Modular Invariance for Vertex Operator Superalgebras

by

Jethro Van Ekeren

B.Sc. Hons. in Mathematics and Physics,
University of Waikato 2008

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2012

Author ....
Department of Mathematics
April 19, 2012

Certified by ....
Victor Kac
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Bjorn Poonen
Chairman, Department Committee on Graduate Theses

Modular Invariance for Vertex Operator Superalgebras

by
Jethro Van Ekeren

Submitted to the Department of Mathematics
on April 19, 2012, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

We generalize Zhu's theorem on modular invariance of characters of vertex operator algebras (VOAs) to the setting of vertex operator superalgebras (VOSAs) with rational, rather than integer, conformal weights. To recover $SL_2(\mathbb{Z})$-invariance, it turns out to be necessary to consider characters of twisted modules. Initially we assume our VOSA to be rational, then we replace rationality with a different (weaker) condition. We regain $SL_2(\mathbb{Z})$-invariance by including certain 'logarithmic' characters. We apply these results to several examples. Next we define and study 'higher level twisted Zhu algebras' associated to a VOSA. Using a novel construction we compute these algebras for some well known VOAs.

Thesis Supervisor: Victor Kac
Title: Professor of Mathematics

Acknowledgments

First and foremost my thanks go to my Ph.D. advisor Victor Kac for his constant guidance and encouragement, the value to me of which cannot be overstated. His friendliness and his supply of interesting mathematical questions made the work both enjoyable and stimulating.

I want to take this chance to thank Pavel Etingof and Roman Bezrukavnikov for some comments and information related to the material in Chapter 4. Although they did not make it into the thesis, these comments were helpful to me. I am especially grateful to David Vogan in this regard, he gave me a lot of thoughtful advice related to the material in Section 3.5.

I benefited from several sources of funding: the Fulbright Scholarship, the MIT Presidential Fellowship, and the RA funding that I received through Victor Kac's grant.

I am lucky to have made good friends during my time at MIT. Spending time with my flatmates, hikes with other maths students, and weekly lunches with my academic sister Uhi Rinn have all been great. Of course I would not be here today without the moral support of my family, which began long before I had any idea where I would end up. Finally I wish to thank Drew, for teaching me to program in Python, for his good company, and for everything.

Contents

Chapter 1

Introduction

One of the most surprising patterns in the representation theory of infinite dimensional algebras is the appearance of modular forms as representation characters. The connection was first unearthed by Kac and Peterson in 1984 [18] in connection with affine Kac-Moody algebras. Let $\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_-$ be a finite dimensional simple Lie algebra with Killing form $(\cdot,\cdot)$. The corresponding affine Kac-Moody algebra is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}\delta$$

with Lie brackets

$$[at^m, bt^n] = [a,b]t^{m+n} + m(a,b)\delta_{m,-n}K, \quad K \text{ central and } [\delta, at^m] = -mat^m.$$

The Cartan subalgebra of $\hat{\mathfrak{g}}$ is $\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}\delta$, and we define $\hat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm + \mathfrak{g} \otimes_{\mathbb{C}} t^{\pm 1}\mathbb{C}[t^{\pm 1}]$. The Verma module with highest weight $\Lambda \in \hat{\mathfrak{h}}^*$ is

$$M(\Lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+)} \mathbb{C}v_\Lambda$$

where $hv_\Lambda = \Lambda(h)v_\Lambda$ for $h \in \hat{\mathfrak{h}}$ and $nv_\Lambda = 0$ for $n \in \hat{\mathfrak{n}}_+$. The unique irreducible quotient of $M(\Lambda)$ is denoted $L(\Lambda)$. The vector K acts on $M(\Lambda)$ and $L(\Lambda)$ as the scalar $k = \Lambda(K)$ which is called the level of Λ.

If $\Pi = \{\alpha_1^\vee, \ldots \alpha_r^\vee\}$ is a set of simple coroots for $\mathfrak{g}$ then $\Pi \cup \{\alpha_0^\vee = K - \theta^\vee\}$ (where $\theta^\vee$ is the highest coroot of $\mathfrak{g}$) is a set of simple coroots for $\hat{\mathfrak{g}}$. As in the finite dimensional case, the nicest $\hat{\mathfrak{g}}$-modules are the $L(\Lambda)$ for Λ a dominant integral weight, that is $\Lambda(\alpha^\vee) \in \mathbb{Z}_+$ for each simple coroot $\alpha^\vee$. These modules are called integrable.

The character of $L(\Lambda)$ (or of any $\hat{\mathfrak{h}}$-diagonal module) is the formal exponential generating function for the dimensions of the weight spaces,

$$\operatorname{Ch} L(\Lambda) = \sum_{\lambda \in \mathfrak{h}^*} \dim L(\Lambda)_\lambda e^\lambda.$$

If we specialize the character by applying it to the fixed element $\delta \in \mathfrak{h}$, we obtain a series

$$\widetilde{\mathrm{Ch}}_{L(\Lambda)}(q) = \sum_{\lambda \in \mathfrak{h}^*} \dim L(\Lambda)_\lambda q^{\langle \lambda, \delta \rangle},$$

in $q = e^{2\pi i \tau \delta}$, convergent for $|q| < 1$. Let I_k denote the finite set of dominant integral highest weights of level $k \in \mathbb{Z}_+$. Kac and Peterson showed (among other things) that the vector space

$$\mathrm{Span}\{q^{s_\Lambda} \widetilde{\mathrm{Ch}}_{L(\Lambda)}(q) | \Lambda \in I_k\}, \quad \text{where } s_\Lambda = \frac{(\lambda|\lambda + 2\rho)}{2(k + h^\vee)} - \frac{1}{24} \frac{k \dim \mathfrak{g}}{2(k + h^\vee)}$$

is invariant under the standard weight 0 action of the modular group $SL_2(\mathbb{Z})$

$$[f \cdot A](\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \text{where } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Above λ denotes the projection of Λ to $\mathfrak{h}^*$ along δ, and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Kac and Peterson used the Weyl-Kac character formula to derive explicit formulas for the characters in terms of Jacobi theta functions, then the modular invariance follows from explicit transformation laws for the latter.

One cute application [15] of this result is in explaining the observation, due to McKay, that the coefficients of the cube root of the modular J-function

$$J(\tau)^{1/3} = q^{-1/3}\left[1 + 248q + 4124q^2 + 34752q^3 + \cdots\right],$$

appear to be simply related to dimensions of irreducible representations of E_8, e.g., $4124 = 3875 + 248$ and $34752 = 30380 + 3875 + 2 \cdot 248 + 1$. The function $J(\tau)^{1/3}$ is the normalized character of the unique integrable $\hat{E}_8$-module V at level 1. On the other hand, δ commutes with the elements at^0 which form a copy of E_8, so the graded pieces of V are E_8-modules. This explains the connection.

