrak{S}_{w^{-1}}(x, y)$ [Man, Corollary 2.4.2]. □

3.3 Double Schubert functors.

3.3.1 Super linear algebra preliminaries.

Let $V = V_0 \oplus V_1$ be a free $\mathbf{Z}/2$-graded module over a (commutative) ring R with $V_0 = \langle e_1, \dots, e_n \rangle$ and $V_1 = \langle e'_1, \dots, e'_m \rangle$, and let $\mathfrak{gl}(m|n) = \mathfrak{gl}(V)$ be the Lie superalgebra of endomorphisms of V. Let $\mathfrak{b}(m|n) \subset \mathfrak{gl}(m|n)$ be the standard Borel subalgebra of upper triangular matrices with respect to the ordered basis $\langle e'_m, \dots, e'_1, e_1, \dots, e_n \rangle$. We will mainly deal with the case $m = n$, in which case we write $\mathfrak{b}(n) = \mathfrak{b}(n|n)$, and if it is clear from context, we will drop the n and simply write $\mathfrak{b}$. Also, let $\mathfrak{b}(n)_0 = \mathfrak{gl}(V)_0 \cap \mathfrak{b}(n)$ be the even degree elements in $\mathfrak{b}(n)$, and again, we will usually denote this by $\mathfrak{b}_0$. We also write $\mathfrak{h}(n) \subset \mathfrak{b}(n)$ for the Cartan subalgebra of diagonal matrices (this is a Lie algebra concentrated in degree 0). Let $\varepsilon'_n, \dots, \varepsilon'_1, \varepsilon_1, \dots, \varepsilon_n$ be the dual basis vectors to the standard basis of $\mathfrak{h}(n)$. For notation, write $(a_n, \dots, a_1 | b_1, \dots, b_n)$ for $\sum_{i=1}^n (a_i \varepsilon'_i + b_i \varepsilon_i)$. The even and odd roots of $\mathfrak{b}(n)$ are $\Phi_0 = \{\varepsilon'_j - \varepsilon'_i,\ \varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$ and $\Phi_1 = \{\varepsilon'_i - \varepsilon_j \mid 1 \le i, j \le n\}$, respectively. The even and odd simple roots are $\Delta_0 = \{\varepsilon'_{i+1} - \varepsilon'_i,\ \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$ and $\Delta_1 = \{\varepsilon'_1 - \varepsilon_1\}$.

Given a highest weight representation W of $\mathfrak{b}(n)$, we have a weight decomposition $W = \bigoplus_\lambda W_\lambda$ as a representation of $\mathfrak{h}(n)$. Let Λ be the highest weight of W. Then every weight λ appearing in the weight decomposition can be written in the form $\Lambda - \sum n_\alpha \alpha$ where α ranges over the simple roots of $\mathfrak{b}(n)$ and $n_\alpha \in \mathbf{Z}_{\ge 0}$. For such a λ, set $\omega(\lambda) = (-1)^{\sum n_\alpha \deg \alpha}$. Then we define the **character** and **supercharacter** of W as

$$\operatorname{char} W = \sum_\lambda (\dim W_\lambda) e^\lambda, \quad \operatorname{sch} W = \sum_\lambda \omega(\lambda)(\dim W_\lambda) e^\lambda. \tag{3.3.1}$$

Here the e^λ are formal symbols with the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$.

We will need $\mathbf{Z}/2$-graded analogues of the divided and exterior powers (see [Wey, §2.4] for the dual versions of our definitions). Let $F = F_0 \oplus F_1$ be a free R-supermodule. Let D denote the divided power functor, let $\bigwedge$ denote the exterior power functor, and let Sym denote the symmetric power functor. Then $\bigwedge^i F$ and $\mathrm{D}^i F$ are $\mathbf{Z}$-graded modules with terms given by

$$(\bigwedge^i F)_d = \bigwedge^{i-d} F_0 \otimes \mathrm{Sym}^d F_1, \quad (\mathrm{D}^i F)_d = \mathrm{D}^{i-d} F_0 \otimes \bigwedge^d F_1. \tag{3.3.2}$$

We can define a coassociative $\mathbf{Z}$-graded comultiplication $\Delta\colon \mathrm{D}^{i+j} F \to \mathrm{D}^i F \otimes \mathrm{D}^j F$ as follows. On degree d, pick $0 \le a \le i$ and $0 \le b \le j$ such that $a + b = d$. Then we have the composition $\Delta_{a,b}$

$$\begin{aligned}(\mathrm{D}^{i+j} F)_d = \mathrm{D}^{i+j-a-b} F_0 \otimes \bigwedge^{a+b} F_1 \\ \xrightarrow{\Delta' \otimes \Delta'} \mathrm{D}^{i-a} F_0 \otimes \mathrm{D}^{j-b} F_0 \otimes \bigwedge^a F_1 \otimes \bigwedge^b F_1 \\ \cong \mathrm{D}^{i-a} F_0 \otimes \bigwedge^a F_1 \otimes \mathrm{D}^{j-b} F_0 \otimes \bigwedge^b F_1 = (\mathrm{D}^i F)_a \otimes (\mathrm{D}^j F)_b,\end{aligned} \tag{3.3.3}$$

where Δ' is the usual comultiplication, and we define Δ on the degree d part to be $\sum_{a+b=d} \Delta_{a,b}$.

Similarly, we can define an associative $\mathbf{Z}$-graded multiplication $m\colon \bigwedge^i F \otimes \bigwedge^j F \to \bigwedge^{i+j} F$ as follows. For degrees a and b, we have

$$\begin{aligned}(\bigwedge^i F)_a \otimes (\bigwedge^j F)_b = \bigwedge^{i-a} F_0 \otimes \mathrm{Sym}^a F_1 \otimes \bigwedge^{j-b} F_0 \otimes \mathrm{Sym}^b F_1 \\ \cong \bigwedge^{i-a} F_0 \otimes \bigwedge^{j-b} F_0 \otimes \mathrm{Sym}^a F_1 \otimes \mathrm{Sym}^b F_1 \\ \xrightarrow{m' \otimes m'} \bigwedge^{i+j-a-b} F_0 \otimes \mathrm{Sym}^{a+b} F_1 = (\bigwedge^{i+j} F)_{a+b},\end{aligned} \tag{3.3.4}$$

where m' is the usual multiplication.

3.3.2 Constructions.

Define a flag of $\mathbf{Z}/2$-graded submodules

$$V^\bullet : V^{-n} \subset \cdots \subset V^{-1} \subset V^1 \subset \cdots \subset V^n \tag{3.3.5}$$

such that V^{-1} consists of all of the odd elements of V^n. We will say that the flag is **split** if each term and each quotient is a free module. Fix a permutation $w \in \Sigma_n$. Let $r_k = r_k(w)$, respectively $c_j = c_j(w)$, be the number of boxes in the kth row, respectively jth column, of $D(w)$. Define $\chi_{k,j}$ to be 1 if $(k, j) \in D(w)$ and 0 otherwise. Consider the map

$$\begin{aligned}\bigotimes_{k=1}^{n-1} \mathrm{D}^{r_k} V^k \xrightarrow{\otimes \Delta} \bigotimes_{k=1}^{n-1} \bigotimes_{j=1}^{n-1} \mathrm{D}^{\chi_{k,j}} V^k \cong \bigotimes_{j=1}^{n-1} \bigotimes_{k=1}^{n-1} \mathrm{D}^{\chi_{k,j}} V^k \\ \xrightarrow{\otimes m} \bigotimes_{j=1}^{n-1} \bigwedge^{c_j} V^{w^{-1}(j)} \xrightarrow{\otimes \pi} \bigotimes_{j=1}^{n-1} \bigwedge^{c_j} (V^{w^{-1}(j)} / V^{-j-1}),\end{aligned} \tag{3.3.6}$$

where $\otimes\pi$ denotes the product of projection maps. Note that $\mathrm{D}^1 V^k = V^k$ and $\mathrm{D}^0 V^k = R$, so that the multiplication above makes sense. Then its image $\mathfrak{S}_w(V^\bullet)$ is the **$\mathbf{Z}/2$-graded Schubert functor**, or **double Schubert functor**. By convention, the empty tensor product is R, so that if

w is the identity permutation, then $\mathfrak{S}_w(V^\bullet) = R$.

This definition is clearly functorial: given an even map of flags $f \colon V^\bullet \to W^\bullet$, i.e., $f(V^k) \subset W^k$ for $-n \le k \le n$, we have an induced map $f \colon \mathfrak{S}_w(V^\bullet) \to \mathfrak{S}_w(W^\bullet)$.

We will focus on the case when $V^{-i} = \langle e'_n, e'_{n-1}, \dots, e'_i \rangle$ and $V^i = V^{-1} + \langle e_1, e_2, \dots, e_i \rangle$, so that $\mathfrak{S}_w = \mathfrak{S}_w(V^\bullet)$ is a $\underline{(}n)$-module.

Remark 3.3.7. One could dually define the double Schubert functor as the image of (dual) exterior powers mapping to symmetric powers. One has to be careful, because the $\mathbf{Z}/2$-graded version of exterior powers are not self-dual. For the dual of our definition, one uses $(\bigwedge^i F)_d = \bigwedge^{i-d} F_0 \otimes \mathrm{D}^d F_1$. We have chosen our definitions to be consistent with [KP]. This will be especially convenient for Theorem 3.3.13. □

Remark 3.3.8. We could also define $\mathfrak{S}_D(V^\bullet)$ for an arbitrary diagram D which does not necessarily come from a permutation. This is relevant in [KP, §4], whose proof we use in Theorem 3.3.13. However, since the details will go through without significant changes, we will have no need to elaborate on this point. □

Lemma 3.3.9. *Let $w \in \Sigma_n$ and $v \in \Sigma_m$ be two permutations. Define a new permutation $u \in \Sigma_{n+m}$ by $u(i) = w(i)$ for $i = 1, \dots, n$ and $u(n+j) = v(j)$ for $j = 1, \dots, m$. Also, define a permutation $v' \in \Sigma_{n+m}$ by $v'(i) = i$ for $i = 1, \dots, n$ and $v'(n+j) = v(j)$ for $j = 1, \dots, m$. Then*

$$\mathfrak{S}_u(V^\bullet) \cong \mathfrak{S}_w(V^\bullet) \otimes \mathfrak{S}_{v'}(V^\bullet).$$

Proof. This follows from the definition of double Schubert functors, the fact that $D(u) = D(w) \cup D(v')$, and the fact that no two cells of $D(w)$ and $D(v')$ lie in the same row or column. □

Example 3.3.10. Consider $n = 3$ and $w = 321$. Then $r_1 = 2$, $r_2 = 1$, $c_1 = 2$, and $c_2 = 1$. We need to calculate the image of the map

$$\mathrm{D}^2 V^1 \otimes V^2 \xrightarrow{\Delta \otimes 1} (V^1 \otimes V^1) \otimes V^2 \xrightarrow{t_{2,3}} (V^1 \otimes V^2) \otimes V^1 \xrightarrow{m \otimes 1} \bigwedge^2 (V^3/V^{-2}) \otimes V^2/V^{-3},$$

where $t_{2,3}$ is the map that switches the second and third parts of the tensor product. Write x, y for e_1, e_2, and x', y' for e'_1, e'_2. We can ignore e_3 and e'_3 since they will not appear in the image. We can write $\mathrm{D}^2 V^1 = \langle x^2, x \otimes x', x \otimes y', x' \wedge y' \rangle$ and $V^2 = \langle x, y, x', y' \rangle$. Then we have

$$\begin{aligned}
m(t_{2,3}(\Delta(x^2 \otimes x))) &= m(t_{2,3}(x \otimes x \otimes x)) = 0 \\
m(t_{2,3}(\Delta(x^2 \otimes y))) &= m(t_{2,3}(x \otimes x \otimes y)) = (x \wedge y) \otimes x \\
m(t_{2,3}(\Delta(x^2 \otimes x'))) &= m(t_{2,3}(x \otimes x \otimes x')) = (x \otimes x') \otimes x \\
m(t_{2,3}(\Delta(x^2 \otimes y'))) &= m(t_{2,3}(x \otimes x \otimes y')) = 0
\end{aligned}$$

$$\begin{aligned}
m(t_{2,3}(\Delta(x \otimes x' \otimes x))) &= m(t_{2,3}((x \otimes x' + x' \otimes x) \otimes x)) = (x \otimes x') \otimes x \\
m(t_{2,3}(\Delta(x \otimes x' \otimes y))) &= m(t_{2,3}((x \otimes x' + x' \otimes x) \otimes y)) = (x \wedge y) \otimes x' + (y \otimes x') \otimes x \\
m(t_{2,3}(\Delta(x \otimes x' \otimes x'))) &= m(t_{2,3}((x \otimes x' + x' \otimes x) \otimes x')) = (x \otimes x') \otimes x' + x'^2 \otimes x \\
m(t_{2,3}(\Delta(x \otimes x' \otimes y'))) &= m(t_{2,3}((x \otimes x' + x' \otimes x) \otimes y')) = 0
\end{aligned}$$

$$m(t_{2,3}(\Delta(x\otimes y'\otimes x))) = m(t_{2,3}((x\otimes y' + y'\otimes x)\otimes x)) = 0$$
$$m(t_{2,3}(\Delta(x\otimes y'\otimes y))) = m(t_{2,3}((x\otimes y' + y'\otimes x)\otimes y)) = (x\wedge y)\otimes y'$$
$$m(t_{2,3}(\Delta(x\otimes y'\otimes x'))) = m(t_{2,3}((x\otimes y' + y'\otimes x)\otimes x')) = (x\otimes x')\otimes y'$$
$$m(t_{2,3}(\Delta(x\otimes y'\otimes y'))) = m(t_{2,3}((x\otimes y' + y'\otimes x)\otimes y')) = 0$$

$$m(t_{2,3}(\Delta(x'\wedge y'\otimes x))) = m(t_{2,3}((x'\otimes y' - y'\otimes x')\otimes x)) = (x\otimes x')\otimes y'$$
$$m(t_{2,3}(\Delta(x'\wedge y'\otimes y))) = m(t_{2,3}((x'\otimes y' - y'\otimes x')\otimes y)) = (y\otimes x')\otimes y'$$
$$m(t_{2,3}(\Delta(x'\wedge y'\otimes x'))) = m(t_{2,3}((x'\otimes y' - y'\otimes x')\otimes x')) = x'^2\otimes y'$$
$$m(t_{2,3}(\Delta(x'\wedge y'\otimes y'))) = m(t_{2,3}((x'\otimes y' - y'\otimes x')\otimes y')) = 0 \qquad \square$$

Here is a combinatorial description of the map (3.3.6). The elements of $\bigotimes_{k=1}^{n-1} \mathrm{D}^{r_k} V^k$ can be thought of as labelings of $D = D(w)$ such that in row k, only the labels $n', (n-1)', \dots, 1', 1, \dots, k$ are used, such that there is at most one use of i' in a given row, and such that the entries in each row are ordered in the usual way (i.e., $n' < (n-1)' < \cdots < 1' < 1 < \cdots < k$). Let Σ_D be the permutation group of D. We say that $\sigma \in \Sigma_D$ is **row-preserving** if each box and its image under σ are in the same row. Denote the set of row-preserving permutations as $\mathrm{Row}(D)$. Let T be a labeling of D that is row-strict with respect to the marked letters. Let $\mathrm{Row}(D)_T$ be the subgroup of $\mathrm{Row}(D)$ that leaves T fixed, and let $\mathrm{Row}(D)^T$ be the set of cosets $\mathrm{Row}(D)/\mathrm{Row}(D)_T$. Given $\sigma \in \mathrm{Row}(D)^T$, and considering the boxes as ordered from left to right, let $\alpha(T,\sigma)_k$ be the number of inversions of σ among the *marked letters* in the kth row, and define $\alpha(T,\sigma) = \sum_{k=1}^{n-1} \alpha(T,\sigma)_k$. Note that this number is independent of the representative chosen since T is row strict with respect to the marked letters. Then the comultiplication sends T to $\sum_{\sigma\in\mathrm{Row}(D)^T}(-1)^{\alpha(T,\sigma)}\sigma T$ where σT is the result of permuting the labels of T according to σ.