The more famous observation of McKay is that the coefficients of the J-function itself

$$J(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots,$$

are related to dimensions of irreducible representations of the monster sporadic finite simple group $\mathbb{M}$, e.g., $196884 = 196883 + 1$ and $21493760 = 21296876 + 196883 + 1$. This observation is much more mysterious than the one involving E_8 and its explanation was eventually given using the theory of vertex operator algebras (VOAs) developed by Borcherds in 1986 [3]. Borcherds used a 'monstrous' VOA $V^\natural$, constructed by Frenkel, Lepowsky and Meurman [11], as an intermediary to prove that the monster group $\mathbb{M}$ acts on a certain 'monstrous' Lie algebra. This Lie algebra is an example of a generalized Kac-Moody algebra (another class of algebras introduced by Borcherds), and for these there is an analogue of the Weyl-Kac character formula. This provides the connection with modular forms.

In 1996 a more direct link between VOAs and modular forms was made by Zhu [29].

He showed that for a vertex operator algebra V satisfying certain basic conditions, the characters of certain V-modules (analogues of highest weight modules) span a finite dimensional $SL_2(\mathbb{Z})$-invariant family.

Before stating Zhu's theorem, we need to state the definitions of VOA, module over a VOA, etc. And before doing that it pays to give a little informal motivation. The commutative algebra is a familiar notion, it may be thought of as a vector space V with a map $L : V \longrightarrow \operatorname{End} V$ of left multiplication which satisfies $L_u L_v = L_v L_u$ for all $u, v \in V$. In the presence of a unit element this equality implies associativity of the product as well. Vertex algebras are an analogue (in fact a generalization) of commutative algebras in which L is replaced by a 'function-valued product' $Y(\cdot, z) : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$. The series $Y(u, z)$ may be infinite in both directions, but it is important to impose the 'quantum field condition' that $Y(u, z)b \in V((z))$. Commutativity should be some sort of equality between $Y(u, z)Y(v, w)$ and $Y(v, w)Y(u, z)$, but direct equality of formal power series turns out to be too stringent a condition to allow interesting examples. In fact we require

$$(z - w)^N [Y(u, z), Y(v, w)] = 0$$

for all $u, v \in V$, for some $N \in \mathbb{Z}_+$ depending on u and v. This condition on Y is called locality. It is convenient to now give a full definition of vertex algebra and VOA.

Definition 1.0.1. A *vertex algebra* [17] is a quadruple $(V, |0\rangle, T, Y)$ where V is a vector space, $|0\rangle$ is a vector in V called the *vacuum vector*, $T \in \operatorname{End} V$, and $Y : V \longrightarrow (\operatorname{End} V)[[z^{\pm 1}]]$ is an injective linear map such that $Y(u, z)$ is a quantum field for each $u \in V$. The map Y is called the *state-field correspondence*, and is written

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u_{(n)} z^{-n-1}.$$

The operators $u_{(n)}$ are called the *Fourier modes* of u, and the operation $\cdot_{(n)} \cdot : V \otimes V \longrightarrow V$ is called the n^{th} *product.* The following axioms are to be satisfied.

- $Y(|0\rangle, z) = I_V$ and $T|0\rangle = 0$.
- $Y(Tu, z) = \partial_z Y(u, z)$.
- For all $u, v \in V$, there exists $N \in \mathbb{Z}_+$ such that
$$(z - w)^N [Y(u, z), Y(v, w)] = 0.$$

A *vertex operator algebra* (VOA) is a vertex algebra together with a distinguished Virasoro element $\omega \in V$, satisfying the following axioms.[1]

[1]This naming convention is pretty standard now, although Borcherds used the term 'vertex algebra' and included the Virasoro vector.

- If $Y(\omega, z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ then the operators L_n satisfy the commutation relations of the Virasoro algebra, i.e.,

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c,$$

where $c \in \mathbb{R}$ is an invariant called the *central charge* of V.

- $L_{-1} = T$.

- L_0 is diagonalizable on V with integer eigenvalues, and the eigenspaces are finite dimensional. The L_0-eigenvalue of an eigenvector $u \in V$ is called the *conformal weight* Δ_u of u. Also $\Delta_{|0\rangle} = 0$ and $\Delta_\omega = 2$.

- The eigenvalues of L_0 on V are bounded below.

From the VOA axioms it is possible to derive [17] the *Borcherds identity*

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+} \binom{m}{j} (u_{(n+j)}v)_{(m+k-j)}x \\ = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \left[u_{(m+n-j)}v_{(k+j)} - (-1)^n v_{(k+n-j)}u_{(m+j)}\right] x\end{aligned} \tag{1.0.1}$$

for all $u, v, x \in V$, $m, k, n \in \mathbb{Z}$. A convenient re-indexing of the modes, called the conformal weight indexing, is defined by $u_n = u_{(n+\Delta_u-1)}$ (for u of pure conformal weight, then extended to all $u \in V$ linearly). Hence

$$Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-\Delta_u}.$$

From the axioms we may derive the relation $[L_0, u_k] = -ku_k$ for all $u \in V$, so that $u_k : V_m \longrightarrow V_{m-k}$ where $V_k = \{u \in V | \Delta_u = k\}$.

Definition 1.0.2. A *V-module* is a vector space M together with a state-field correspondence $Y^M : V \longrightarrow (\mathrm{End}\, M)[[z, z^{-1}]]$,

$$Y^M(u, z) = \sum_{n\in\mathbb{Z}} u^M_{(n)} z^{-n-1}$$

satisfying the quantum field property. The following axioms are to be satisfied.

- $Y^M(|0\rangle, z) = I_M$.

- For all $u, v \in V$, $x \in M$, $m, k, n \in \mathbb{Z}$, equation (1.0.1) holds.

A V-module M is said to be *positive energy* if

- $M = \oplus_{j\in\mathbb{R}_+} M_j$ is $\mathbb{R}$-graded by finite dimensional pieces.

- $u_n^M M_j \subseteq M_{j-n}$ for all $u \in V$, $n \in \mathbb{Z}$, $j \in \mathbb{R}_+$.

Now we indicate why the Virasoro algebra made a sudden appearance in the definitions above. If Y really is to be regarded as a 'function-valued product' then it should be coordinate independent, i.e., some extra structure should be put on V incorporating changes of the z coordinate. Such a structure is provided by the Virasoro action.