For the multiplication map, we can interpret the columns as being alternating in the *unmarked letters* and symmetric in the *marked letters*. We write $m(T)$ for the image of T under this equivalence relation. Therefore, the map (3.3.6) can be defined as

$$T \mapsto \sum_{\sigma\in\mathrm{Row}(D)^T} (-1)^{\alpha(T,\alpha)} m(\sigma T). \tag{3.3.11}$$

3.3.3 A basis and a filtration.

In order to prove properties of $\mathfrak{S}_w$, we will construct a filtration by submodules, which is based on the filtration of the single Schubert functors introduced by Kraśkiewicz and Pragacz [KP].

Let $w \in \Sigma_n$ be a nonidentity permutation. Consider the set of pairs (α,β) such that $\alpha < \beta$ and $w(\alpha) > w(\beta)$. Choose (α,β) to be maximal with respect to the lexicographic ordering. Let $\gamma_1 < \cdots < \gamma_k$ be the numbers such that $\gamma_t < \alpha$ and $w(\gamma_t) < w(\beta)$, and such that $\gamma_t < i < \alpha$ implies that $w(i) \notin \{w(\gamma_t), w(\gamma_t)+1, \dots, w(\beta)\}$. Then we have the following identity of double Schubert polynomials

$$\mathfrak{S}_w = \mathfrak{S}_v \cdot (x_\alpha - y_{w(\beta)}) + \sum_{t=1}^{k} \mathfrak{S}_{\psi_t}, \tag{3.3.12}$$

where $v = wt_{\alpha,\beta}$ and $\psi_t = wt_{\alpha,\beta}t_{\gamma_t,\alpha}$. Here $t_{i,j}$ denotes the transposition which switches i and j. See, for example, [Man, Exercise 2.7.3]. The formula in (3.3.12) will be called a **maximal transition** for w. Define the **index** of a permutation u to be the number $\sum_k (k-1)\#\{j > k \mid u(k) > u(j)\}$.

Note that the index of ψ_t is smaller than the index of w.

When $w = s_i$ is a simple transposition, $v = 1$ is the identity, $k = 1$, and $\psi_1 = s_{i-1}$. See Example 3.4.11 for more details regarding the filtration in this case.

Theorem 3.3.13. *Let $V^\bullet$ be a split flag as in (3.3.5). Given a nonidentity permutation $w \in \Sigma_n$, let (3.3.12) be the maximal transition for w. Then there exists a functorial $\mathfrak{b}$-equivariant filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset F' \subset F = \mathfrak{S}_w(V^\bullet)$$

such that $F/F' \cong \mathfrak{S}_v(V^\bullet) \otimes V^\alpha/V^{\alpha-1}$, $F'/F_k \cong \mathfrak{S}_v(V^\bullet) \otimes V^{-w(\beta)}/V^{-w(\beta)-1}$, and $F_t/F_{t-1} \cong \mathfrak{S}_{\psi_t}(V^\bullet)$ for $t = 1, \dots, k$.

Proof. For notation, write $W^i = V^i/V^{-w(i)-1}$, and let $p \colon W^\beta \to V^\beta/V^{\alpha-1}$ be the projection map. Define the $\mathfrak{b}$-equivariant morphism φ' by the composition

$$\bigwedge^{c_{w(1)}} W^1 \otimes \cdots \otimes \bigwedge^{c_{w(\alpha)}} W^\alpha \otimes \cdots \otimes \bigwedge^{c_{w(\beta)}} W^\beta \otimes \cdots \xrightarrow{1\otimes\cdots\otimes 1\otimes\cdots\otimes\Delta\otimes\cdots}$$
$$\bigwedge^{c_{w(1)}} W^1 \otimes \cdots \otimes \bigwedge^{c_{w(\alpha)}} W^\alpha \otimes \cdots \otimes \bigwedge^{c_{w(\beta)}-1} W^\beta \otimes W^\beta \otimes \cdots \xrightarrow{T}$$
$$\left(\bigwedge^{c_{w(1)}} W^1 \otimes \cdots \otimes \bigwedge^{c_{w(\beta)}-1} W^\beta \otimes \cdots \otimes \bigwedge^{c_{w(\alpha)}} W^\alpha \otimes \cdots \right) \otimes W^\beta$$

where T is the map which switches the order of the tensor product in the way prescribed. Let $\varphi = (1 \otimes p) \circ \varphi'$. We set $F' = \ker \varphi$. Let C_w and C_v be the $\mathfrak{b}$-cyclic generators of $\mathfrak{S}_w(V^\bullet)$ and $\mathfrak{S}_v(V^\bullet)$, respectively. These are given by BSLs where the ith row only has the label i. By maximality of the pair (α, β) (with respect to the property $\alpha < \beta$ and $w(\alpha) > w(\beta)$), the lowest box in column $w(\beta)$ is in row α. Hence, restricting φ to $F = \mathfrak{S}_w(V^\bullet)$, we get $\varphi(C_w) = C_v \otimes e_\alpha$. This gives an isomorphism $F/F' \to \mathfrak{S}_v(V^\bullet) \otimes V^\alpha/V^{\alpha-1}$.

Let $X \in \mathfrak{b}$ be the matrix defined by $X(e_\alpha) = e'_{w(\beta)}$ and $X(e_i) = 0$ for $i \neq \alpha$ and $X(e'_j) = 0$ for all j. We claim that the rightmost box in row α has column index $w(\beta)$. If not, then there is a box $(\alpha, w(\beta')) \in D(w)$ with $w(\beta') > w(\beta)$ and $\beta' > \alpha$. If $\beta' < \beta$, then $(\beta', \beta) > (\alpha, \beta)$ which contradicts the maximality of (α, β). Otherwise, if $\beta' > \beta$, we have $(\alpha, \beta') > (\alpha, \beta)$ which also contradicts maximality. This contradiction proves the claim. The claim implies that $\varphi'(X(C_w)) = C_v \otimes e'_{w(\beta)}$, and hence $X(C_w) \in \ker \varphi$. Letting F'' be the $\mathfrak{b}$-submodule generated by $X(C_w)$, we get an isomorphism $F''/\ker \varphi' \cong \mathfrak{S}_v(V^\bullet) \otimes V^{-w(\beta)}/V^{-w(\beta)-1}$.

Using the notation of [KP, §4] with the obvious changes (see also [KP, Remark 5.3]), let $F_t = \sum_{r \le t} S_{\mathcal{I}_r}(V^\bullet)$.[2] The proofs from [KP, §4] of the fact that there is a surjection $F_t/F_{t-1} \to \mathfrak{S}'_{\psi_t}(V^\bullet)$ in the ungraded case extend to the $\mathbf{Z}/2$-graded case. We just need to show that these surjections are actually isomorphisms and that $F' = F''$. Since this is all defined over $\mathbf{Z}$ and obtained for arbitrary R via extension of scalars, it is enough to prove the corresponding statements when R is a field of arbitrary characteristic.

We will use the proof of Step 3 in [KP, §4]. The key steps there involve a tensor product identity [KP, Lemma 1.8], using the maximal transitions, and verifying the theorem for the simple transpositions. The tensor product identity in our case is Lemma 3.3.9, and the maximal transitions still exist. Also, the fact that the statement is valid for simple transpositions can be seen directly, or see Example 3.4.11. The rest of the proof goes through using the definitions $d_w = \dim_R \mathfrak{S}_w(V^\bullet)$

[2]There is a typo in the definition of $\mathcal{F}_t$ in [KP] regarding $\le$ versus $<$.

and $z_w = \mathfrak{S}_w(1,-1)$. So we have defined the desired filtration, and the functoriality is evident from the constructions. ☐

Given any labeling T, denote its weight by $w(T) = (a_{-n}, \dots, a_{-1} | a_1, \dots, a_n)$, where a_i is the number of times that the label i is used, and a_{-i} is the number of times that the label i' is used. We define a **dominance order** $\geq$ by $(a_{-n}, \dots, a_{-1} | a_1, \dots, a_n) \geq (a'_{-n}, \dots, a'_{-1} | a'_1, \dots, a'_n)$ if $\sum_{i=-n}^{k} a_i \geq \sum_{i=-n}^{k} a'_i$ for all $-n \leq k \leq n$.

Theorem 3.3.14. *Assume that the flag $V^\bullet$ is split. The images of the BSLs under* (3.3.6) *form a basis over R for $\mathfrak{S}_w$.*

Proof. Since the BSLs are defined when $R = \mathbf{Z}$, and are compatible with extension of scalars, it is enough to show that the statement is true when $R = K$ is an infinite field of arbitrary characteristic, so we will work in this case.

We can show linear independence of the BSLs following the proof of [FGRS, Theorem 7.2]. Combined with Theorem 3.3.13, this will show that they form a basis. First, we note that $\mathfrak{S}_w$ has a weight decomposition since it is a highest weight module of $\mathfrak{b}(n)$, so we only need to show linear independence of the BSLs in each weight space. Second, if T is column-strict, row-strict, and satisfies the flagged conditions, then $T \neq \pm\sigma T$ in $\mathfrak{S}_w$ whenever $\sigma \in \mathrm{Row}(D)_T$. This implies that if T is a BSL, then the image of T under (3.3.6) is nonzero.

First write $L = \ell(w)$. We assign to T a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_L}$ of w following the method in Lemma 3.2.5 by induction on L. If T contains unmarked letters, let M be the largest such label, and let i_L denote the smallest row index which contains M in a border cell. Let $s_{i_1} \cdots s_{i_{L-1}}$ be the reduced decomposition assigned to the labeling $T \setminus (i_L, w(i_L))$ of $w s_{i_L}$, so that we get the reduced decomposition $s_{i_1} \cdots s_{i_L}$ for w. If T contains no unmarked letters, let $s_{i_1^*} \cdots s_{i_L^*}$ be the reduced decomposition associated to the labeling T^* of w^{-1} and then assign the reduced decomposition $s_{i_L^*} \cdots s_{i_1^*}$ to T. So we can write this reduced decomposition as $s^*(T)s(T)$ where $s^*(T)$, respectively $s(T)$, corresponds to the transpositions coming from removing marked, respectively unmarked, letters. We will totally order reduced decompositions as follows: $s_{i_1} \cdots s_{i_L} < s_{i'_1} \cdots s_{i'_L}$ if there exists a j such that $i_j < i'_j$ and $i_k = i'_k$ for $j+1 \leq k \leq L$. We say that $s^*(T')s(T') \leq s^*(T)s(T)$ if either $s(T') < s(T)$ (the ordering for reduced decompositions), or $s(T') = s(T)$ and $s^*(T')^{-1} < s(T)^{-1}$ (the inverse means write the decomposition backwards).

Taking into account the description (3.3.11), we show that if $m(T') = \pm m(\sigma T)$ where T' and T are BSLs and $\sigma \in \mathrm{Row}(D)$, then $s^*(T')s(T') \leq s^*(T)s(T)$. Note that since we assume that T and T' have the same weights, we have $s^*(T')s(T') = s^*(T)s(T)$ if and only if $T = T'$. The BSLs are linearly independent in $\bigotimes_k \mathrm{D}^{r_k} V^k$, so by induction on $\leq$, we see that the coefficients of any linear dependence of their images in $\mathfrak{S}_w$ must all be zero.

So suppose that $m(T') = \pm m(\sigma T)$ holds and choose representatives T' and σT that realize this equality. First suppose that T contains an unmarked letter. Then so does T', and let M be the largest such one. Write $s(T) = s_{i_r} \cdots s_{i_L}$ and $s(T') = s_{i'_r} \cdots s_{i'_L}$. Since m only affects entries within the same column, the M in row i'_L is moved to some row with index $\leq i'_L$ because M occupies a border cell. By definition of i_L, all instances of M in σT lie in rows with index $\geq i_L$ since σ is row-preserving. Hence the equality $m(T') = m(\sigma T)$ implies that $i'_L \leq i_L$.

If $i'_L < i_L$, there is nothing left to do. So suppose that $i'_L = i_L$. Then M lies in the same border cell b in both T and T'. Hence $T' \setminus b = \sigma(T \setminus b)$, and we conclude by induction. So we only need to handle the case that T (and hence T') do not contain any unmarked letters. In this case, we pass to T^* and T'^*, and the above shows that $s^*(T')^{-1} \leq s^*(T)^{-1}$, so we are done. ☐

The above proof does not establish how one can write the image of an arbitrary labeling as a linear combination of the images of the BSLs. Such a straightening algorithm is preferred, but we have not been successful in finding one, so we leave this task as an open problem.

Problem 3.3.15. *Find an algorithm for writing the image of an arbitrary labeling of $D(w)$ as a linear combination of the images of the BSLs of $D(w)$.*

Corollary 3.3.16. *Identify $x_i = -e^{\varepsilon_i}$ and $y_i = -e^{\varepsilon'_i}$ for $1 \le i \le n$. Then*

$$\operatorname{char} \mathfrak{S}_w = \mathfrak{S}_w(-x, y), \quad \operatorname{sch} \mathfrak{S}_w = \mathfrak{S}_w(x, y).$$

Corollary 3.3.17. *Choose an ordering of the set of permutations below w in the weak Bruhat order: $1 = v_1 \prec v_2 \prec \cdots \prec v_N = w$ such that $v_i \prec v_{i+1}$ implies that $\ell(v_i) \le \ell(v_{i+1})$. Then there exists a $\mathfrak{b}$-equivariant filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathfrak{S}_w$$

such that

$$F_i/F_{i-1} \cong \mathfrak{S}'_{v_i} \otimes \mathfrak{S}''_{wv_i^{-1}}$$

as $\mathfrak{b}_0$-modules.

Proof. Let $\mathfrak{S}'$, respectively $\mathfrak{S}''$, denote the usual Schubert functor which uses only unmarked, respectively marked, letters. Let $W_v = \mathfrak{S}'_v \otimes \mathfrak{S}''_{wv^{-1}}$. Theorem 3.3.14 implies that we have a $\mathfrak{b}_0$-equivariant decomposition $\mathfrak{S}_w = \bigoplus_{v \le_W w} W_v$. Let $F_i = \bigoplus_{j \le i} W_{v_j}$. Then F_i is a $\mathfrak{b}_0$-submodule, and applying an element of $\mathfrak{b} \setminus \mathfrak{b}_0$ to W_{v_j} can only give elements in W_{v_k} where $\ell(v_k) < \ell(v_j)$. So F_i is in fact a $\mathfrak{b}$-submodule, and we have the desired filtration. □

3.4 Schubert complexes.

Now we can use the above machinery to define Schubert complexes. We start with the data of two split flags $F_0^\bullet : 0 = F_0^0 \subset F_0^1 \subset \cdots \subset F_0^n = F_0$ and $F_1^\bullet : F_1^{-n} \subset F_1^{-n+1} \subset \cdots \subset F_1^{-1} = F_1$, and a map $\partial \colon F_0 \to F_1$ between them. Given the flag for F_0, we pick an ordered basis $\{e_1, \dots, e_n\}$ for it such that $e_i \in F_0^i \setminus F_0^{i-1}$. Similarly, we pick an ordered basis $\{e'_1, \dots, e'_n\}$ for F_1 such that $e'_i \in F_1^{-i} \setminus F_1^{-i-1}$. Given these bases, we can represent ∂ as a matrix. This matrix representation will be relevant for the definition of certain ideals later.