Roughly speaking, the positive half of the Virasoro algebra $\text{Vir}_+ = \oplus_{n\in\mathbb{Z}_+}\mathbb{C}L_n$ is isomorphic to the Lie algebra of the group $\text{Aut}\, D$ of automorphisms of the formal disc $D = \text{Spec}\,\mathbb{C}[[z]]$. Let X be a compact complex algebraic curve. For each $x \in X$ let D_x denote the formal disc centered on x, $D_x^\times \cong \text{Spec}\,\mathbb{C}((z))$ the punctured formal disc, and Coord_x the set of formal coordinates on D_x which is an $\text{Aut}\, D$-torsor. One may exponentiate the Vir_+-action on V to an $\text{Aut}\, D$-action, and then perform an associated bundle construction to produce a vector bundle $\mathcal{V}$ over X whose fibers are

$$\mathcal{V}_x = \text{Coord}_x \times_{\text{Aut}\, D} V.$$

The vertex operation now manifests as a collection of maps

$$\begin{aligned} \mathcal{Y}_x^\vee : \Gamma(D_x^\times, \mathcal{V} \otimes \Omega_X) &\to \text{End}\,\mathcal{V}_x \\ u \otimes z^n dz &\mapsto u_{(n)} = \text{Res}_z\, Y(u,z) z^n dz \end{aligned}$$

(here Ω_X is the bundle of 1-forms on X), satisfying appropriately formulated commutativity and associativity conditions. To define $\mathcal{Y}_x^\vee$ here we used a choice of coordinate z but the map itself is intrinsic, it does not depend on the choice of coordinate. The preceding constructions are explained nicely (and rigorously) in the book of Frenkel and Ben-Zvi [10], to which we refer the reader for more details. The actual definition a VOA is basically the minimal structure necessary to get all this to work. It is a vector space V with an operation $Y(u,z)$ satisfying locality, a unit element, an action of the Virasoro algebra, and some compatibility conditions between these structures.

One may ask what the negative half of the Virasoro algebra action is for. The full answer is subtle but in particular the operator $T = L_{-1}$ allows one to define a flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_X$, given explicitly as $\nabla = L_{-1} + dz$. It is natural then to consider the cohomology $h = \mathcal{V} \otimes \Omega_X / \text{Im}(\nabla)$ of the complex

$$0 \to \mathcal{V} \to \mathcal{V} \otimes \Omega_X \to 0,$$

and from this emerges an object of central importance in Zhu's work – the space of *conformal blocks*. The space of conformal blocks is defined to be

$$C(X, V, x) = \Gamma(X \backslash \{x\}, h)^*,$$

and is essentially the dual of the coinvariants of the action of $\Gamma(X\backslash\{x\}, \mathcal{V} \otimes \Omega_X) \subseteq \Gamma(D_x^\times, \mathcal{V} \otimes \Omega_X)$ on $\mathcal{V}_x$.

Let X be the torus $T_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ now (for $\tau \in \mathcal{H}$). It turns out that in this case the bundle $\mathcal{V}$ is trivial, i.e., $\mathcal{V} = V \times T_\tau$. It is well known that Ω_X is trivial too, so sections of $\mathcal{V} \otimes \Omega_X$ regular away from 0 are $u \otimes f(z)dz$ where $f(z)$ is a linear combination of the Weierstrass function $\wp(z, \tau)$, its derivatives, and the constant function (see Section 2.1 for the definition of $\wp$). This means that $C(T_\tau, V, 0)$ is the space of linear maps $V \to \mathbb{C}$ that annihilate all elements of the form

$$\mathrm{Res}_z\, Y(u, z)vdz = u_{(0)}v \quad \text{and} \quad \mathrm{Res}_z\, \wp(z, \tau)Y(u, z)vdz \tag{1.0.2}$$

for $u, v \in V$. From deeper theory (see [10] for details) one finds that the spaces of conformal blocks $C(T_\tau, V, 0)$ fit together, as the moduli parameter τ is varied, to form a vector bundle over the moduli space of pointed tori, and that this bundle carries another flat connection. A section of this bundle is essentially a function $S(u, \tau) : V \times \mathcal{H} \to \mathbb{C}$ and the condition on a section of being horizontal with respect to the connection turns into the differential equation

$$(2\pi i)^2 q \frac{d}{dq} S(u, \tau) = -S(\mathrm{Res}_z\, [\zeta(z, \tau) - zG_2(\tau)]\, L(z)udz, \tau) \tag{1.0.3}$$

(again refer to Section 2.1 regarding notation). One may write down a formal definition of a conformal block as a function $S : V \otimes_{\mathbb{C}} \mathrm{Hol}\, \mathcal{H} \to \mathbb{C}$ annihilating the vectors (1.0.2) and satisfying equation (1.0.3). This is essentially what Zhu did in [29]; he wrote down a definition encapsulating a different list of properties of conformal blocks, but it amounts to the same thing.

The definition of a conformal block is formulated intrinsically in terms of complex tori, and the moduli space of complex tori is more or less $SL_2(\mathbb{Z})\backslash\mathcal{H}$. It is therefore not surprising that conformal blocks transform nicely under $SL_2(\mathbb{Z})$. More precisely

Proposition 1.0.1 (Zhu). *For each $A \in SL_2(\mathbb{Z})$ let*

$$[S \cdot A](u, \tau) = (c\tau + d)^{-k} S(u, A\tau) \quad \textit{for } u \textit{ of conformal weight } k.$$

The space of conformal blocks is invariant under this $SL_2(\mathbb{Z})$-action, indeed it constitutes a representation of $SL_2(\mathbb{Z})$.

The hard part of Zhu's theorem is to link the conformal blocks associated to V to characters of V-modules. In fact Zhu proved the following lemma (see the remarks below regarding undefined terms).

Proposition 1.0.2 (Zhu). *Let V be a C_2-cofinite, rational vertex operator algebra. Let M be a positive energy V-module and put*

$$S_M(u, q) = \mathrm{Tr}_M\, u_0^M q^{L_0 - c/24}.$$

If we put $q = e^{2\pi i \tau}$ then this series converges (for each $u \in V$) to a holomorphic function $S_M(u, \tau)$ of $\tau \in \mathcal{H}$. Regarded as a function on $V \otimes_{\mathbb{C}} \mathrm{Hol}(\mathcal{H})$, each trace function *$S_M$ is a conformal block with respect to the Zhu VOA structure on V, and the trace functions associated to irreducible M span the space $\mathcal{C}$ of such conformal blocks.*

A VOA V is said to be rational if its category of positive energy modules is semisimple. In particular there are finitely many irreducible positive energy modules, and their trace functions span the space of all trace functions of positive energy modules. Next, V is C_2-cofinite if the span $V_{(-2)}V$ of all vectors $u_{(-2)}v$ for $u, v \in V$ has finite codimension in V. This condition ensures several nice regularity properties on V, in particular that the space $\mathcal{C}$ of conformal blocks of V is finite dimensional.

The 'Zhu VOA structure' on V is a second VOA structure $(V, |0\rangle, \tilde{\omega}, Y[\cdot, z])$ on V constructed by Zhu from the original one. The vacuum vector is unchanged, the new Virasoro vector is

$$\tilde{\omega} = (2\pi i)^2(\omega - \tfrac{c}{24}|0\rangle),$$

and the new state-field correspondence is

$$Y[u, z] = e^{2\pi i \Delta_u z} Y(u, e^{2\pi i z} - 1).$$

We typically use $u_{([n])}$ and $u_{[n]}$ to denote Fourier modes, and ∇_u to denote the conformal weight, in the Zhu structure, i.e.,

$$Y[u, z] = \sum_{n \in \mathbb{Z}_+} u_{([n])} z^{-n-1} = \sum_{n \in \mathbb{Z}_+} u_{[n]} z^{-n-\nabla_u}.$$

The central notion in the proof of Proposition 1.0.2 is an associative algebra introduced by Zhu, now referred to as the Zhu algebra Zhu(V) of V. It governs a lot of the representation theory of V, its main properties being summarized in the following theorem.

Theorem 1.0.3 (Zhu).

- *There is a restriction functor Ω from the category of positive energy V-modules to the category of* Zhu(V)*-modules. It sends M to its lowest graded piece (M_0 without loss of generality) with the action $[u] * x = u_0^M x$ for $u \in V$ and $x \in M_0$.*
- *There is an induction functor L going in the other direction, and we have $\Omega(L(N)) \cong N$ for any* Zhu$_g$(V)*-module N.*
- *Ω and L are inverse bijections between the sets of irreducible modules in each category.*

A conformal block $S(u, \tau)$ can be written as a series in q whose coefficients are linear functions on V. The leading coefficient descends to a function f on Zhu(V) (which is defined as a quotient of V). It turns out that f is symmetric, i.e., $f(u*v) = f(v*u)$.

Rationality of V implies semisimplicity of Zhu(V). A semisimple algebra is a direct sum of matrix algebras and so symmetric functions on a semisimple algebra A are linear combinations of functions of the form $a \mapsto \mathrm{Tr}_N\, a$ where N is an irreducible A-module. Using this, as well as the induction functor L, Zhu was able to write S in terms of the trace functions S_M for irreducible V-modules M.

The conclusion from all this, which now goes by the name of Zhu's theorem, is that if we fix $u \in V$ with $\nabla_u = k$, then the functions $S_M(u, \tau)$ span a finite dimensional vector space invariant under the weight k action of $SL_2(\mathbb{Z})$. In particular $\nabla_{|0\rangle} = 0$ and $|0\rangle_0 = 1$, so the characters

$$\mathrm{Ch}_M(q) = q^{-c/24} \,\mathrm{Tr}_M \, q^{L_0}$$

constitute a modular invariant family. In Section 3.5 we explain how this plays out in the affine Kac-Moody case, and we discuss a generalization to non integer level k.

1.0.1 Results of this thesis

Zhu's theorem requires three main facts about the VOA V, first that it be rational, second that it be C_2-cofinite, and third is a condition that we did not stress above – that the conformal weights in V be integers. This third condition is also assumed for the underlying geometric constructions of Frenkel and Ben Zvi. The construction of the bundle $\mathcal{V}$ involves exponentiating the action of Vir_+ on V, and to do this it is necessary that the L_0-eigenvalues, i.e., the conformal weights, be integers.

These conditions on V rule out many interesting examples, for which various kinds of partial modular invariance are evident. In this thesis we study several of these examples, and generalize Zhu's theorem to cases where one or more of the conditions listed above are relaxed. Also, in general we consider not VOAs but their natural supersymmetric generalization to vertex operator superalgebras (see Section 1.1 for definitions regarding vector superspaces, etc). Again there are many interesting examples of these.

1.0.2 Allowing non integer conformal weights

Our full definition of vertex operator superalgebra (VOSA) is given in Section 1.1, for now it suffices to say that it is the same as Definition 1.0.1 except that $V = V_{\bar{0}} + V_{\bar{1}}$ is a vector superspace, in the penultimate bullet point the word 'integer' is replaced by 'rational', and the locality axiom is replaced by the usual super analogue

$$(z - w)^N \left[Y(u, z)Y(v, w) - p(u, v)Y(v, w)Y(u, z)\right] = 0,$$

where $p(u, v) = (-1)^{p(u)p(v)}$ and $p(u) \in \{0, 1\}$ is the parity of u (again, see Section 1.1 for full definitions).

The most natural definition of V-module for us comes about after a slight reformulation of Definition 1.0.2. In terms of the conformal weight indexing of the Fourier modes, the Borcherds identity becomes

$$\sum_{j\in\mathbb{Z}_+}\binom{m+\Delta_u-1}{j}(u_{(n+j)}v)_{m+k}x$$
$$=\sum_{j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\left[u_{m+n-j}v_{k+j-n}-p(u,v)(-1)^n v_{k-j}u_{m+j}\right]x \tag{1.0.4}$$

for all $u, v, x \in V$, $m \in -[\Delta_u]$, $k \in -[\Delta_v]$ and $n \in \mathbb{Z}$ (note the appearance of $p(u, v)$ because we are now in the super case).

Definition 1.0.3. A *V-module* is a vector superspace M together with a state-field correspondence $Y^M : V \to (\operatorname{End} M)[[z, z^{-1}]]$,

$$Y^M(u,z)=\sum_{n\in\mathbb{Z}}u_n^M z^{-n-\Delta_u}$$

satisfying the quantum field property. The following axioms are to be satisfied.

- $Y^M(|0\rangle, z) = I_M$.
- For all $u, v \in V$, $x \in M$, $m, k, n \in \mathbb{Z}$, equation (1.0.4) holds.

A V-module M is said to be *positive energy* if

- $M = \oplus_{j\in\mathbb{R}_+} M_j$ is $\mathbb{R}_+$-graded by finite dimensional pieces.
- $u_n^M M_j \subseteq M_{j-n}$ for all $u \in V$, $n \in \mathbb{Z}$, $j \in \mathbb{R}_+$.

Note that because the indices in equation (1.0.4) do not lie in $\mathbb{Z}$ necessarily, V may no longer be a V-module. Of course when the conformal weights of all vectors in V are integers this definition reduces to the usual one.