Equivalently, we can give $F_1^\bullet$ as a quotient flag $F_1 = G^n \twoheadrightarrow G^{n-1} \twoheadrightarrow \cdots \twoheadrightarrow G^1 \twoheadrightarrow G^0 = 0$, so that the correspondence is given by $F_1^{-i} = \ker(G^n \twoheadrightarrow G^{i-1})$. Note that $F_1^{-i}/F_1^{-i-1} = \ker(G^i \twoheadrightarrow G^{i-1})$. We assume that each quotient has rank 1. Then we form a flagged $\mathbf{Z}/2$-graded module F with even part F_0 and odd part F_1. The formation of divided and exterior products commutes with the differential ∂ by functoriality, so we can form the **Schubert complex** $\mathfrak{S}_w(F)$ for a permutation $w \in \Sigma_n$.

Proposition 3.4.1. *The ith term of $\mathfrak{S}_w(F)$ has a natural filtration whose associated graded is*

$$\bigoplus_{\substack{v \le_W w \\ \ell(v) = i}} \mathfrak{S}_v(F_0) \otimes \mathfrak{S}_{wv^{-1}}(F_1).$$

Proof. This is a consequence of Corollary 3.3.17. □

Proposition 3.4.2. *Let $\partial\colon F_0 \to F_1$ be a map. With the notation as in Theorem 3.3.13, there is a functorial $\mathfrak{b}$-equivariant filtration of complexes*

$$0 = C_0 \subset C_1 \subset \cdots \subset C_k \subset C' \subset C = \mathfrak{S}_w(\partial)$$

such that $C/C' \cong \mathfrak{S}_v(\partial)[-1] \otimes F_0^\alpha / F_0^{\alpha-1}$, $C'/C_k \cong \mathfrak{S}_v(\partial) \otimes F_1^{-w(\beta)}/F_1^{-w(\beta)+1}$, and $C_t/C_{t-1} \cong \mathfrak{S}_{\psi_t}(\partial)$ for $t = 1, \dots, k$.

Proof. The filtration of Theorem 3.3.13 respects the differentials since everything is defined in terms of multilinear operations. The grading shift of C/C' follows from the fact that the F_0 terms have homological degree 1. □

Corollary 3.4.3. *Let $\partial\colon F_0 \to F_1$ be a flagged isomorphism. Then $\mathfrak{S}_w(\partial)$ is an exact complex whenever $w \neq 1$.*

Proof. This is an immediate consequence of Proposition 3.4.2 using induction on length and index, and the long exact sequence on homology: when $w = s_i$, exactness is obvious. □

3.4.1 Flag varieties and K-theory.

Throughout this section, we use [F3] as a reference. The reader may wish to see [F3, Appendix B.1, B.2] for the conventions used there.

We will need some facts about the geometry of flag varieties. Let V be a vector space with ordered basis $\{e_1, \dots, e_n\}$. Then the complete flag variety $\mathbf{Flag}(V)$ can be identified with $\mathbf{GL}(V)/B$ where B is the subgroup of upper triangular matrices with respect to the given basis. For a permutation $w \in \Sigma_n$, we define the **Schubert cell** Ω_w to be the B-orbit of the flag

$$\langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \dots, e_{w(n-1)} \rangle \subset V.$$

Then Ω_w is an affine space of dimension $\ell(w)$ (see [Man, §3.6]), and the flag variety is a disjoint union of the Ω_w. The **Schubert variety** X_w is the closure of Ω_w. Alternatively,

$$X_w = \{W_\bullet \in \mathbf{Flag}(V) \mid \dim(W_p \cap \langle e_1, \dots, e_q \rangle) \geq r_w(p,q)\}.$$

Recall from Section 3.2.1 that $r_w(p,q) = \#\{i \leq p \mid w(i) \leq q\}$. Given a matrix ∂ and a permutation w, let $I_w(\partial)$ be the ideal generated by the $(r_w(p,q)+1) \times (r_w(p,q)+1)$ minors of the upper left $p \times q$ submatrix of ∂. It is clear that $I_v \subseteq I_w$ if and only if $v \leq w$. In the case that ∂ is a generic matrix of variables over some coefficient ring R, let $X(w)$ be the variety defined by $I_w(\partial) \subset R[\partial_{i,j}]$. We refer to the ideals $I_w(\partial)$ as **Schubert determinantal ideals**, and the varieties $X(w)$ as **matrix Schubert varieties**. Given a permutation w, we say that a cell α in the diagram $D(w)$ is a **southeast corner** if the cells to the immediate right of α and immediately below α do not belong to $D(w)$.

Theorem 3.4.4. *Let ∂ be a generic matrix defined over a field, and let w be a permutation.*
(a) *$I_w(\partial)$ is generated by the minors coming from the submatrices whose lower right corner is a southeast corner of $D(w)$.*
(b) *$I_w(\partial)$ is a prime ideal of codimension $\ell(w)$.*
(c) *$X(w)$ is a normal variety.*

Proof. See [MS, Chapter 15] for (a) and (b). For (c), we can realize $X(w)$ as a product of an affine space with an open subset of a Schubert variety in the complete flag variety (see Step 2 of the proof of Theorem 3.4.8 for more details), so it is enough to know that Schubert varieties are normal. This is proven in [RR, Theorem 3].

See also [KM, Theorem 2.4.3] for more about the relationship of local properties for Schubert varieties and local properties of a product of matrix Schubert varieties with affine spaces. □

Given any scheme X, we let $\mathrm{K}(X)$ denote the K-theory of coherent sheaves on X. This is the free Abelian group generated by the symbols $[\mathcal{F}]$ for each coherent sheaf $\mathcal{F}$ modulo the relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for each short exact sequence of the form

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

Given a finite complex $\mathcal{C}_\bullet$ of coherent sheaves, we set $[\mathcal{C}_\bullet] = \sum_i (-1)^i [\mathcal{C}_i] = \sum_i (-1)^i [\mathrm{H}_i(\mathcal{C}_\bullet)]$. If X is nonsingular and finite-dimensional, then $\mathrm{K}(X)$ has a ring structure given by

$$[\mathcal{F}][\mathcal{F}'] = \sum_{i=0}^{\dim X} (-1)^i [\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}')].$$

Now suppose that X is an equidimensional smooth quasi-projective variety over K. For $k \geq 0$, let $F^k\mathrm{K}(X)$ be the subgroup of $\mathrm{K}(X)$ generated by coherent sheaves whose support has codimension at least k, and set $\mathrm{gr}^k \mathrm{K}(X) = F^k\mathrm{K}(X)/F^{k+1}\mathrm{K}(X)$. This filtration is compatible with the ring structure on $\mathrm{K}(X)$ [Gro, Théorème 2.12, Corollaire 1], and we set $\mathrm{gr}\,\mathrm{K}(X) = \bigoplus_{k \geq 0} \mathrm{gr}^k \mathrm{K}(X)$ to be the associated graded ring.

Let $\mathrm{A}^*(X)$ be the Chow ring of X. We identify this with the direct sum of Chow groups $\mathrm{A}_*(X)$ of X via the isomorphism $c \mapsto c \cap [X]$. Let $\varphi\colon \mathrm{A}^*(X) \to \mathrm{gr}\,\mathrm{K}(X)$ be the functorial morphism of graded rings which for a closed subvariety $V \subseteq X$ sends $[V]$ to $[\mathcal{O}_V]$. If $\mathcal{F}$ is a coherent sheaf whose support has codimension at least k, then we have $\varphi(Z^k(\mathcal{F})) = [\mathcal{F}]$ as elements of $\mathrm{gr}^k \mathrm{K}(X)$ where

$$Z^k(\mathcal{F}) = \sum_{\operatorname{codim} V = k} m_V(\mathcal{F})[V], \tag{3.4.5}$$

and $m_V(\mathcal{F})$ is the length of the stalk of $\mathcal{F}$ at the generic point of V. We will need to know later that φ becomes an isomorphism after tensoring with $\mathbf{Q}$. See [F3, Example 15.1.5, 15.2.16] for more details. For a finite complex of vector bundles $C_\bullet$ such that $[C_\bullet] \in F^k\mathrm{K}(X)$, we use $[C_\bullet]_k$ to denote the corresponding element of $\mathrm{gr}^k \mathrm{K}(X)$.

Lemma 3.4.6. *The identity* $\varphi(\mathfrak{S}_w(x,y)) = [C_\bullet]_{\ell(w)}$ *holds.*

Proof. For a line bundle L of the form $\mathcal{O}(D)$ where D is an irreducible divisor, we have $c_1(L) \cap [X] = [D]$ [F3, Theorem 3.2(f)]. Hence

$$\varphi(c_1(L) \cap [X]) = (1 - [L^\vee])_1 \in \mathrm{gr}^1 \mathrm{K}(X)$$

by the short exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

So the same formula holds for all L by linearity, and

$$\varphi(x_i) = 1 - [E_i/E_{i-1}], \quad \varphi(y_j) = 1 - [\ker(F_j \twoheadrightarrow F_{j-1})].$$

Let a and b be a new set of variables. We have $\mathfrak{S}_w(a,b) = \sum_{u \leq_W w} \mathfrak{S}_u(a)\mathfrak{S}_{uw^{-1}}(-b)$ by (3.2.2).

Doing the transformation $a_i \mapsto x_i - 1$ and $b_j \mapsto y_j - 1$, we get

$$\varphi(\mathfrak{S}_w(a,b)) = \sum_{u \le_W w} (-1)^{\ell(u)} \mathfrak{S}_u(E)\mathfrak{S}_{uw^{-1}}(F).$$

By Proposition 3.4.1, this sum is $[\mathfrak{S}_w(\partial)]$ (the change from uw^{-1} to wu^{-1} is a consequence of the fact that F_1 in Proposition 3.4.1 contains only odd elements). So it is enough to show that the substitution $a_i \mapsto a_i + 1$, $b_j \mapsto b_j + 1$ leaves the expression $\mathfrak{S}_w(a,b)$ invariant. This is clearly true for $\mathfrak{S}_{w_0}(x,y) = \prod_{i+j \le n}(x_i - y_j)$, and holds for an arbitrary permutation because the divided difference operators (see (3.2.1)) applied to a substitution invariant function yield a substitution invariant function. □

The flag variety $\mathbf{Flag}(V)$ is smooth, and its K-theory is freely generated as a group by the structure sheaves $[\mathcal{O}_{X_w}]$ (see [F3, Examples 1.9.1, 15.2.16]). Also, the irreducible components of any B-equivariant subvariety of $\mathbf{Flag}(V)$ must be Schubert varieties. There is a tautological flag of subbundles

$$0 = \mathcal{R}_0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_{n-1} \subset \mathcal{R}_n = V \times \mathbf{Flag}(V)$$

on $\mathbf{Flag}(V)$, where the fiber of $\mathcal{R}_i$ over a flag $W_\bullet$ is the space W_i. Setting $x_i = -c_1(\mathcal{R}_i/\mathcal{R}_{i-1})$, the Schubert polynomial $\mathfrak{S}_w(x_1, \dots, x_n)$ represents the Poincaré dual of the Schubert variety $X_{w_0 w}$ (see, for example, [Man, Theorem 3.6.18]).

Corollary 3.4.7. *Let V be an n-dimensional vector space and let $\mathcal{C}$ be the Schubert complex associated with the permutation w and the identity map of $V \times \mathbf{Flag}(V)$ to itself, where the subspace flag consists of the tautological subbundles and the quotient flag consists of trivial vector bundles. Then $[\mathcal{C}]_{\ell(w)} = [\mathcal{O}_{X_{w_0 w}}]$ in $\operatorname{gr}^{\ell(w)} \mathrm{K}(\mathbf{Flag}(V))$.*

Proof. Both quantities agree with $\varphi(\mathfrak{S}_w(x,0))$ where $x_i = -c_1(\mathcal{R}_i/\mathcal{R}_{i-1})$ for $i = 1, \dots, n$. □

3.4.2 Generic acyclicity of Schubert complexes.

Theorem 3.4.8. *Let $A = K[\partial_{i,j}]$ be a polynomial ring over a field K, and let $\partial \colon F_0 \to F_1$ be a generic map of variables between two free A-modules.*

(a) *The Schubert complex $\mathfrak{S}_w(\partial)$ is acyclic, and resolves a Cohen–Macaulay module M of codimension $\ell(w)$ supported in $I_{w^{-1}}(\partial) \subseteq A$.*

(b) *The restriction of M to $X(w^{-1})$ is a line bundle outside of a certain codimension 2 subset.*

(c) *The Schubert complex defined over the integers is acyclic.*

Before we begin the proof, let us outline the strategy. The main idea is to use the filtration given by Proposition 3.4.2 and work by induction. The main difficulty is the fact that there is a homological shift in the filtration, which only allows one to conclude that $\mathrm{H}_i(\mathfrak{S}_w(\partial)) = 0$ for $i > 1$ (see (3.4.9)). Hence the class of $C = \mathfrak{S}_w(\partial)$ in an appropriate Grothendieck group is $[\mathrm{H}_0(C)] - [\mathrm{H}_1(C)]$. To make this expression more useful, we work with a sheaf version $\mathcal{C}$ of C over a flag variety, where the K-theory possesses a nice basis. To get a handle on $[\mathcal{C}]$, we work with an associated graded of K-theory and show that the top degree terms of $[\mathrm{H}_0(\mathcal{C})]$ and $[\mathcal{C}]$ agree. Finally, we show that the support of $\mathrm{H}_1(\mathcal{C})$ must be a proper closed subset of the support of $\mathcal{C}$, and we use this to show that $\mathrm{H}_1(\mathcal{C})$ must be 0.

Proof. We will prove the statement by induction first on $\ell(w)$ and second on the index of w (see Section 3.3.3 for definitions). The case $w = 1$ is immediate. Using the notation of Proposition 3.4.2, it is immediate that C' is acyclic by induction and the long exact sequence on homology. Hence we only need to analyze the short exact sequence

$$0 \to C' \to C \to \mathfrak{S}_v(\partial)[-1] \otimes F_0^\alpha / F_0^{\alpha-1} \to 0.$$

The induced long exact sequence is

$$0 \to \mathrm{H}_1(C) \to \mathrm{H}_0(\mathfrak{S}_v(\partial)) \otimes F_0^\alpha / F_0^{\alpha-1} \to \mathrm{H}_0(C') \to \mathrm{H}_0(C) \to 0, \tag{3.4.9}$$

so we have to show that $\mathrm{H}_1(C) = 0$, and that the support of $\mathrm{H}_0(C) = M$ is $P = I_{w^{-1}}(\partial)$. We proceed in steps.

Step 1. We first show that the length of $\mathrm{H}_0(C)_P$ restricted to $X(w^{-1})$ is at most 1.

The short exact sequence

$$0 \to C_k \to C' \to \mathfrak{S}_v(\partial) \otimes \langle e'_{w(\beta)} \rangle \to 0$$

induces the sequence

$$0 \to \mathrm{H}_0(C_k) \to \mathrm{H}_0(C') \to \mathrm{H}_0(\mathfrak{S}_v(\partial)) \otimes \langle e'_{w(\beta)} \rangle \to 0.$$

By induction on the filtration in Proposition 3.4.2, the support of $\mathrm{H}_0(C_k)$ is in the union of the $X(\psi_t^{-1})$, and hence does not contain $X(w^{-1})$. So localizing at P, we get an isomorphism

$$\mathrm{H}_0(C')_P \cong \mathrm{H}_0(\mathfrak{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle.$$

So we can restrict this isomorphism to $X(w^{-1})$. Localizing (3.4.9) at P and then restricting to $X(w^{-1})$, we get a surjection

$$\mathrm{H}_0(\mathfrak{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle \to \mathrm{H}_0(C)_P \to 0.$$

By induction, (b) gives that the first term has length 1 over the generic point of $X(w^{-1})$, so $\mathrm{length}(\mathrm{H}_0(C)_P) \leq 1$.

Step 2. We show that the length of $\mathrm{H}_0(C)_P$ restricted to $X(w^{-1})$ is exactly 1.