Let V be a C_2-cofinite, rational VOSA with rational conformal weights. We retain the formal definition of conformal blocks $\mathcal{C}$ from the last section, despite the geometric underpinnings being less clear in this case. Conformal blocks reduce to supersymmetric functions (i.e., functions f satisfying $f(u * v) = p(u, v)f(v * u)$) on a modified Zhu algebra introduced by De Sole and Kac [4]. The naive analogue of lemma 1.0.2 is now almost, but not quite true. We may write down supertrace functions $\operatorname{STr}_M u_0^M q^{L_0-c/24}$ associated to positive energy V-modules M in the sense of Definition 1.0.3. They lie in $\mathcal{C}$ but they do not span it. To explain why, recall the classification of semisimple associative superalgebras into the following two types:

- Type I: $\operatorname{End}(\mathbb{C}^{m|k})$ for $m + k \geq 1$ (here $\mathbb{C}^{m|k}$ is the vector superspace with basis consisting of m even vectors and k odd vectors),
- Type II: $Q_n = \operatorname{End}(\mathbb{C}^n)[\xi]/(\xi^2 = 1)$ where ξ is an odd indeterminate.

(the latter superalgebras are sometimes called 'queer superalgebras'). A supersymmetric function on the Type I superalgebra $\operatorname{End}(\mathbb{C}^{m|k})$ is a scalar multiple of the supertrace $\mathrm{STr}_{\mathbb{C}^{m|k}}$. A supersymmetric function on the Type II superalgebra Q_n is a scalar multiple of $\varphi : a \mapsto \mathrm{Tr}_{\mathbb{C}^n}(a\xi)$ [14], and the naive supertrace STr_{Q_n} is identically zero. We expect to require supertrace functions, and also V-module analogues of the queer supertrace, to exhaust $\mathcal{C}$. This is true, but to avoid repetition we postpone the rigorous statement, recovering it instead as a special case of the results below.

We would also like to say something about the character $\mathrm{Tr}_V\, q^{L_0 - c/24}$ of V itself, even though V is not a V-module. In fact V is something called a *twisted* V-module, so we are led inexorably to consider Zhu's theorem in the context of twisted V-modules. In general let g be an automorphism of V, then a *g-twisted V-module* is a vector superspace M with fields $Y^M(u,z) = \sum_n u_n^M z^{-n-\Delta_u}$ where the sum is over $n \in \epsilon_u + \mathbb{Z}$ (instead of $\mathbb{Z}$) for some real number ϵ_u depending on u and g. See Definition 1.1.2 for the precise definition. Positive energy g-twisted V-modules are defined as usual and we say that V is g-rational if the usual definition of rationality holds with 'module' replaced by 'g-twisted module'. Dong, Li, and Mason [8] proved twisted versions of Propositions 1.0.1 and 1.0.2, still in the case of V a VOA with integer conformal weights.

Let G be a finite group of automorphisms of V and let $g, h \in G$ commute. Following [8] we consider a modified space $\mathcal{C}(g,h)$ of conformal blocks associated to V, g and h (see Definition 2.2.2). The following analogue of Lemma 1.0.1 now holds.

Lemma 1.0.4. *For $S \in \mathcal{C}(g,h)$ and $A \in \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{smallmatrix}\right)$ define $S \cdot A$ by*

$$[S \cdot A](u, \tau) = (c\tau + d)^{-k} S(u, A\tau) \quad \text{for } u \text{ of Zhu weight } k. \qquad (1.0.5)$$

Then we have

$$\left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{smallmatrix}\right) : \mathcal{C}(g,h) \to \mathcal{C}(g^{\mathsf{a}} h^{\mathsf{c}}, g^{\mathsf{b}} h^{\mathsf{d}}). \qquad (1.0.6)$$

The definition of $S \cdot A$ involves $(\mathsf{c}\tau + \mathsf{d})^{-k}$ for $k \notin \mathbb{Z}$. We define the latter as a principal value, see Section 2.2. For $A = \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{smallmatrix}\right)$ let $\gamma_A(\tau) = \mathsf{c}\tau + \mathsf{d}$. We have

$$\gamma_B(\tau)^{-k} \gamma_A(B\tau)^{-k} \propto \gamma_{AB}(\tau)^{-k}$$

for all $A, B \in SL_2(\mathbb{Z})$, where the constant of proportionality is a root of unity. There is equality for all A, B only when $k \in \mathbb{Z}$. Therefore equation (1.0.5) defines a projective representation of $SL_2(\mathbb{Z})$ of weight k on the conformal blocks restricted to a given element u of pure Zhu weight k, i.e., on $\{S(u,\tau) | S \in \oplus_{g,h \in G} \mathcal{C}(g,h)\}$. But in general the direct sum $\oplus_{g,h \in G} \mathcal{C}(g,h)$ is neither a representation, nor a projective representation, of $SL_2(\mathbb{Z})$.

Suppose M is a g-twisted V-module. We may give the vector space M a different g-twisted V-module structure (denoted $h \cdot M$) by setting

$$Y^{h \cdot M}(u,z) = Y^M(h(u), z)$$

(we need g and h to commute for this to actually be a g-twisted V-module). Suppose $h \cdot M$ is equivalent to M, i.e., there exists an intertwining isomorphism $\gamma : M \to M$

such that

$$h(u)_n^M = \gamma^{-1} u_n^M \gamma \quad \text{for all } u \in V \text{ and } n \in \epsilon_u + \mathbb{Z}. \tag{1.0.7}$$

We call such g-twisted V-modules *h-invariant.* For brevity we let $\mathsf{PEMod}(g, V)$ denote the set of g-twisted positive energy V-modules and $\mathsf{PEMod}_h(g, V)$ the subset of h-invariant modules.

There is a modified Zhu algebra $\text{Zhu}_g(V)$ (again, the one introduced by De Sole and Kac in [4], generalizing constructions of Zhu [29] and of Dong, Li and Mason [6]) whose irreducible modules are naturally in bijection with the irreducible modules in $\mathsf{PEMod}(g, V)$. The automorphism h descends to an automorphism of $\text{Zhu}_g(V)$ which permutes its simple components. Let A be a h-invariant simple component of $\text{Zhu}_g(V)$, let N be the unique irreducible A-module, and put $M = L(N)$. Then $M \in \mathsf{PEMod}_h(g, V)$ and in fact this establishes a bijection between irreducible objects in $\mathsf{PEMod}_h(g, V)$ and h-invariant simple components of $\text{Zhu}_g(V)$.