Without loss of generality, we may extend ∂ to a generic $2n \times 2n$ matrix by embedding it in the upper left corner. Since $w \in \Sigma_n$, these new variables do not affect the Schubert complex when we interpret w as a permutation of Σ_{2n} by having it fix $\{n+1, \dots, 2n\}$, so we will refer to them as **irrelevant variables**. Now consider the Schubert complex $\mathcal{C}$ on the complete flag variety Z of a vector space of dimension $2n$, where the even flag is given by the tautological flag of vector bundles $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_n$, and the odd flag is given by the trivial vector bundles $V_i = \langle e_1, \dots, e_i \rangle$. We identify Z with a quotient $\mathbf{GL}(V_{2n})/B$. Restricting to the unique open B-orbit $\Omega = \Omega_{w_0}$ of Z (which is an affine space), we return to the current situation with some of the irrelevant variables of ∂ specialized to 1 and some specialized to 0. So to finish this step, we only need to show that $\mathrm{length}(\mathrm{H}_0(\mathcal{C})_P) = 1$.

Let $w_0 \in \Sigma_{2n}$ be the long word. Identify V_i over Ω with $F_1^{-2n-1+i}$. Then the intersection

$X_{w_0 w} \cap \Omega$ is defined by the ideal $I_{w^{-1}}(\partial)$:

$$\begin{aligned} \dim(W_p \cap V_q) \geq r_{w_0 w}(p,q) &\iff \dim(W_p \cap F_1^{-2n-1+q}) \geq r_{w_0 w}(p,q) \\ &\iff \operatorname{rank}(W_p \to F_1/F_1^{-2n-1+q}) \leq p - r_{w_0 w}(p,q) \\ &\iff \operatorname{rank}(W_p \to F_1/F_1^{-1-q}) \leq p - r_{w_0 w}(p, 2n-q) = r_w(p,q), \end{aligned}$$

and the map $W_p \to F_1/F_1^{-1-q}$ is given by the upper left $q \times p$ submatrix of ∂.

From the earlier discussion, $[\mathcal{C}] = [\mathrm{H}_0(\mathcal{C})] - [\mathrm{H}_1(\mathcal{C})]$. By Corollary 3.4.7, the top dimension term of $[\mathcal{C}]$ is $[\mathcal{O}_{X_{w_0 w}}]$. So $\operatorname{length}(\mathrm{H}_0(\mathcal{C})_P) - \operatorname{length}(\mathrm{H}_1(\mathcal{C})_P) = 1$. We showed above that the first length is at most 1, which means that it must be 1, and the stalk of $\mathrm{H}_1(\mathcal{C})$ at the generic point of $X_{w_0 w}$ must be 0.

Step 3. The annihilator of $\mathrm{H}_0(C)$ properly contains $I_{v^{-1}}(\partial)$.

We have that $D(w) = D(v) \cup \{(\alpha, w(\beta))\}$, and $(\alpha, w(\beta))$ is a southeast corner of $D(w)$: no boxes of $D(w)$ lie directly below or to the right of it. This means in particular that $I_{w^{-1}}$ is generated by $I_{v^{-1}}$ and the $(r+1) \times (r+1)$ minors of the upper $w(\beta) \times \alpha$ submatrix of ∂, where $r = r_w(\alpha, w(\beta))$. We will show that a minor in $I_{w^{-1}}$ which is not in $I_{v^{-1}}$ annihilates $\mathrm{H}_0(C)$.

The module $\mathrm{H}_0(C)$ is generated by the BSLs of $D(w)$ that only contain marked letters. We have that $w(\beta) - r$ is the number of boxes in $D(w)$ in the αth row. Let $J = \{w(\beta) - r, \dots, w(\beta)\}$ and let I be an $(r+1)$-subset of $\{1, \dots, \alpha\}$. Set $M_{J,I}$ to be the minor of ∂ consisting of the rows indexed by J and the columns indexed by I. We will show that $M_{J,I}$ annihilates $\mathrm{H}_0(C)$.

Given a label j, and a labeling T of the first $\alpha - 1$ rows of $D(w)$, let $T(j)$ be the labeling of $D(w)$ that agrees with T for the first $\alpha - 1$ rows, and in which the ith box in the αth row (going from left to right) has the label i' for $i = 1, \dots, w(\beta) - r - 1$, and the box $(\alpha, w(\beta))$ has the label j. Let $d\colon C_1 \to C_0$ denote the differential. Then $d(T(j)) = \sum_{k=1}^{w(\beta)} \partial_{k,j} T(k')$. Note that $T(k') = 0$ whenever $1 \leq k < w(\beta) - r$ since in this case the label k' appears in the bottom row twice. Since $\alpha > r$, the α equations

$$\sum_{k=w(\beta)-r}^{w(\beta)} \partial_{k,j} T(k') = 0 \text{ for } j = 1, \dots, \alpha$$

in $\mathrm{H}_0(C)$ show that $M_{J,I}$ annihilates $T(k')$ for $1 \leq k \leq w(\beta)$.

It remains to show that $M_{J,I}$ annihilates the elements T where the labels in the first $w(\beta) - r - 1$ boxes of the αth row of T are allowed to take values in $\{(w(\beta) - r)', \dots, w(\beta)'\}$. It is enough to show how to vary the entries one box at a time by decreasing their values (remembering that $i' < j'$ if $i > j$). So fix a column index c which contains the ith box in row α and choose $j > i$. Let T_j denote the labeling obtained from T by changing the label in (α, c) from i' to j'. Let $X \in \mathfrak{b}$ be the matrix which sends the basis vector e'_i to e'_j and kills all other basis vectors. Then $X \cdot T$ is equal to T_j plus other terms whose labels in the αth row are the same as those of T, and hence are annihilated by $M_{J,I}$. Since the actions of $\mathfrak{b}$ and A commute with one another, we conclude that $M_{J,I}$ annihilates T_j.

Step 4. We show that $\mathrm{H}_1(C) = 0$.

By examining different open affine charts of Z, Step 3 shows that the support of $\mathrm{H}_0(\mathcal{C})$ is a proper subset of $X_{w_0 v}$. The argument in Step 2 implies that the same is true for $\mathrm{H}_1(\mathcal{C})$ since the structure sheaves of the Schubert varieties form a basis for $\mathrm{K}(Z)$. So the codimension of the support of $\mathrm{H}_1(\mathcal{C})$ is at least $\ell(w)$. Name the differentials in the complex $d_i\colon \mathcal{C}_i \to \mathcal{C}_{i-1}$. Restrict to an open affine set. Let r_i be the rank of d_i, and set $I(d_i)$ to be the ideal generated by the $r_i \times r_i$ minors of d_i. Let Q be a prime ideal which does not contain $\sqrt{I(d_1)} = \mathrm{Ann}(\mathrm{H}_0(\mathcal{C}))$. Then $\mathrm{H}_0(\mathcal{C})_Q = 0$, which makes the localization $(d_1)_Q\colon (\mathcal{C}_1)_Q \to (\mathcal{C}_0)_Q$ a split surjection. Let $\mathcal{C}_1'$ be the quotient of a splitting of $(d_1)_Q$, so that we have a free resolution

$$0 \to (\mathcal{C}_{\ell(w)})_Q \to \cdots \to (\mathcal{C}_2)_Q \to \mathcal{C}_1'$$

of $\mathrm{H}_1(\mathcal{C})_Q$. Hence the projective dimension of $\mathrm{H}_1(\mathcal{C})_Q$ is at most $\ell(w) - 1$. In general, localizing can only increase the codimension of a module (if we interpret the codimension of 0 to be ∞), so $\operatorname{codim} \mathrm{H}_1(\mathcal{C})_Q \geq \ell(w)$. This is also equal to the depth of its annihilator since Z is nonsingular. So the inequality

$$\operatorname{proj.dim} \mathrm{H}_1(\mathcal{C})_Q < \operatorname{depth} \operatorname{Ann} \mathrm{H}_1(\mathcal{C})_Q$$

contradicts [Eis2, Corollary 18.5] unless $\mathrm{H}_1(\mathcal{C})_Q = 0$. This implies that $\sqrt{I(d_2)_Q}$ is the unit ideal, which means that $\sqrt{I(d_2)} \not\subseteq Q$. Hence we conclude that any prime ideal which contains $\sqrt{I(d_2)}$ also contains $\sqrt{I(d_1)}$. Since a radical ideal is the intersection of the prime ideals containing it, we conclude that $\sqrt{I(d_1)} \subseteq \sqrt{I(d_2)}$. We also get the inclusions

$$\sqrt{I(d_2)} \subseteq \sqrt{I(d_3)} \subseteq \cdots \subseteq \sqrt{I(d_{\ell}(w))}$$

since the rest of the homology of $\mathcal{C}$ vanishes [Eis2, Corollary 20.12], so $\operatorname{depth} I(d_i) \geq \operatorname{depth} I(d_1) \geq \ell(w)$ for all i. We conclude the acyclicity of $\mathcal{C}$ using the Buchsbaum–Eisenbud criterion [Eis2, Theorem 20.9] (the complex is acyclic at the generic point, and the rank of a map over an integral domain stays the same upon passing to its field of fractions, so the rank conditions of this criterion are satisfied).

Step 5. We show that the restriction of $M = \mathrm{H}_0(C)$ to $X(w^{-1})$ is a line bundle, and that its support is exactly $X(w^{-1})$.

Since the projective dimension of M is 1 more than the projective dimension of $\mathrm{H}_0(\mathfrak{S}_v(\partial))$, the codimension of its support can increase by at most 1 by the Auslander–Buchsbaum formula [Eis2, Theorem 19.9]. Thus if we can show that P is contained in the annihilator of M, then it must be equal to its annihilator. We have already done this by showing that the stalk of M at the generic point of $X(w^{-1})$ is nonzero. Thus the codimension and projective dimension of M coincide, which means that it is Cohen–Macaulay by the Auslander–Buchsbaum formula. So (a) is proven.

Let Q be the prime ideal associated with a codimension 1 subvariety of $X(w^{-1})$. To prove (b), we only need to show that M_Q is generated by 1 element. Since $X(w^{-1})$ is normal (Theorem 3.4.4(c)), the local ring $R = \mathcal{O}_{X(v^{-1}),Q}$ is a discrete valuation ring, and hence regular. Furthermore, we have established already that M is Cohen–Macaulay, so M_Q is a free R-module by the Auslander–Buchsbaum formula. So M is free in some open neighborhood around Q. Since M is generated by a single element generically (after further localization), we conclude that M_Q must also be generated by 1 element.

Now (c) follows since we have shown acyclicity over an arbitrary field. □

Corollary 3.4.10. *Let X be an equidimensional Cohen–Macaulay scheme, and let $\partial\colon E \to F$ be a map of vector bundles on X. Let $E_1 \subset \cdots \subset E_n = E$ and $F^{-n} \subset \cdots \subset F^{-1} = F$ be split flags of* *subbundles. Let $w \in \Sigma_n$ be a permutation, and define the degeneracy locus*

$$D_w(\partial) = \{x \in X \mid \operatorname{rank}(\partial_x \colon E_p(x) \to F/F^{-q-1}(x)) \leq r_w(p,q)\},$$

where the ideal sheaf of $D_w(\partial)$ is locally generated by the minors given by the rank conditions. Suppose that $D_w(\partial)$ has codimension $\ell(w)$.

(a) *The Schubert complex $\mathfrak{S}_w(\partial)$ is acyclic, and the support of its cokernel $\mathcal{L}$ is $D_w(\partial)$.*

(b) *The degeneracy locus $D_w(\partial)$ is Cohen–Macaulay.*

(c) *The restriction of $\mathcal{L}$ to $D_w(\partial)$ is a line bundle outside of a certain codimension 2 subset.*

Proof. The statement is local, so we can replace X by $\operatorname{Spec} R$ where R is a local Cohen–Macaulay ring. In this case, $D_w(\partial)$ is defined by the ideal $I_{w^{-1}}(\partial)$. Let ∂^g denote the generic matrix, and let $(C_\bullet, d_\bullet)$ be the complex over $\mathbf{Z}[\partial^g_{i,j}]$ as in Theorem 3.4.8. We get $(C'_\bullet, d'_\bullet) = \mathfrak{S}_w(\partial)$ by specializing the variables $\partial^g_{i,j}$ to elements of R and base changing to R. Let r_i be the rank of d_i, and let $I(d_i)$ be the ideal generated by the $r_i \times r_i$ minors of d_i. By [Eis2, Corollary 20.12], $\sqrt{I(d_1)} = \sqrt{I(d_2)} = \cdots = \sqrt{I(d_{\ell(w)})}$ since C is acyclic and since $\operatorname{depth} I_{w^{-1}}(\partial^g) = \ell(w)$.

Specializing ∂ to elements of R, the same equalities hold when replacing d_i with d'_i. Noting that $I(d'_1) = \operatorname{Ann} \operatorname{coker} d'_1 \supseteq I_{w^{-1}}(\partial)$, we get that

$$\operatorname{depth} I(d'_1) \geq \operatorname{depth} I_{w^{-1}}(\partial) = \operatorname{codim} I_{w^{-1}}(\partial) = \ell(w)$$

by the assumptions that R is Cohen–Macaulay and that $D_w(\partial)$ has codimension $\ell(w)$. Hence $\operatorname{depth} I(d'_i) \geq \ell(w)$ for $i = 1, \dots, \ell(w)$, which means that C' is acyclic by the Buchsbaum–Eisenbud acyclicity criterion [Eis2, Theorem 20.9]. Finally, since the length of the Schubert complex is $\ell(w)$, we conclude that the depth of the cokernel must be $\ell(w)$ by the Auslander–Buchsbaum formula. So in fact $\operatorname{Ann} \operatorname{coker} d'_1 = I_{w^{-1}}(\partial)$, which implies that the support of the cokernel is $D_w(\partial)$. This establishes (a) and (b).

Now (c) follows from Theorem 3.4.8(b). □

3.4.3 Examples.

Example 3.4.11. Let s_i denote the simple transposition that switches i and $i+1$. For $w = s_i$, the maximal transition (3.3.12) simplifies to $(\alpha, \beta) = (i, i+1)$, $v = 1$, $k = 1$, and $\psi_1 = s_{i-1}$. This is also evident from the fact that $\mathfrak{S}_{s_i}(x,y) = x_1 + \cdots + x_i - y_1 - \cdots - y_i$.

Let F_0 and F_1 be vector spaces of dimension n with $n \geq i$. Given a map $\partial \colon F_0 \to F_1$ with distinguished bases $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ (coming from a flag of F_0 and a quotient flag of F_1), respectively, the associated Schubert complex $\mathfrak{S}_{s_i}(\partial)$ is obtained from ∂ by taking the upper left $i \times i$ submatrix of ∂.

The filtration of Proposition 3.4.2 can be described as follows. First, it should look like

$$0 = C_0 \subset C_1 \subset C' \subset C = \mathfrak{S}_{s_i}(\partial)$$

where $C_1 \cong \mathfrak{S}_{s_{i-1}}(\partial)$, $C'/C_1 \cong F_1^{-i}/F_1^{-i+1}$ and $C/C' \cong (F_0^i/F_0^{i-1})[-1]$.

Then C' is the subcomplex $\langle e_1, \dots, e_{i-1}\rangle \to \langle e'_1, \dots, e'_i\rangle$ of C, so the quotient is then F_0^i/F_0^{i-1} concentrated in degree 1. Finally, C_1 is the subcomplex $\langle e_1, \dots, e_{i-1}\rangle \to \langle e'_1, \dots, e'_{i-1}\rangle$ which is isomorphic to $\mathfrak{S}_{s_{i-1}}(\partial)$ and the quotient C'/C_1 is F_1^{-i}/F_1^{-i+1} as required. □

Here is a combinatorial description of the differentials in the Schubert complex for a flagged isomorphism. We will work with just the tensor product complex $\bigotimes_{k=1}^{n-1} \mathrm{D}^{r_k(w)}(F)$. Then the basis elements of its terms are row-strict labelings. The differential sends such a labeling to the signed sum of all possible ways to change a single unmarked letter to the corresponding marked letter. If T' is obtained from T by marking a letter in the ith row, then the sign on T' is $(-1)^n$, where n is the number of unmarked letters of T in the first $i-1$ rows.