Theorem 1.0.5 (First Main Theorem). *Let V be a C_2-cofinite VOSA graded by rational conformal weights. Let G be a finite group of automorphisms of V and suppose that $\text{Zhu}_g(V)$ is semisimple for each $g \in G$. Fix commuting $g, h \in G$ and let $M \in \mathsf{PEMod}_h(g, V)$ be irreducible. We define the formal supertrace function $S_M(a, q)$ as follows.*

$$S_M(u, q) = \begin{cases} \text{STr}_M\, u_0^M \gamma q^{L_0 - c/24} & \text{if } A \text{ is of Type I,} \\ & \text{or if } A \text{ is of Type II and } h(\xi) = -\xi, \\ \text{Tr}_M\, u_0^M \gamma q^{L_0 - c/24} & \text{if } A \text{ is of Type II and } h(\xi) = \xi. \end{cases} \tag{1.0.8}$$

Here A is the h-invariant component of $\text{Zhu}_g(V)$ corresponding to M, and in each case $\gamma : M \to M$ is chosen to satisfy equation (1.0.7). Then

- *The supertrace function (1.0.8) converges to a holomorphic function $S_M(u, \tau)$ of $\tau \in \mathcal{H}$, where $q = e^{2\pi i \tau}$.*
- *The functions S_M for irreducible $M \in \mathsf{PEMod}_h(g, V)$ form a basis of $\mathcal{C}(g, h)$.*

In Section 2.6 we write down an explicit choice of γ for each of the cases. This theorem, combined with Lemma 1.0.4, is our generalization of Zhu's modularity theorem to the present setting.

The following corollary gives information on the modular transformation properties of the supercharacter of the VOSA V itself.

Corollary 1.0.6. *Let V be as above, let $g_0 = e^{2\pi i L_0}$ and let G be the group of automorphisms of V generated by g_0, which is finite. Then the supercharacter $\text{SCh}_V\, q^{L_0 - c/24}$ lies in the span of the h-twisted supercharacters*

$$\text{STr}_M\, \gamma q^{L_0 - c/24}$$

(where γ is as above) of the irreducible $M \in \mathsf{PEMod}_h(g, V)$, as g, h range over G. This span is a finite dimensional space of functions invariant under the weight 0 action of $SL_2(\mathbb{Z})$.

The set-up and proof of Theorem 1.0.5 occupies Chapter 2. In Chapter 3 we give five examples. The first example, the neutral free fermion VOSA, illustrates the appearance of queer superalgebras and modular forms of half integer weight. The second example, the charged free fermions VOSA, illustrates that more or less arbitrary conformal weights can appear, and twisted modules are necessary to restore full $SL_2(\mathbb{Z})$-invariance. We then consider the VOSA associated to an integral lattice, we see that characters and supercharacters of irreducible modules and σ-twisted modules are altogether $SL_2(\mathbb{Z})$-invariant as expected (this is well-known and also follows from a special case of our theorem proved by Dong and Zhao [9]). For our fourth example we consider the $N = 1$ superconformal VOSAs. Using Theorem 1.0.5 we are able to completely determine one of the supertrace functions (of queer type), the result is a modular form of weight 3/2. An attempt at explicit computation leads to a connection with a well known identity of Jacobi. Finally we consider affine Kac-Moody VOAs at admissible rational level (in particular the case $\mathfrak{g} = \mathfrak{sl}_2$). We explain how results deduced from Theorem 1.0.5 connect with formulas proved by Kac and Wakimoto using different methods.

1.0.3 Generalization to non rational vertex operator superalgebras

Let V be a C_2-cofinite, but not necessarily rational, VOSA and let G be a finite group of automorphisms of V. The spaces $\mathcal{C}(g,h)$ of conformal blocks are finite dimensional and $SL_2(\mathbb{Z})$-invariant. It is natural to wonder if $\mathcal{C}(g,h)$ might be described in terms of some generalization of supertrace functions attached to positive energy V-modules. This is the subject of Chapter 4.

Let B be an associative algebra now and let P be a finitely generated projective right B-module. There is an isomorphism

$$\operatorname{End}_B P \to P^* \otimes_{\mathbb{C}} P,$$

which we compose with the evaluation map

$$\begin{aligned} P \otimes_{\mathbb{C}} P^* \twoheadrightarrow P \otimes_B P^* &\to B/[B,B] \\ \phi \otimes x &\mapsto \phi(x) \end{aligned}$$

to define the well-known 'abstract trace' $\operatorname{Tr}_P : \operatorname{End}_B P \to B/[B,B]$. Here $[B,B]$ is the span of all elements $ab - ba$ for $a, b \in B$. As for the familiar trace we have $\operatorname{Tr}_P(fg) = \operatorname{Tr}_P(gf)$.

In Chapter 4 we generalize this construction to the case of a finite dimensional superalgebra B carrying a finite order automorphism h to define abstract 'h-supertrace' $\operatorname{STr}_P^{(h)}$ taking values in $B/[B,B]_h$ where $[B,B]_h$ is the span of elements of the form $ah(b) - p(a,b)ba$ for $a, b \in B$.

Now consider Z_g the free rank 1 left module over $A_g = \operatorname{Zhu}_g(V)$ and let $B_g = (\operatorname{End}_{A_g} P)^{\text{op}} \cong A_g$ which acts on Z_g from the right. $\operatorname{STr}_{Z_g}^{(h)}$ provides a natural identification of h-supersymmetric functions on A_g, i.e., elements of $(A_g/[A_g, A_g])^*$, with those on B_g. The induced g-twisted V-module $L(Z_g)$ has the structure of a right B_g-module in a natural way, this is clear after seeing its construction (see Section 5.2.2 for the construction in a more general setting). We now wish to define the $(B_g/[B_g, B_g]_h)$-valued supertrace function as

$$S_{g,h}(a,\tau) = \mathrm{STr}^{(h)}_{L(Z_g)} \, a_0 q^{L_0 - c/24}.$$

Note that $L(Z_g)$ is not finitely generated, but it is graded by pieces that are. Unfortunately it is not clear if $L(Z_g)$ is a projective B_g-module in general. For some interesting cases it can be verified that it is. We have the following theorem.

Theorem 1.0.7 (Second Main Theorem). *Let V be a C_2-cofinite VOSA graded by rational conformal weights. Let G be a finite group of automorphisms of V. For commuting $g, h \in G$, consider $L(Z_g)$ as above and suppose it is a projective right B_g-module. Then*

- *$S_{g,h}$ converges to a $B_g/[B_g, B_g]_h$-valued holomorphic function $S_{g,h}(u,\tau)$ of $\tau \in \mathcal{H}$, where $q = e^{2\pi i\tau}$.*
- *The functions $\phi \circ S_{g,h}(a,\tau)$ for $\phi \in (B_g/[B_g, B_g]_h)^*$ span $\mathcal{C}(g,h)$, and so the collection of all such functions is modular invariant in the sense of Lemma 1.0.4.*

The meaning of q^{L_0} in this case deserves comment. In the rational case we deal only with irreducible V-modules and by a standard argument L_0 acts semisimply on these. So q^{L_0} is defined to be q^k on the eigenspace with eigenvalue k. In the present case the action of L_0 may have nontrivial Jordan blocks. The graded pieces of $L(Z_g)$ are finite dimensional and on any one we have the Jordan decomposition $L_0 = L_0^{\text{ss}} + L_0^{\text{nilp}}$ into commuting semisimple and nilpotent parts. Since $q = e^{2\pi i\tau}$ we define q^{L_0} as $q^{L_0^{\text{ss}}} e^{2\pi i\tau L_0^{\text{nilp}}}$. Therefore modular forms with logarithmic q-series appear naturally in connection with non rational VOSAs.