Example 3.4.12. Consider the permutation $w = 1423$. Then $D(w) = \{(2,2),(2,3)\}$, and if we use the identity matrix I, $\mathfrak{S}_w(I)$ looks like

$$\begin{array}{lll}
\boxed{2}\boxed{2} \mapsto \boxed{2}\boxed{2'}, & \boxed{2}\boxed{2'} \mapsto 0, & \boxed{1'}\boxed{2'} \mapsto 0, \\
\boxed{2}\boxed{1} \mapsto \boxed{1}\boxed{2'} + \boxed{2}\boxed{1'}, & \boxed{1}\boxed{2'} \mapsto \boxed{1'}\boxed{2'}, & \boxed{1'}\boxed{3'} \mapsto 0, \\
\boxed{1}\boxed{1} \mapsto \boxed{1}\boxed{1'}, & \boxed{2}\boxed{1'} \mapsto -\boxed{1'}\boxed{2'}, & \boxed{2'}\boxed{3'} \mapsto 0. \\
 & \boxed{1}\boxed{1'} \mapsto 0, & \\
 & \boxed{1}\boxed{3'} \mapsto \boxed{1'}\boxed{3'}, & \\
 & \boxed{2}\boxed{3'} \mapsto \boxed{2'}\boxed{3'}, &
\end{array}$$

If we use a generic map $e_1 \mapsto ae'_1 + be'_2 + ce'_3$ and $e_2 \mapsto de'_1 + ee'_2 + fe'_3$ (the images of e_3 and e_4 are irrelevant, and it is also irrelevant to map to e'_4) instead, then the complex looks like

$$0 \to A^3 \xrightarrow{\begin{pmatrix} e & b & 0 \\ 0 & e & b \\ d & a & 0 \\ 0 & d & a \\ 0 & f & c \\ f & c & 0 \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} d & a & -e & -b & 0 & 0 \\ 0 & 0 & -f & -c & a & d \\ -f & -c & 0 & 0 & b & e \end{pmatrix}} A^3 \to M \to 0$$

The cokernel M is Cohen–Macaulay of codimension 2 over $A = K[a,b,c,d,e,f]$. □

Example 3.4.13. Consider the permutation $w = 2413$. Then $D(w) = \{(1,1),(2,1),(2,3)\}$, and if we use the identity matrix I, $\mathfrak{S}_w(I)$ looks like (note the negative signs which come from the fact that we are working with an image of a tensor product of two divided power complexes)

$$\begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{2}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{2}\end{array} - \begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{2'}\end{array}, \quad \begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{1}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{1}\end{array} - \begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{1'}\end{array}$$

$$\begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{1'}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{1'}\end{array} + \begin{array}{ccc}\boxed{1'}&&\\\boxed{1}&&\boxed{2'}\end{array}, \quad \begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{2'}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{2'}\end{array}, \quad \begin{array}{ccc}\boxed{1}&&\\\boxed{2}&&\boxed{3'}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{3'}\end{array}$$

$$\begin{array}{ccc}\boxed{1'}&&\\\boxed{1}&&\boxed{1}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{1}&&\boxed{1'}\end{array}, \quad \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{2}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{2'}\end{array}, \quad \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{1}\end{array} \mapsto \begin{array}{ccc}\boxed{1'}&&\\\boxed{1}&&\boxed{2'}\end{array} + \begin{array}{ccc}\boxed{1'}&&\\\boxed{2}&&\boxed{1'}\end{array}$$

$$\begin{array}[b]{|c|}\hline 1' \\ \hline 1 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 1' \\ \hline\end{array} \mapsto 0, \qquad \begin{array}[b]{|c|}\hline 1' \\ \hline 1 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 2' \\ \hline\end{array} \mapsto \begin{array}[b]{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 2' \\ \hline\end{array}, \qquad \begin{array}[b]{|c|}\hline 1' \\ \hline 1 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 3' \\ \hline\end{array} \mapsto \begin{array}[b]{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 3' \\ \hline\end{array}$$

$$\begin{array}[b]{|c|}\hline 1' \\ \hline 2 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 1' \\ \hline\end{array} \mapsto -\begin{array}[b]{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 2' \\ \hline\end{array}, \qquad \begin{array}[b]{|c|}\hline 1' \\ \hline 2 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 2' \\ \hline\end{array} \mapsto 0, \qquad \begin{array}[b]{|c|}\hline 1' \\ \hline 2 \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 3' \\ \hline\end{array} \mapsto 0$$

$$\begin{array}[b]{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 2' \\ \hline\end{array} \mapsto 0, \qquad \begin{array}[b]{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}\ \begin{array}[b]{|c|}\hline 3' \\ \hline\end{array} \mapsto 0$$

Using a generic matrix defined by $e_1 \mapsto ae'_1 + be'_2 + ce'_3$ and $e_2 \mapsto de'_1 + ee'_2 + fe'_3$ (the other coefficients are irrelevant) instead of the identity matrix gives the following complex

$$0 \to A^2 \xrightarrow{\begin{pmatrix} -d & -a \\ -e & -b \\ -f & -c \\ 0 & -d \\ a & 0 \\ 0 & a \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & a & 0 & d \\ e & -d & 0 & b & 0 & e \\ f & 0 & -d & c & 0 & f \\ a & 0 & 0 & 0 & d & a \\ 0 & a & 0 & 0 & e & b \\ 0 & 0 & a & 0 & f & c \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} -b & a & 0 & -e & d & 0 \\ -c & 0 & a & -f & 0 & d \end{pmatrix}} A^2 \to M \to 0$$

Its cokernel M is Cohen–Macaulay of codimension 3 over $A = K[a, b, c, d, e, f]$. □

3.5 Degeneracy loci.

3.5.1 A formula of Fulton.

Suppose we are given a map $\partial \colon E \to F$ of vector bundles of rank n on a scheme X, together with a flag of subbundles $E_1 \subset E_2 \subset \cdots \subset E_n = E$ and a flag of quotient bundles $F = F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$ such that $\operatorname{rank} E_i = \operatorname{rank} F_i = i$. We assume that the quotients E_i/E_{i+1} are locally free. For a permutation w, define

$$D_w(\partial) = \{x \in X \mid \operatorname{rank}(\partial_x \colon E_p(x) \to F_q(x)) \leq r_w(p, q)\}.$$

Then $\operatorname{codim} D_w(\partial) \leq \ell(w)$. Define Chern classes $x_i = -c_1(E_i/E_{i-1})$ and $y_i = -c_1(\ker(F_i \twoheadrightarrow F_{i-1}))$.

Theorem 3.5.1 (Fulton)**.** *Suppose that X is an equidimensional Cohen–Macaulay scheme of finite type over a field K and $D_w(\partial)$ has codimension $\ell(w)$. Then the identity*

$$[D_w(\partial)] = \mathfrak{S}_w(x, y) \cap [X]$$

holds in the Chow group $\mathrm{A}_{\dim(D_w(\partial))}(X)$.

See [F1, §8] for a more general statement which does not enforce a codimension requirement on $D_w(\partial)$ or assume that X is Cohen–Macaulay. In order to state the connection between the Schubert complex and Fulton's formula, we will need the following lemma which was observed in [Pra, Appendix 6].

Lemma 3.5.2. *Let X be an equidimensional smooth scheme of finite type over a field K, and let D be an irreducible closed subscheme of X of codimension k. Let $C_\bullet$ be a finite complex of vector bundles on X and let $\alpha \in \mathrm{A}^k(X)$. If*

$$\operatorname{supp} C_\bullet = X \setminus \{x \in X \mid (C_\bullet)|_x \text{ is an exact complex}\}$$

is contained in D*, and* $\varphi(\alpha) = [C_\bullet]_k$*, then* $c[D] = \alpha$ *for some* $c \in \mathbf{Q}$*.*

For completeness (and since we have changed notation from [Pra]), we will reproduce the proof.

Proof. Let $i\colon D \to X$ and $j\colon X \setminus D \to X$ be the inclusions. Let the subscript $(-)_\mathbf{Q}$ denote tensoring with $\mathbf{Q}$. Then the diagram (of Abelian groups)

$$\begin{array}{ccccccc} \mathrm{A}^*(D)_\mathbf{Q} & \xrightarrow{i_*} & \mathrm{A}^*(X)_\mathbf{Q} & \xrightarrow{j^*_\mathrm{A}} & \mathrm{A}^*(X \setminus D)_\mathbf{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_\mathbf{Q}} & & \downarrow{\scriptstyle \varphi_\mathbf{Q}} & & \\ & & \operatorname{gr}\mathrm{K}(X)_\mathbf{Q} & \xrightarrow{j^*_\mathrm{K}} & \operatorname{gr}\mathrm{K}(X \setminus D)_\mathbf{Q} & & \end{array}$$

commutes by functoriality of φ, and the first row is exact [F3, Proposition 1.8]. Since $\operatorname{supp}(C_\bullet) \subseteq D$, we have $j^*_\mathrm{K}([C_\bullet]) = 0$, and since $\varphi_\mathbf{Q}(\alpha) = [C_\bullet]$, we conclude that $j^*_\mathrm{A}(\alpha) = 0$ because $\varphi_\mathbf{Q}$ is an isomorphism [F3, Example 15.2.16(b)]. Since we assumed that $\alpha \in \mathrm{A}^k(X)$ we have $\alpha = i_*(\beta)$ for some $\beta \in \mathrm{A}^0(D)_\mathbf{Q}$. But D is irreducible, and hence β is some rational multiple of $[D]$. □

We will verify Theorem 3.5.1 in the case that X is a smooth quasi-projective variety. The general case can be reduced to this case using a "universal construction" and Chow's lemma (see [F1, §8]).

Proof of Theorem 3.5.1. We will use Lemma 3.5.2 with $D = D_w(\partial)$, $C_\bullet = \mathfrak{S}_w(\partial)$, and $\alpha = \mathfrak{S}_w(x, y)$ using the notation from the beginning of this section. We know that $\operatorname{supp} C_\bullet \subseteq D$ and that the codimension of D is $\ell(w) = \deg \alpha$ by Corollary 3.4.10. So in order to conclude Theorem 3.5.1, we need to check that $\varphi(\alpha) = [C_\bullet]$, which is the content of Lemma 3.4.6. Finally, it remains to show that the constant given by Lemma 3.5.2 is 1. This follows from (3.4.5) and Corollary 3.4.10(c). □

3.5.2 Some remarks.

Remark 3.5.3. The previous constructions do not require that the flags be complete, so that one can omit certain subbundles or quotient bundles as desired. The appropriate generalization would be to use partial flag varieties, but we have omitted such generality to keep the notation simpler. □

Remark 3.5.4. A permutation $w \in \Sigma_n$ is **Grassmannian** if it has at most one descent, i.e., there exists r such that $w(1) < w(2) < \cdots < w(r) > w(r+1) < \cdots < w(n)$. Suppose that w is **bigrassmannian**, which means that w and w^{-1} are Grassmannian permutations. This is equivalent to saying that $D(w)$ is a rectangle. In this case, the double Schubert polynomial $\mathfrak{S}_w(x, y)$ is a multi-Schur function for the partition $D(w)$ (one can use [Man, Proposition 2.6.8] combined with (3.2.2)). The degeneracy locus $D_w(\partial)$ can then be described by a single rank condition of the map $\partial\colon E \to F$, so the degeneracy locus formula of Fulton specializes to the Thom–Porteous formula mentioned in the introduction. So in principle, the action of $\mathfrak{b}$ on $\mathfrak{S}_w(\partial)$ should extend to an action of a general linear superalgebra, but it is not clear why this should be true without appealing to Schur polynomials. □

Remark 3.5.5. The Schubert complex only gives a formula for the structure sheaf of the given degeneracy locus in the associated graded of K-theory. A formula for the structure sheaf in the actual K-theory is given in [FL, Theorem 3] using the so-called Grothendieck polynomials, but the formula is not obtained by constructing a complex, so it would be interesting to try to construct these complexes. The degeneracy loci for bigrassmannian permutations are determinantal varieties, and the resolutions in characteristic 0 are explained in [Wey, §6.1]. We should point out that the terms of the resolutions may change with the characteristic, see [Wey, §6.2]. □

Remark 3.5.6. We have seen that the modules which are the cokernels of generic Schubert complexes have linear minimal free resolutions. These modules can be thought of as a "linear approximation" to the ideal which defines the matrix Schubert varieties, which in general have rich and complicated minimal free resolutions. More precisely, we have shown that matrix Schubert varieties possess maximal Cohen–Macaulay modules with linear resolutions. In general, the question of whether or not every graded ring possesses such a module is open (see [ES1, p.543] for further information).

Furthermore, such modules can be obtained geometrically, as outlined in [Wey, Chapter 6, Exercises 34–36] for the case of generic determinantal varieties and their symmetric and skew-symmetric analogues, which we will denote by $D \subset \mathbf{A}^N$. The idea is to find a projective variety V and a subbundle $Z \subset V \times \mathbf{A}^N$ such that the projection $Z \to D$ is a desingularization. In each case, one can find a vector bundle on Z whose pushforward to $\mathbf{A}^N$ provides the desired module supported on D. The proof that its minimal free resolution is linear involves some sheaf cohomology calculations. It would be interesting to try to do this for matrix Schubert varieties, which are our affine models of Fulton's degeneracy loci. The desingularizations of matrix Schubert varieties one might try to use could be given by some analogue of Bott–Samelson varieties. The problem would then be to find the appropriate vector bundle and do the relevant sheaf cohomology calculations. It is the latter part that seems to be complicated. □

Chapter 4

Shapes of free resolutions over local rings

4.1 Introduction

Let M be a finitely generated module over a local ring R. From its minimal free resolution

$$0 \leftarrow M \leftarrow R^{b_0} \leftarrow R^{b_1} \leftarrow R^{b_2} \leftarrow \cdots$$

we obtain the **Betti sequence** $b^R(M) := (b_0, b_1, b_2, \dots)$ of M. Questions about the possible behavior of $b^R(M)$ arise in many different contexts (see [PS] for a recent survey). For instance, the Buchsbaum–Eisenbud–Horrocks Rank Conjecture proposes lower bounds for each $b_i^R(M)$, at least when R is regular, and this conjecture is related to multiplicative structures on resolutions [BE, p. 453], vector bundles on punctured discs [Har, Problem 24], and equivariant cohomology of products of spheres ([Car1] and [Car2, Conjecture II.8]). When R is not regular, there are even more questions about the possible behavior of $b^R(M)$ [Avr, §4].

Here we consider the qualitative behavior of these sequences; we define the **shape** of the free resolution of M as the Betti sequence $b^R(M)$ viewed *up to scalar multiple.* Instead of asking if there exists a module M with a given Betti sequence, say $\mathbf{v} = (18, 20, 4, 4, 20, 18)$, we ask if there exists a Betti sequence $b^R(M)$ with the same shape as $\mathbf{v}$, i.e., whether $b^R(M)$ is a scalar multiple of $\mathbf{v}$. In a sense, this approach is orthogonal to questions like the Buchsbaum–Eisenbud–Horrocks Rank Conjecture, which focus on the *size* of a free resolution.

In this article, we show that this shift in approach, which was motivated by ideas of [BS1], provides a clarifying viewpoint on Betti sequences over local rings. First, we completely classify shapes of resolutions when R is regular. To state the result, we let $\mathbf{V} = \mathbf{Q}^{n+1}$ be a vector space with standard basis $\{\epsilon_i\}_{i=0}^n$.