The approach above was inspired by the paper [2] of Arike, which was written as a clarification of the work [25] of Miyamoto. For a C_2-cofinite VOA V with integer conformal weights Miyamoto gave an explicit description of a collection of positive energy V-modules and a collection of 'pseudotrace functions' associated to each one, which together span the space $\mathcal{C}$ of conformal blocks. Miyamoto's description is made in terms of a sequence of 'higher level Zhu algebras' $\mathrm{Zhu}_n(V)$ for $n \in \mathbb{Z}_+$, introduced by Dong, Li and Mason [7], generalizing the ordinary Zhu algebra $\mathrm{Zhu}(V) = \mathrm{Zhu}_0(V)$. To remove our assumption about projectivity of V-modules in Theorem 1.0.7, we need to pass to a generalization of Miyamoto's theorem, and this requires a generalization of higher level Zhu algebras.

In Chapter 5 we introduce the higher level twisted Zhu algebra $\mathrm{Zhu}_{P,g}(V)$ associated to a VOSA V with an automorphism g, and $P \in \mathbb{R}_+$. We prove the following theorem.

Theorem 1.0.8 (Third Main Theorem).

- *There is a restriction functor Ω_P from the category* $\mathsf{PEMod}(g,V)$ *of positive energy g-twisted V-modules to the category of* $\mathrm{Zhu}_{P,g}(V)$*-modules. It sends M to its P^{th} graded piece M_P with the action $[u]*x = u_0^M x$ for $u \in V$ and $x \in M_P$.*
- *There is an induction functor L^P going in the other direction, and we have*

$$\Omega_P(L^P(N)) \cong N$$

  for any $\mathrm{Zhu}_{P,g}(V)$*-module N.*
- *Ω_P and L^P are inverse bijections between the sets of irreducible modules in each category.*

We then give an equivalent construction of $\mathrm{Zhu}_{P,g}(V)$ to complement the first. Using the second description we are able to explicitly compute higher Zhu algebras for some well known VOAs, namely the universal Virasoro VOA Vir^c and the universal affine Kac-Moody VOAs $V^k(\mathfrak{g})$.

We stop short of actually using the algebras $\mathrm{Zhu}_{P,g}(V)$ to generalize Miyamoto's theorem to our setting (that is, twisted modules over a VOSA with non integer conformal weights). The tools are in place but the theorem itself is work in progress. In Chapter 6 we do use $\mathrm{Zhu}_{P,g}(V)$ to prove two propositions which are used in Chapter 4, and which will become part of a generalization of Miyamoto's theorem.

1.1 Basic definitions

We use the notation $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. All vector spaces and superspaces are over $\mathbb{C}$.

A vector *superspace* U is a vector space graded by $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. We call $U_{\bar{0}}$ and $U_{\bar{1}}$ the even and odd components of U respectively. A subspace of a vector superspace is always $\mathbb{Z}/2\mathbb{Z}$-graded. We use the following notations: $p(u) = \alpha$ for homogeneous $u \in U_\alpha$, $\sigma_U(u) = (-1)^{p(u)}$, and $p(u,v) = (-1)^{p(u)p(v)}$.

An *associative superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra. For example $\operatorname{End} U$ is an associative superalgebra where $(\operatorname{End} U)_\alpha = \{X \in \operatorname{End} U | X U_\beta \subseteq U_{\alpha+\beta}\}$. The commutator of operators X and Y on a vector superspace is defined to be $[X,Y] = XY - p(X,Y)YX$. A module over an associative superalgebra A is a vector superspace M and a homomorphism $A \to \operatorname{End} M$. Two A-modules are equivalent if there is a $\mathbb{Z}/2\mathbb{Z}$-graded linear isomorphism between them intertwining with the action of A. We will assume such equivalences to be even unless otherwise stated.

We write $\mathbb{C}^{m|k}$ for the vector superspace with a basis consisting of m even vectors and k odd vectors. The *supertrace* of an operator $X \in \operatorname{End} U$ is $\mathrm{STr}_U X = \mathrm{Tr}_{U_{\bar{0}}} X - \mathrm{Tr}_{U_{\bar{1}}} X$. In general $\mathrm{STr}_U [X,Y] = 0$.

We write $U[z]$ for the ring of polynomials in z with coefficients in the vector superspace U, $U[[z]]$ for the ring of formal power series, and $U((z))$ for the ring of Laurent series, i.e., expressions $\sum_{n\in\mathbb{Z}} a_n z^n$ in which finitely many negative powers of z occur. The space $U[[z^{\pm 1}]]$ of *formal distributions* is the set of expressions $\sum_{n\in\mathbb{Z}} a_n z^n$ with no restriction on the coefficients a_n. Finally $U\{\{z\}\} = \oplus_{r\in\mathbb{R}} z^r U[[z^{\pm 1}]]$. Extension to several variables is obvious, but note that $U((z))((w)) \neq U((w))((z))$.

We write $\partial_z f(z)$ for the z-derivative of $f(z)$, also $z^{(n)}$ for $z^n/n!$, and $[z^n] : f(z)$ for f_n the z^n coefficient of $f(z)$.

A convenient index convention for formal distributions is $f(z) = \sum_{n\in\mathbb{Z}} f_{(n)} z^{-n-1}$. The *formal residue* operation $\operatorname{Res}_z(\cdot)dz : U[[z^{\pm 1}]] \longrightarrow U$ is defined by $\operatorname{Res}_z f(z)dz = [z^{-1}] : f(z) = f_{(0)}$. We have

$$\operatorname{Res}_z \partial_z f(z)dz = 0 \quad \text{and} \quad \operatorname{Res}_z f(z)\partial_z g(z)dz = -\operatorname{Res}_z g(z)\partial_z f(z)dz.$$

Let $f(z) \in U((z))$ and $g(w) \in w\mathbb{C}^\times + w^2\mathbb{C}[[w]]$. The substitution of $z = g(w)$ into $f(z)$ gives a well-defined element $f(g(w)) \in U((w))$, and we have the formal change of variable formula

$$\operatorname{Res}_z f(z)dz = \operatorname{Res}_w f(g(w))\partial_w g(w)dw.$$

The *formal delta function* $\delta(z,w) \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ is defined by

$$\delta(z,w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}.$$

The operators

$$\begin{aligned} & i_{z,w} : U[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}] \longrightarrow U((z))((w)) \\ \text{and} \quad & i_{w,z} : U[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}] \longrightarrow U((w))((z)) \end{aligned}$$

denote expansion of an element as Laurent series in the domains $|z| > |w|$ and $|w| > |z|$, respectively. For example,

$$i_{z,w}(z-w)^{-1} = \sum_{j\in\mathbb{Z}_+} z^{-j-1}w^j \quad \text{and} \quad i_{w,z}(z-w)^{-1} = -\sum_{j\in\mathbb{Z}_+} z^j w^{-j-1}.$$

An $\operatorname{End} U$-valued formal distribution $f(z)$ is called a *quantum field* if $f(z)u \in U((z))$ for each $u \in U$.