Theorem 4.1.1. *Let R be an n-dimensional regular local ring, $\mathbf{v} := (v_i)_{i=0}^n \in \mathbf{V}$, and $0 \le d \le n$. Then the following are equivalent:*

(i) *There exists a finitely generated R-module M of depth d such that $b^R(M)$ has shape $\mathbf{v}$, i.e., there exists $\lambda \in \mathbf{Q}_{>0}$ such that $b^R(M) = \lambda\mathbf{v}$.*

(ii) *There exist $a_{-1} \in \mathbf{Q}_{\ge 0}$ and $a_i \in \mathbf{Q}_{>0}$ for $i \in \{0, \dots, n-d-1\}$ such that*

$$\mathbf{v} = a_{-1}\epsilon_0 + \sum_{i=0}^{n-d-1} a_i(\epsilon_i + \epsilon_{i+1}).$$

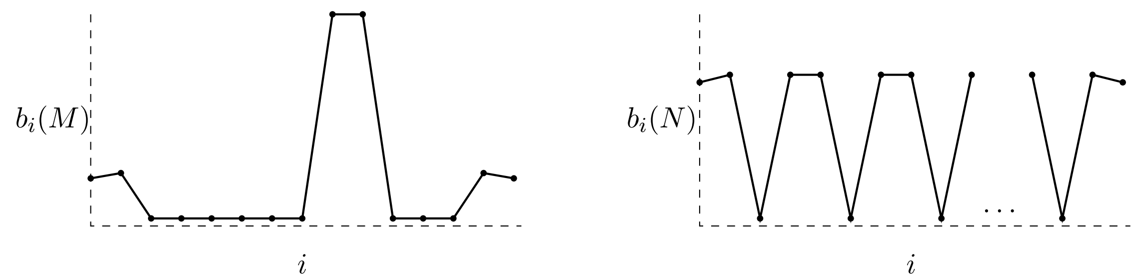


Figure 4-1: On the left, we illustrate the shape of $\mathbf{v} = (1 - \frac{\delta}{2}, 1, \delta, \delta, \delta, \delta, \delta, \delta, 4, 4, \delta, \delta, \delta, 1, 1 - \frac{\delta}{2})$ where $0 < \delta \ll 1$ is a rational number. On the right, we illustrate an oscillating shape, as in Example 4.3.3. Each arises as the shape of some minimal free resolution.

If $a_{-1} = 0$ in (ii)*, then M can also be chosen to be Cohen–Macaulay.*

This demonstrates that there are almost no bounds on the shape of a minimal free R-resolution. While showing that (i) implies (ii) is straightforward, the converse is more interesting, as it leads to examples of free resolutions with unexpected behavior. For instance, let $R = \mathbf{Q}[[x_1, \dots, x_{14}]]$, fix some $0 < \delta \ll 1$, and let $\mathbf{v} = (1 - \frac{\delta}{2}, 1, \delta, \delta, \delta, \delta, \delta, \delta, 4, 4, \delta, \delta, \delta, 1, 1 - \frac{\delta}{2})$. Plotting its entries, the shape of $\mathbf{v}$ is shown in Figure 4-1. As $\mathbf{v}$ satisfies Theorem 4.1.1(ii), there exists a finite length R-module M whose minimal free resolution has this shape. Similar pathological examples abound. As mentioned above, our work is inspired by the Boij–Söderberg perspective that the numerics of minimal free resolutions over a graded polynomial ring S are easier to understand if one works up to scalar multiple. They introduced the cone of Betti diagrams over S and provided conjectures about the structure of this cone. Their conjectures were proven and extended in a series of papers [BS1, BS2, EFW, ES2]. (See also [ES4] for a survey.)

To provide a local version of Boij–Söderberg theory, we study the **cone of Betti sequences** $\mathrm{B}_{\mathbf{Q}}(R)$, which we define to be the convex cone spanned by all points $b^R(M) \in \mathbf{V}$, where M is a finitely generated R-module. Theorem 4.1.1 implies that the closure of $\mathrm{B}_{\mathbf{Q}}(R)$ is spanned by the rays corresponding to ϵ_0 and $(\epsilon_i + \epsilon_{i+1})$ for $i = 0, \dots, n-1$. The point $(\epsilon_i + \epsilon_{i+1})$ can be interpreted as the Betti sequence of the non-minimal complex $(R^1 \overset{\sim}{\longleftarrow} R^1)$, where the copies of R lie in homological positions i and $i+1$. Since this is not itself a minimal free resolution, it follows that $\mathrm{B}_{\mathbf{Q}}(R)$ is not a closed cone, in contrast with the graded case. The facet equation description of $\mathrm{B}_{\mathbf{Q}}(R)$ is also simpler than in the graded case: by Proposition 4.3.1 below, all facets are given by partial Euler characteristics.

Our proof of Theorem 4.1.1 relies on a limiting technique that is possible because we study Betti sequences in R only up to scalar multiple; the introduction of the rational points of $\mathrm{B}_{\mathbf{Q}}(R)$, which can be thought of as formal $\mathbf{Q}$-Betti sequences, enables the use of this technique. To produce the necessary limiting sequences, we first produce local analogues of the Eisenbud–Schreyer pure resolutions, as we have precise control over their Betti numbers.

We emphasize here the fact that $\mathrm{B}_{\mathbf{Q}}(R)$ depends only on the dimension of R. In particular, the result is the same for both equicharacteristic and mixed characteristic rings.

Hypersurface rings

We also examine the shapes of minimal free resolutions over the simplest singular local rings: hypersurface rings. Given a regular local ring $(R, \mathfrak{m}_R)$, we say that Q is a **hypersurface ring of**

R if $Q = R/\langle f\rangle$ and $f \in \mathfrak{m}_R^2$.

Unlike the regular local case, free resolutions are not necessarily finite in length over a hypersurface ring. Hence Betti sequences $b^Q(M)$ lie in an infinite dimensional vector space $\mathbf{W} := \prod_{i=0}^{\infty} \mathbf{Q}$. We let $\{\epsilon_i\}$ denote the coordinate vectors of $\mathbf{W}$ and we write elements of $\mathbf{W}$ as possibly infinite sums $\sum_{i=0}^{\infty} a_i\epsilon_i$. We also view $\mathbf{V}$ as a subspace of $\mathbf{W}$ in the natural way.

The key tool for studying free resolutions over a hypersurface ring is the **standard construction** (which is briefly reviewed in §4.4). Given a Q-module M, this builds a (generally non-minimal) Q-free resolution of M from the minimal R-free resolution of M. The numerics of this free resolution of M are easy to understand in terms of $b^R(M)$. Define $\Phi\colon \mathbf{W} \to \mathbf{W}$ by

$$\Phi(v_0, v_1, v_2, \dots) := (v_0, v_1, v_0 + v_2, v_1 + v_3, v_0 + v_2 + v_4, \dots).$$

The standard construction for M yields a (generally non-minimal) resolution $G_\bullet$ with Betti sequence $b^Q(G_\bullet) = \Phi(b^R(M))$.

Due to this close connection between free resolutions over R and over Q, it is tempting to conjecture that the numerics of $\mathrm{B}_{\mathbf{Q}}(Q)$ should be controlled by the cone $\mathrm{B}_{\mathbf{Q}}(R)$ and the map Φ. However, additional ingredients are clearly required. First, the sequence $\Phi(b^R(M))$ always has infinite length, whereas there do exist minimal free resolutions over Q with finite projective dimension. Second, if an R-module M is annihilated by some polynomial f, then it automatically has rank 0 as an R-module. Thus we should only be interested in applying Φ to modules of rank 0.

The following theorem shows that all minimal free resolutions over hypersurface rings of R are controlled by correcting precisely these two factors.

Theorem 4.1.2. *Let $(R, \mathfrak{m}_R)$ be an n-dimensional regular local ring, let $\overline{R}$ be an $(n-1)$-dimensional regular local ring, and fix $\mathbf{w} := (w_i)_{i=0}^{\infty} \in \mathbf{W}$. Then the following are equivalent:*

(i) *There exists $f \in \mathfrak{m}_R$, a positive integer λ, and a finitely generated $R/\langle f\rangle$-module M such that $b^{R/\langle f\rangle}(M) = \lambda\mathbf{w}$.*

(ii) *There exists an R-module M_1 of rank 0 and an $\overline{R}$-module M_2 such that $\mathbf{w} = \Phi(b^R(M_1)) + b^{\overline{R}}(M_2)$.*

This demonstrates that, except for eventual periodicity, there are essentially no bounds on the shape of a minimal free resolution over a hypersurface ring of R. As in the regular local case, this leads to examples of free resolutions with surprising behavior. For instance, fix any $\delta > 0$ and let $R = \mathbf{Q}[[x_1, \dots, x_{14}]]$. Applying Theorem 4.1.1, there exist M_1 and M_2 so that $\mathbf{w} = \Phi(b^R(M_1)) + b^{\overline{R}}(M_2)$, where

$$\mathbf{w} := (\tfrac{\delta}{2}, 4, 4, \delta, \delta, \delta, \delta, \delta, \delta, \delta, \delta, 1, 1, \delta, 6 + \tfrac{\delta}{2}, 6, 6, 6, \dots).$$

Since $\mathbf{w}$ satisfies Theorem 4.1.2(ii), there exists a module M over a hypersurface ring of R whose minimal free resolution has this shape.

We now make the connection with local Boij–Söderberg theory explicit.

Definition 4.1.3. The **total hypersurface cone** $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ is the closure in $\mathbf{W}$ of the union $\bigcup_{f\in\mathfrak{m}_R} \mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)$.

We show in Remark 4.4.4 that the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ may also be realized as a limit of cones

$$\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)} = \lim_{t\to\infty} \mathrm{B}_{\mathbf{Q}}(R/\langle f_t\rangle) \subseteq \mathbf{W} \tag{4.1.4}$$

for any sequence $(f_t \in \mathfrak{m}_R^t)_{t\geq 1}$.

The following result provides an extremal rays description of this cone.

Proposition 4.1.5. *The cone* $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ *is an* $(n+1)$*-dimensional subcone of* $\mathbf{W}$ *spanned by the following list of* $(n+2)$ *extremal rays:*

(i) *the ray spanned by* ϵ_0*,*

(ii) *the rays spanned by* $(\epsilon_i + \epsilon_{i+1})$ *for* $i \in \{0, \ldots, n-2\}$*, and*

(iii) *the rays spanned by*

$$\sum_{i=n-2}^{\infty} \epsilon_i \quad \text{and} \quad \sum_{i=n-1}^{\infty} \epsilon_i.$$

The proofs of Theorem 4.1.2 and Proposition 4.1.5 rely on two types of asymptotic arguments. First, as in the proof of Theorem 4.1.1, we study sequences of formal **Q**-Betti sequences. Second, we use that the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ is itself a limit, as illustrated in (4.1.4).

In Proposition 4.4.2, we also describe the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ in terms of defining hyperplanes. In addition, we observe that, as in the description of $\mathrm{B}_{\mathbf{Q}}(R)$, most of the extremal rays of $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ do not correspond to actual minimal free resolutions. Note that, based on (4.1.4), the cone $\mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)$ is closely approximated by $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$, at least when the Hilbert–Samuel multiplicity of $R/\langle f\rangle$ is large.

We end by considering the more precise question of completely describing $\mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)$ for a fixed $f \in \mathfrak{m}_R$. The following conjecture claims that the cone $\mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)$ depends only on the dimension and multiplicity of the hypersurface ring $R/\langle f\rangle$.

Conjecture 4.1.6. *Let* Q *be a hypersurface ring of embedding dimension* n *and multiplicity* d*. Then* $\mathrm{B}_{\mathbf{Q}}(Q)$ *is an* $(n+1)$*-dimensional cone, and its closure is defined by the following* $(n+2)$ *extremal rays:*

(i) *the ray spanned by* ϵ_0*,*

(ii) *the rays spanned by* $(\epsilon_i + \epsilon_{i+1})$ *for* $i = \{0, \ldots, n-2\}$*, and*

(iii) *the rays spanned by*

$$\tfrac{d-1}{d}\epsilon_{n-2} + \sum_{i=n-1}^{\infty} \epsilon_i \quad \text{and} \quad \tfrac{1}{d}\epsilon_{n-2} + \sum_{i=n-1}^{\infty} \epsilon_i.$$

Proposition 4.5.1 proves one direction of this conjecture, by showing that $\mathrm{B}_{\mathbf{Q}}(Q)$ belongs to the cone spanned by the proposed extremal rays. We also prove Conjecture 4.1.6 when $\mathrm{edim}(Q) = 2$. Observe also that Proposition 4.1.5 is essentially the $d = \infty$ version of this conjecture.

Notation

Throughout this chapter R will be a regular local ring and Q will be a quotient ring of R. If M is an R-module or a Q-module, then $e(M)$ is the Hilbert–Samuel multiplicity of M and $\mu(M)$ is the minimal number of generators for M. Given a surjection $R^{\mu(M)} \to M$, we denote the kernel by $\Omega(M)$, and in general, we set $\Omega^j(M) = \Omega^1(\Omega^{j-1}(M))$, with the convention $\Omega^0(M) = M$, and we call $\Omega^j(M)$ the j**th syzygy module** of M.

4.2 Passage of graded pure resolutions to a regular local ring

To prove Theorem 4.1.1, we produce a collection of Betti sequences that converge to each extremal ray of $\overline{\mathrm{B}_{\mathbf{Q}}(R)}$. The key step in constructing these sequences is the construction of local analogues of the pure resolutions of Eisenbud and Schreyer.

Let $S = \mathbf{Z}[x_1, \ldots, x_n]$. Fix $d = (d_0, \ldots, d_s) \in \mathbf{Z}^{s+1}$ with $d_i < d_{i+1}$ and $s \leq n$. By [BEKS2, Remark 10.2] and [ES2, §5], we may construct an S-module $M(d)$ that is a generically perfect S-module of codimension s (and hence, $M(d) \otimes_{\mathbf{Z}} \Bbbk$ is a Cohen–Macaulay module of codimension s for every field $\Bbbk$.)

Proposition 4.2.1. *Let R be an n-dimensional regular local ring. Let $S \to R$ be any map sending $x_1, \ldots, x_n$ to an R-regular sequence. Then $M(d) \otimes_S R$ is a Cohen–Macaulay R-module of codimension s, and the Betti sequence of $M(d) \otimes_S R$ is a scalar multiple of*

$$\mathbf{v}(d) := \left(\frac{1}{\prod_{i \neq 0} |d_i - d_0|}, \frac{1}{\prod_{i \neq 1} |d_i - d_1|}, \ldots, \frac{1}{\prod_{i \neq s} |d_i - d_s|}, 0, \ldots, 0 \right) \in \mathbf{V}.$$

Proof. We have noted above that $M(d)$ is a generically perfect S-module of codimension s. It follows from [BV, Theorem 3.9] that $M(d) \otimes_S R$ is Cohen–Macaulay and of the same codimension as $M(d)$. In addition, by [BV, Theorem 3.5], tensoring a minimal S-free resolution of $M(d)$ with R gives a minimal R-free resolution of $M(d) \otimes_S R$. The formula for $\mathbf{v}(d)$ then follows from the Herzog–Kühl equations [HK, Theorem 1]. □

4.3 Cone of Betti sequences for a regular local ring

Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring. Let $\mathbf{V} := \bigoplus_{i=0}^{n} \mathbf{Q}$, with basis $\{\epsilon_i\}$, where $0 \leq i \leq n$. For $i = 0, \ldots, n-1$, set $\rho_i := \epsilon_i + \epsilon_{i+1}$, and set $\rho_{-1} := \epsilon_0$. For all $i \leq j$, we define the partial Euler characteristic functionals

$$\begin{aligned} \chi_{[i,j]} &:= \epsilon_i^* - \epsilon_{i+1}^* + \cdots + (-1)^{j-i} \epsilon_j^* \\ &= \sum_{\ell=i}^{j} (-1)^{\ell - i} \epsilon_\ell^*. \end{aligned}$$

For a ring R, we set $\overline{\mathrm{B}_{\mathbf{Q}}(R)}$ to be the closure of the cone $\mathrm{B}_{\mathbf{Q}}(R) \subseteq \mathbf{V}$, which we describe now.

Proposition 4.3.1. *For any n-dimensional regular local ring R, the following three $(n+1)$-dimensional cones are equal:*

(i) *the closure $\overline{\mathrm{B}_{\mathbf{Q}}(R)}$ of the cone of Betti sequences.*
(ii) *the cone spanned by the rays $\mathbf{Q}_{\geq 0} \langle \rho_{-1}, \rho_0, \rho_1, \ldots, \rho_{n-1} \rangle$.*
(iii) *the intersection of the halfspaces defined by $\chi_{[j,n]} \geq 0$ for $j \in \{0, \ldots, n\}$.*

Proof. The work here lies in showing that (ii) is contained in (i); this is where we use a limiting argument. We first verify the straightforward containments. The rays of (ii) satisfy the inequalities of (iii) because

$$\chi_{[j,n]}(\rho_i) = \begin{cases} 0 & \text{if } j \neq i+1, \\ 1 & \text{if } j = i+1. \end{cases}$$

Conversely, if $\mathbf{v} \in \mathbf{V}$ satisfies all of the inequalities, then we can write $\mathbf{v} = \sum_{i=-1}^{n-1} \chi_{[i+1,n]}(\mathbf{v}) \cdot \rho_i$, which lies in (ii). So we have shown the equivalence of (ii) and (iii).

To see that the functionals of (iii) are nonnegative on $\overline{\mathrm{B}_{\mathbf{Q}}(R)}$, it suffices to consider a point of the form $b^R(M)$. In this case, $\chi_{[i,n]}(b^R(M)) = \operatorname{rank} \Omega^i(M)$ for $i \geq 0$. This implies that $\mathrm{B}_{\mathbf{Q}}(R)$ lies in (iii), and hence so does its closure.

It thus suffices to check that the rays ρ_i in (ii) belong to $\overline{\mathrm{B}_{\mathbf{Q}}(R)}$. Since $\rho_{-1} = \beta(R^1)$, we have $\rho_{-1} \in \mathrm{B}_{\mathbf{Q}}(R)$. To show that $\rho_j \in \overline{\mathrm{B}_{\mathbf{Q}}(R)}$ for $j \geq 0$, we use a limiting argument. Such an argument is necessary because the vectors ρ_j do not belong to $\mathrm{B}_{\mathbf{Q}}(R)$ due to their non-minimal structure (at least when $j > 0$). Adopt the notation of §4.2 and define $\mathbf{v}_j(d)$ to be the unique scalar multiple of $\mathbf{v}(d)$ such that $\mathbf{v}(d)_j = 1$. Based on the formula for $\mathbf{v}(d)$ from Proposition 4.2.1, view $\mathbf{v}_j$ as a map from $\mathbf{Z}^{n+1} \to \mathbf{V}$ (with poles) defined by the formula

$$\mathbf{v}_j(d_0, \dots, d_n) = \left(\frac{\prod_{i \neq j} |d_i - d_j|}{\prod_{i \neq 0} |d_i - d_0|}, \frac{\prod_{i \neq j} |d_i - d_j|}{\prod_{i \neq 1} |d_i - d_1|}, \dots, \frac{\prod_{i \neq j} |d_i - d_j|}{\prod_{i \neq n} |d_i - d_n|} \right) \in \mathbf{V}.$$

And now for the crucial choice, which is explored further in Example 4.3.2. For each j, consider the sequence $\{d^{j,t}\}_{t \geq 0}$ defined by $d^{j,t} := (0, t, 2t, \dots, jt, jt+1, (j+1)t+1, \dots, (n-1)t+1)$. In other words,

$$d_k^{j,t} = \begin{cases} kt & \text{if } k \leq j, \\ (k-1)t + 1 & \text{if } k > j. \end{cases}$$

We claim that $\rho_j = \lim_{t \to \infty} \mathbf{v}_j(d^{j,t})$. This would imply, by Proposition 4.2.1, that $\rho_i \in \overline{\mathrm{B}_{\mathbf{Q}}(R)}$, thus completing the proof. To prove this claim, we observe that the jth coordinate function of $\mathbf{v}_j$ equals 1 and $\mathbf{v}_j(d)$ lies in the hyperplane defined by $\chi_{[0,n]} = 0$. So it suffices to prove that the ℓth coordinate function of $\mathbf{v_j}$ goes to 0 for all $\ell \neq j, j+1$. We directly compute

$$\lim_{t \to \infty} \mathbf{v}_j(d^{j,t})_\ell = \lim_{t \to \infty} \frac{\prod_{i \neq j} |d_i^{j,t} - d_j^{j,t}|}{\prod_{i \neq \ell} |d_i^{j,t} - d_\ell^{j,t}|} = \lim_{t \to \infty} \frac{O(t^{n-1})}{O(t^n)} = 0. \qquad \square$$

Example 4.3.2. If $n = 4$, then $d^{1,t} = (0, t, t+1, 2t+1, 3t+1)$. Over $S = \Bbbk[x_1, \dots, x_4]$ with the standard grading, this degree sequence corresponds to the Betti diagram

$$\beta^S(M(d^{1,t})) = \begin{bmatrix} \beta_0^{1,t} & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & \beta_1^{1,t} & \beta_2^{1,t} & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & - & \beta_3^{1,t} & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & - & - & \beta_4^{1,t} \end{bmatrix} \begin{matrix} \left.\right\} t-1 \text{ rows} \\ \left.\right\} t-1 \text{ rows} \\ \left.\right\} t-1 \text{ rows} \end{matrix}$$

where there are gaps of $t - 3$ rows of zeroes between the various nonzero entries. Notice that as $t \to \infty$, this Betti diagram gets longer. It is thus necessary to consider the total Betti numbers β_i (i.e., to forget about the individual graded Betti numbers $\beta_{i,j}$) before it makes sense to consider a limit. $\square$

Proof of Theorem 4.1.1. First we show that (i) implies (ii). Let M be any module of depth d such that $b^R(M) = \lambda \mathbf{v}$. Since $\chi_{[i,n]}(b^R(M)) = \operatorname{rank} \Omega^i(M)$ for $i \geq 0$, the Auslander–Buchsbaum formula implies that this is strictly positive for $i = 1, \dots, n-d$ and 0 for $i > n-d$. The proof of Proposition 4.3.1 then shows that $b^R(M)$ has the desired form.

Next we show that (ii) implies (i). If there exists any M such that $b^R(M) = \mathbf{v}$, then the Auslander–Buchsbaum formula implies that M has depth d. It thus suffices to produce a module M with the desired Betti sequence. We may also assume that the coefficient a_{-1} of ρ_{-1} equals 0.

Let C denote the cone spanned by $\rho_0, \dots, \rho_{n-d-1}$, so that $\mathbf{v}$ now belongs to the interior of C. The proof of Proposition 4.3.1 illustrates that for each $i = 0, \dots, n-d-1$, we can construct ρ_i as the limit of Betti sequences of Cohen–Macaulay modules of codimension $n-d$. Since we can construct every extremal ray of C via such a sequence, it follows that every interior point of C can be written as a $\mathbf{Q}$-convex combination of the Betti sequences of Cohen–Macaulay R-modules of codimension $n-d$. In particular, $\mathbf{v}$ has this property, and hence $\mathbf{v} \in \mathrm{B}_{\mathbf{Q}}(R)$, as desired. This construction also implies the final sentence of the theorem, as we have written $\mathbf{v}$ as the sum of Betti sequences of Cohen–Macaulay modules of codimension $n-d$. ☐

Example 4.3.3 (Oscillation of Betti numbers)**.** Let $n = \dim R$ be congruent to 1 mod 3. Let $0 < \delta \ll 1$ be a rational number and set

$$a'_i := \begin{cases} 0 & \text{if } i = -1, \\ 1 - \frac{\delta}{2} & \text{if } i \geq 0 \text{ and } i \equiv 0 \pmod 3, \\ \frac{\delta}{2} & \text{if } i \geq 0 \text{ and } i \equiv \pm 1 \pmod 3. \end{cases}$$

Let $\mathbf{v}' := \sum_i a'_i \rho_i$, so that the entries of $\mathbf{v}'$ oscillate between 1 and δ. Then there exists a finite length R-module N such that $b^R(N)$ is a scalar multiple of $\mathbf{v}'$. See Figure 4-1. ☐

Remark 4.3.4. For a finite length module, the Buchsbaum–Eisenbud–Horrocks Rank Conjecture proposes that $b_i(M) \geq \binom{n}{i}$ for $i = 0, 1, \dots, n$. It is natural to seek a sharper lower bound B_i that depends on the number of generators of M and the dimension of the socle of M. For B_1 we may set $B_1(b_0, b_n) := b_0 - 1 + n$, and then $b_1 \geq B_1(b_0, b_n)$; something similar holds for B_{n-1}. However, Theorem 4.1.1 implies that when $i \neq 1, n-1$ there is no such linear bound. This follows immediately from the fact that, for any $0 < \delta \ll 1$, there is a resolution with shape $(1, 1+\frac{\delta}{2}, \delta, \dots, \delta, 1+\frac{\delta}{2}, 1)$. ☐

Question 4.3.5. *Are there nonlinear functions $B_i(b_0, b_n)$ such that $b_i(M) \geq B_i(b_0(M), b_n(M))$ for all finite length modules M?*

Remark 4.3.6 (The graded/local comparison)**.** If $S = \Bbbk[x_1, \dots, x_n]$ (with the standard grading) and $R = \Bbbk[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$, then there is a map $\mathrm{B}_{\mathbf{Q}}(S) \to \mathrm{B}_{\mathbf{Q}}(R)$ obtained by "forgetting the grading" and localizing. Theorem 4.1.1 implies that this map is surjective. It would be interesting to understand if a similar statement is true if we replace S by a more general graded ring. ☐

4.4 Betti sequences over hypersurface rings I: the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$

We say that Q is a **hypersurface ring** of a regular local ring $(R, \mathfrak{m})$ if $Q = R/\langle f \rangle$ for some nonzerodivisor $f \in R$. To avoid trivialities, we assume that $f \in \mathfrak{m}^2$. Let $n := \dim R$ and $d := \mathrm{ord}(f)$, i.e., the unique integer d such that $f \in \mathfrak{m}^d \smallsetminus \mathfrak{m}^{d+1}$. The following result is the basis for the "standard construction." See [Sha], [Eis1, §7], or [Avr] for more details.

Theorem 4.4.1 (Eisenbud, Shamash)**.** *Given a Q-module M, let $\mathbf{F}_\bullet \to M$ be its minimal free resolution over R. Then there are maps $s_k \colon \mathbf{F}_\bullet \to \mathbf{F}_{\bullet+2k-1}$ for $k \geq 0$ such that*

(i) *s_0 is the differential of $\mathbf{F}_\bullet$.*
(ii) *$s_0 s_1 + s_1 s_0$ is multiplication by f.*
(iii) *$\sum_{i=0}^{k} s_i s_{k-i} = 0$ for all $k > 1$.*

We note that if R and Q are graded local rings, then the maps s_k can be chosen to be homogeneous. Using the s_k, we may form a new complex $\mathbf{F}'_\bullet$ with terms

$$\mathbf{F}'_i = \bigoplus_{j\geq 0} \mathbf{F}_{i-2j} \otimes_R Q$$

and with differentials given by taking the sum of the maps

$$\mathbf{F}_i \otimes_R Q \xrightarrow{(s_0, s_1, s_2, \dots)} (\mathbf{F}_{i-1} \oplus \mathbf{F}_{i-3} \oplus \mathbf{F}_{i-5} \oplus \cdots) \otimes_R Q.$$

Then $\mathbf{F}'_\bullet \to M$ is a Q-free resolution which need not be minimal.

With $\mathbf{W} = \prod_{i=0}^{\infty} \mathbf{Q}$ and $\epsilon_i \in \mathbf{W}$ the ith coordinate vector, we define $\Phi \colon \mathbf{W} \to \mathbf{W}$ by

$$\Phi(w_0, w_1, \dots) := (w_0, w_1, w_0 + w_2, w_1 + w_3, w_0 + w_2 + w_4, \dots).$$

In other words, the ℓth coordinate function of Φ is given by

$$\Phi_\ell(w_0, w_1, \dots) = \begin{cases} \sum_{i=0}^{\frac{\ell}{2}} w_{2i} & \text{if } \ell \text{ is even,} \\ \sum_{i=0}^{\frac{\ell-1}{2}} w_{2i+1} & \text{if } \ell \text{ is odd.} \end{cases}$$

As in Section 4.3, let $\rho_{-1} := \epsilon_0$ and $\rho_i := \epsilon_i + \epsilon_{i+1}$ for $i \geq 0$.

Free resolutions over a hypersurface ring can be infinite in length, but they are periodic after n steps [Eis1, Corollary 6.2], so that $b_i^Q(M) = b_{i+1}^Q(M)$ for all $i \geq n$ [Eis1, Proposition 5.3]. Thus, if we seek to describe the cone of Betti sequences in the hypersurface case, it is necessary to include some rays with infinite support. We define

$$\tau_i^\infty := \sum_{j=i}^{\infty} \epsilon_j \in \mathbf{W}$$

and note that $\tau_i^\infty = \Phi(\rho_i)$. The rays τ_{n-2}^∞ and τ_{n-1}^∞ will be especially important for us.

We now give a precise description of the total hypersurface cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ from Definition 4.1.3.

Proposition 4.4.2. *The following three $(n+1)$-dimensional cones in $\mathbf{W}$ coincide:*

(i) *The total hypersurface cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$.*
(ii) *The cone spanned by the rays $\mathbf{Q}_{\geq 0}\langle \rho_{-1}, \rho_0, \dots, \rho_{n-2}, \tau_{n-2}^\infty, \tau_{n-1}^\infty \rangle$.*
(iii) *The cone defined by the functionals*

$$\begin{cases} \chi_{[i,j]} \geq 0 & \text{for all } i \leq j \leq n \text{ with } i - j \text{ even,} \\ \chi_{[i,i+1]} = 0 & \text{for all } i \geq n, \text{ and} \\ \chi_{[n-1,n]} \geq 0. \end{cases}$$

Proof. It is straightforward to check that the extremal rays satisfy the desired facet inequalities, and hence we have (ii)⊆(iii). The reverse inclusion is more difficult than the analogous statement in Proposition 4.3.1 because here (ii) is not a simplicial cone. We first identify the boundary facets, and then show that for each boundary facet, one of the listed functionals vanishes on it.

To do this, we use that these rays satisfy a unique linear dependence relation. When n is even, the relation is given by

$$\tau_{n-1}^{\infty} + \rho_{n-3} + \cdots + \rho_{-1} = \tau_{n-2}^{\infty} + \rho_{n-4} + \cdots + \rho_0,$$

and a similar relation holds when n is odd. We now consider subsets of these rays of size n, which we index by the two rays that we omit from the collection. These fall into three categories:
(a) $\{\rho_i, \rho_j\}$ with $i < j$,
(b) $\{\rho_i, \tau_j^{\infty}\}$, and
(c) $\{\tau_{n-2}^{\infty}, \tau_{n-1}^{\infty}\}$.
Any such collection is linearly independent, and hence spans a unique hyperplane of the subspace

$$\{w \in \mathbf{W} \mid w_{n+i} = w_n \text{ for all } i \geq 0\}. \tag{4.4.3}$$

As such, there is a unique up to scalar functional vanishing on each collection; we write $F_{i,j}$ for the corresponding functional in type (a), $G_{i,j}$ for type (b), and H for type (c). In order to show the desired containment, we compute these functionals and determine which correspond to boundary facets of (ii) by evaluating the functionals on their corresponding omitted rays.

To begin, note that if $j < n-2$, then $F_{i,j} = \chi_{[i+1,j]}$. This evaluates to 1 on ρ_i for $i \geq 0$, $(-1)^{j-(i+1)}$ on ρ_j, and 0 on the remaining rays. Hence it determines a boundary facet if and only if $i+1$ and j have the same parity. In addition, for any $i < n-2$, $F_{i,n-2} = \chi_{[n-1,n]}$, which is the last functional in (iii).

Next, observe that $G_{i,n-2} = \chi_{[i+1,n]}$. If $i < n-2$, this evaluates to 1 on ρ_i for $i \geq 0$, $(-1)^{n-(i+1)}$ on τ_{n-2}, and 0 on the remaining rays. Hence in this case, it yields a boundary facet if and only if $i+1$ and n have the same parity. Similarly, $G_{i,n-1} = \chi_{[i+1,n-1]}$ if and only if $i < n-2$, which is a boundary facet only if n and i have the same parity.

Finally, we compute that $G_{n-2,n-2} = \chi_{[n-1,n]}$, $G_{n-2,n-1} = \chi_{[n-1,n]}$, and $H = \chi_{[n,n]}$, which all appear in (iii). As the subspace description (4.4.3) accounts for the remaining functionals, we have established the equivalence of (ii) and (iii).

We next show that (i)$\subseteq$(iii). For this it suffices to check that the functionals in (iii) are nonnegative on points in $\mathrm{B}_{\mathbf{Q}}(Q)$, where $Q = R/\langle f \rangle$ and $f \in \mathfrak{m}$ is arbitrary. We thus reduce to the consideration of a point $\mathbf{w} = b^Q(M)$, where M is a Q-module. In this case,

$$\chi_{[i,j]}(b^Q(M)) = \frac{1}{e(Q)} \left(e(\Omega^i(M)) + (-1)^{i-j} e(\Omega^j(M)) \right),$$

which is certainly nonnegative when i and j have the same parity. It follows from [Eis1, Proposition 5.3, Corollary 6.2] that $\chi_{[i,i+1]}(b^Q(M)) = 0$ for $i \geq n$. Thus it remains to check the inequality $\chi_{[n-1,n]}(b^Q(M)) \geq 0$. Using $\mu(N)$ to denote the minimal number of generators of a module N, we have

$$\chi_{[n-1,n]}(b^Q(M)) = \mu(\Omega^{n-1}(M)) - \mu(\Omega^n(M)).$$

Both of these syzygy modules are maximal Cohen–Macaulay Q-modules. The key difference is that $\Omega^{n-1}(M)$ might have a free summand, whereas $\Omega^n(M)$ does not. Since maximal Cohen–Macaulay modules without free summands over hypersurface rings have a periodic resolution by [Eis1, Theorem 6.1(ii)], it follows that $\chi_{[n-1,n]}(b^Q(M))$ computes the number of free summands in $\Omega^{n-1}(M)$, so it is nonnegative.

To complete the proof, we show that (ii)$\subseteq$(i) by showing that each extremal ray lies in $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$. We first show that ρ_i belongs to $\overline{\mathrm{B}_{\mathbf{Q}}(R/\langle f \rangle)}$ for any f. Choose a regular local subring $R' \subseteq R/\langle f \rangle$ of dimension $n-1$ and an R'-module M'. Then $b^{R/\langle f \rangle}(M' \otimes_{R'} R/\langle f \rangle) = b^{R'}(M')$ because $R/\langle f \rangle$ is finite and flat over R'. In particular, $\overline{\mathrm{B}_{\mathbf{Q}}(R')} \subseteq \overline{\mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)}$. Since $\rho_i \in \overline{\mathrm{B}_{\mathbf{Q}}(R')}$ by Proposition 4.3.1, we have $\rho_i \in \overline{\mathrm{B}_{\mathbf{Q}}(R/\langle f\rangle)}$.

Finally, we must show that τ_{n-2}^{∞} and τ_{n-1}^{∞} belong to $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$. This is where the advantage of working with $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ becomes clear, as it enables a second limiting argument that, roughly speaking, makes the standard construction exact. The key observation is summarized in Lemma 4.4.5 below.

In fact, we now show the more general statement that $\Phi(\rho_i) \in \overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ for $i = 0, \ldots, n-1$. Fix i and let $d^{i,t}$ be the sequence of degree sequences defined in the proof of Proposition 4.3.1. For each t, we choose any polynomial $f_t \in \mathfrak{m}^{d_n^{i,t} - d_0^{i,t}+1}$. We now apply Lemma 4.4.5, along with the fact that Φ is continuous, to conclude that

$$\begin{aligned}\tau_i^{\infty} &= \Phi(\rho_i) \\ &= \Phi\left(\lim_{t\to\infty} b^R(M(d^{i,t}) \otimes_S R)\right) \\ &= \lim_{t\to\infty} \Phi\left(b^R(M(d^{i,t}) \otimes_S R)\right) \\ &= \lim_{t\to\infty} b^{R/\langle f_t\rangle}(M(d^{i,t}) \otimes_S R).\end{aligned}$$

Since $b^{R/\langle f_t\rangle}(M(d^{i,t}) \otimes_S R) \in \overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ for all t, it follows that the final limit lies in $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$. □

Remark 4.4.4. The proof of Proposition 4.4.2 goes through if we replace $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ by the closure of the limit cone $\lim_{t\to\infty} \mathrm{B}_{\mathbf{Q}}(R/\langle f_t\rangle)$, illustrating that these two cones are equal as well. This justifies equation (4.1.4). □

Lemma 4.4.5. *Let M be an R-module that is annihilated by $\mathfrak{m}^{N_0}$ and let $f \in \mathfrak{m}^N$ with $N \gg N_0$. Then*

$$\Phi(b^R(M)) = b^{R/\langle f\rangle}(M).$$

More specifically, let $d = (d_0, \ldots, d_n)$ be a degree sequence, $M(d) \otimes_S R$ be defined as in Proposition 4.2.1, and $f \in \mathfrak{m}^{d_n - d_0 + 1}$. Then

$$\Phi(b^R(M(d) \otimes_S R)) = b^{R/\langle f\rangle}(M(d) \otimes_S R).$$

Proof. Since R is a regular local ring, the minimal R-free resolution of M has finite length. So there are only finitely many j such that the s_j in Theorem 4.4.1 are nonzero, and there is some positive integer P such that the matrix entries in the minimal R-free resolution of M belong to $\mathfrak{m}^P$. To conclude, we need to know that the entries of each s_j belong to the maximal ideal $\mathfrak{m}$. From Theorem 4.4.1(iii), this will be true if it holds for $j = 1$, and this in turn is true if we set $N_0 = P$ and apply Theorem 4.4.1(ii). □

Remark 4.4.6. Assume that $n \geq 3$. By [DRS, Lemma 2.4.2], there are exactly two triangulations of the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$, which we now describe. First, we project from $\mathbf{W}$ onto the first $n+1$ coordinates. This does not change the combinatorial structure of the cone. The hyperplane section of the projection given by $\epsilon_0 + \cdots + \epsilon_n = 1$ is an n-dimensional polytope with vertices ρ_{-1}, $\frac{1}{2}\rho_0$, $\frac{1}{2}\rho_1, \ldots, \frac{1}{2}\rho_{n-2}$, $\frac{1}{3}\tau_{n-2}^{\infty}$, $\frac{1}{2}\tau_{n-1}^{\infty}$.

To express the triangulations, let Δ_r denote the polytope generated by all vertices other than r. If n is odd, then the two triangulations are

$$\{\Delta_{\rho_i} \mid i \text{ odd}, i \neq n-2\} \cup \{\Delta_{\tau_{n-1}^{\infty}}\} \quad \text{or} \quad \{\Delta_{\rho_i} \mid i \text{ even}\} \cup \{\Delta_{\tau_{n-2}^{\infty}}\}.$$

If n is even, then the two triangulations are

$$\{\Delta_{\rho_i} \mid i \text{ odd}\} \cup \{\Delta_{\tau^\infty_{n-2}}\}, \quad \text{or} \quad \{\Delta_{\rho_i} \mid i \text{ even}, i \neq n-2\} \cup \{\Delta_{\tau^\infty_{n-1}}\}. \qquad \square$$

4.5 Betti sequences over hypersurface rings II: A fixed hypersurface

For a regular local ring $(R, \mathfrak{m})$ and $f \in \mathfrak{m}_R$, the cone $\overline{\mathrm{B}_{\mathbf{Q}}(R_\infty)}$ is larger than $\mathrm{B}_{\mathbf{Q}}(Q)$ for the hypersurface ring $Q = R/\langle f\rangle$. In this section, we seek to make this relationship precise. Set $Q := R/\langle f\rangle$ and $d := \mathrm{ord}(f)$, i.e., $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d-1}$. We note that $e(Q) = d$. We define the vectors

$$\tau^d_{n-2} := \left(\tfrac{d-1}{d}\epsilon_{n-2} + \sum_{j=n-1}^{\infty} \epsilon_j\right) \qquad \text{and} \qquad \tau^d_{n-1} := \left(\tfrac{1}{d}\epsilon_{n-2} + \sum_{\ell=n-1}^{\infty} \epsilon_\ell\right).$$

We also define the functionals

$$\xi^d_{[i,j]} := \begin{cases} -\epsilon^*_j + d\chi_{[i,j-1]} & \text{if } i-j \text{ is odd,} \\ (d-1)\epsilon^*_j + d\chi_{[i,j-1]} & \text{if } i-j \text{ is even.} \end{cases}$$

The following proposition gives some partial information about Conjecture 4.1.6.

Proposition 4.5.1. *The following two $(n+1)$-dimensional cones in $\mathbf{W}$ coincide:*

(i) *The cone spanned by the rays $\mathbf{Q}_{\geq 0}\langle \rho_{-1}, \rho_0, \ldots, \rho_{n-2}, \tau^d_{n-2}, \tau^d_{n-1}\rangle$.*

(ii) *The cone defined by the functionals*

$$\begin{cases} \xi^d_{[i,n]} \geq 0 & \text{for all } 0 \leq i \leq n, \\ \chi_{[i,j]} \geq 0 & \text{for all } i \leq j \leq n \text{ and } i-j \text{ even,} \\ \chi_{[i,i+1]} = 0 & \text{for all } i \geq n, \text{ and} \\ \chi_{[n-1,n]} \geq 0. \end{cases}$$

Furthermore, this cone contains $\overline{\mathrm{B}_{\mathbf{Q}}(Q)}$.

Proof. One may check that the cones (i) and (ii) coincide by an argument entirely analogous to that used in the proof of Proposition 4.4.2. It thus suffices to check that the functionals in (ii) are satisfied by all points in $\mathrm{B}_{\mathbf{Q}}(Q)$. By applying Proposition 4.4.2, we immediately reduce to the case of showing that $\xi^d_{[i,n]}$ is nonnegative on any Betti sequence $b^Q(M)$.

Fix a finitely generated Q-module M and a minimal resolution of M: $0 \leftarrow M \leftarrow Q^{b_0} \leftarrow Q^{b_1} \leftarrow \cdots$. To compute $\xi^d_{[i,n]}(b^Q(M))$, we consider the exact sequence

$$0 \longleftarrow \Omega^i(M) \longleftarrow Q^{b_i} \longleftarrow Q^{b_{i+1}} \longleftarrow \cdots \longleftarrow Q^{b_n} \longleftarrow \Omega^{n+1}(M) \longleftarrow 0\,.$$

Assume now that $n-i$ is even and that $i \geq 1$. Taking multiplicities, we obtain the equation

$$e(\Omega^i(M)) + e(Q^{b_{i+1}}) + \cdots + e(Q^{b_{n-1}}) + e(\Omega^{n+1}(M)) = e(Q^{b_i}) + e(Q^{b_{i+2}}) + \cdots + e(Q^{b_n}),$$

which can be rewritten as

$$e(\Omega^i(M)) = d\chi_{[i,n]}\left(b^Q(M)\right) - e(\Omega^{n+1}(M)).$$

Since $\Omega^{n+1}(M)$ is Cohen–Macaulay, $e\left(\Omega^{n+1}(M)\right) \geq \mu\left(\Omega^{n+1}(M)\right) = b^Q_{n+1}(M) = b_n(M)$. Hence

$$e(\Omega^i(M)) \leq d\chi_{[i,n]}\left(b^Q(M)\right) - b_n(M) = \xi^d_{[i,n]}\left(b^Q(M)\right).$$

It follows that $\xi^d_{[i,n]}\left(b^Q(M)\right)$ is nonnegative, as desired.

When $n-i$ is odd and $i \geq 1$, essentially the same argument holds, starting instead from the exact sequence

$$0 \longleftarrow \Omega^i(M) \longleftarrow Q^{b_i} \longleftarrow Q^{b_{i+1}} \longleftarrow \cdots \longleftarrow Q^{b_{n-1}} \longleftarrow \Omega^n(M) \longleftarrow 0\,.$$

The same argument also holds when $i = 0$, after one replaces $e(\Omega^i(M))$ by the number

$$e' := \begin{cases} e(M) & \text{if } \dim(M) = \dim(Q), \\ 0 & \text{otherwise.} \end{cases}$$ □

The opposite inclusion also holds when Q has embedding dimension 2.

Proposition 4.5.2. *If Q is a hypersurface ring of embedding dimension 2, then $\mathrm{B}_{\mathbf{Q}}(Q)$ satisfies Conjecture 4.1.6.*

Proof. By Proposition 4.5.1, it suffices to show that the desired extremal rays lie in $\overline{\mathrm{B}_{\mathbf{Q}}(Q)}$. We may quickly reduce to showing that $\tau^d_0, \tau^d_1 \in \overline{\mathrm{B}_{\mathbf{Q}}(Q)}$. Let $\mathfrak{m}_Q$ denote the maximal ideal of Q, $Q' := Q/\mathfrak{m}_Q^{d-1}$, and $\omega_{Q'}$ be its canonical module. A direct computation confirms that $d\tau^d_1 = b^Q(Q')$ and $d\tau^d_0 = b^Q\left(\omega_{Q'}\right)$. □

Remark 4.5.3 (Codimension 2 complete intersections)**.** For arbitrary quotient rings Q of a regular local ring R, the cone of Betti sequences $\mathrm{B}_{\mathbf{Q}}(Q)$ need not be finite dimensional. For instance, consider $Q = \mathbf{Q}[[x,y]]/\langle f_1, f_2\rangle$ for any regular sequence f_1, f_2 inside $\langle x,y\rangle^2$. Let $\mathbf{T}_\bullet$ be the Tate resolution of the residue field of Q. Since Q is Gorenstein, and hence self-injective, we may construct a doubly infinite acyclic complex $\mathbf{F}_\bullet$ as below:

$$\mathbf{F}_\bullet : \quad \cdots \longleftarrow \mathbf{T}^*_1 \longleftarrow \mathbf{T}^*_0 \longleftarrow \mathbf{T}_0 \longleftarrow \mathbf{T}_1 \longleftarrow \mathbf{T}_2 \longleftarrow \cdots\,.$$

For all $i \geq 0$, let M_i be the kernel of $\mathbf{T}^*_i \to \mathbf{T}^*_{i+1}$, and set $\tau_i := b^Q(M_i)$. The τ_i are linearly independent since $\operatorname{rank} \mathbf{T}_i = i+1$ for all i (see [AB, Example 4.2] for details). So we see that $\mathrm{B}_{\mathbf{Q}}(Q)$ is infinite dimensional. In particular, $\mathrm{B}_{\mathbf{Q}}(Q)$ is spanned by infinitely many extremal rays. □

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