We now give the definition of VOSA that we use throughout the thesis. Note that the locality axiom of Definition 1.0.1 has been replaced by an equation involving formal delta functions, which is a generating function version of the Borcherds identity. See [17] regarding equivalence of these axioms.

Definition 1.1.1. A *vertex operator superalgebra (VOSA)* is a quadruple $(V, |0\rangle, \omega, Y)$ where V is a vector superspace, $|0\rangle$ and ω are even elements of V called the *vacuum vector* and the *Virasoro vector* respectively, and $Y : V \longrightarrow (\operatorname{End} V)[[z^{\pm 1}]]$ is an injective linear map such that $Y(u,z)$ is a quantum field for each $u \in V$. The map Y is called the *state-field correspondence*, and is written

$$Y(u,z) = \sum_{n\in\mathbb{Z}} u_{(n)} z^{-n-1}.$$

The operators $u_{(n)}$ are called the *Fourier modes* of u, and the operation $\cdot_{(n)}\cdot : V \otimes V \to V$ is called the n^{th} *product*. The following axioms are to be satisfied.

- $Y(|0\rangle, z) = I_V$.
- For all $u, v \in V$, $n \in \mathbb{Z}$,
$$\sum_{j \in \mathbb{Z}_+} Y(u_{(n+j)}v, w)\partial_w^{(j)}\delta(z,w)$$
$$= Y(u,z)Y(v,w)i_{z,w}(z-w)^n - p(u,v)Y(v,w)Y(u,z)i_{w,z}(z-w)^n.$$
- If $Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ then the operators L_n satisfy the commutation relations of the Virasoro algebra:
$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}c,$$
where $c \in \mathbb{R}$ is called the *central charge* of V.
- L_0 is diagonalizable on V with rational eigenvalues, and the eigenspaces are finite dimensional. The L_0-eigenvalue of an eigenvector $u \in V$ is called the *conformal weight* Δ_u of u. Also $\Delta_{|0\rangle} = 0$ and $\Delta_\omega = 2$.
- The eigenvalues of L_0 on V are bounded below.
- $Y(L_{-1}u, z) = \partial_z Y(u,z)$ for all $u \in V$.

Sometimes we write $T = L_{-1}$ and $H = L_0$ and refer to these as the 'translation operator' and 'energy operator' of V, respectively.

Let $V_k = \{u \in V | \Delta_u = k\}$. It shall be understood that, whenever we refer to Δ_u in a definition, the definition holds as stated for u of homogeneous conformal weight, and is extended linearly to all $u \in V$. The conformal weight has the following three properties:

$$\Delta_{|0\rangle} = 0, \quad \Delta_{Tu} = \Delta_u + 1, \quad \text{and} \quad \Delta_{u_{(n)}v} = \Delta_u + \Delta_v - n - 1. \qquad (1.1.1)$$

A convenient indexing of the modes, called the conformal weight indexing, is defined by $u_n = u_{(n+\Delta_u-1)}$ (for u of homogeneous conformal weight, then extended to all $u \in V$ linearly). Hence

$$Y(u,z) = \sum_{n \in -[\Delta_u]} u_n z^{-n-\Delta_u},$$

where here and below $[\alpha]$ denotes the coset $\alpha + \mathbb{Z}$ of $\alpha \in \mathbb{R}$ modulo $\mathbb{Z}$.

The second axiom of Definition 1.1.1 is called the *Borcherds identity*. Expressed in terms of modes it becomes

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+}\binom{m+\Delta_u-1}{j}(u_{(n+j)}v)_{m+k}x\\ =\sum_{j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\left[u_{m+n-j}v_{k+j-n}-p(u,v)(-1)^n v_{k-j}u_{m+j}\right]x\end{aligned} \tag{1.1.2}$$

for all $u, v, x \in V$, $n \in \mathbb{Z}$, $m \in -[\Delta_u]$, and $k \in -[\Delta_v]$.

A useful special case of the Borcherds identity is the *commutator formula*

$$[u_m, v_k] = \sum_{j\in\mathbb{Z}_+}\binom{m+\Delta_u-1}{j}(u_{(j)}v)_{m+k}, \tag{1.1.3}$$

obtained by setting $n = 0$ in (1.1.2). The commutator formula together with the final VOSA axiom implies that

$$[L_0, u_k] = -ku_k \quad \text{for all } u \in V. \tag{1.1.4}$$

Let $U \subseteq V$ be a subspace of the VOSA V. Then U is a sub-VOSA of V if $|0\rangle, \omega \in U$ and $u_{(n)}v \in U$ for all $u, v \in U$, $n \in \mathbb{Z}$. U is an ideal if $u_{(n)}v, v_{(n)}u \in U$ for all $v \in V$, $u \in U$, $n \in \mathbb{Z}$. A simple vertex algebra is one with no proper nonzero ideals. A homomorphism $\phi : V_1 \to V_2$ of VOSAs is an even linear map such that $\phi(|0\rangle_1) = |0\rangle_2$, $\phi(\omega_1) = \omega_2$, and $\phi(u_{(n)}v) = \phi(u)_{(n)}\phi(v)$ for all $u, v \in V_1$. Homomorphisms preserve conformal weight. Isomorphism and automorphism are defined in the obvious way.

Definition 1.1.2. Let V be a VOSA and g an automorphism of V. Let $\mu(u)$ denote the g-eigenvalue of an eigenvector $u \in V$. Pull $\mu(u)$ back to a coset $[\epsilon_u]$ in $\mathbb{R}$ modulo $\mathbb{Z}$ via the map $e^{2\pi i x} : \mathbb{R} \to S^1$ (also define ϵ_u to be the largest non positive element of $[\epsilon_u]$). A *g-twisted V-module* is a vector superspace M together with a state-field correspondence $Y^M : V \to (\operatorname{End} M)\{\{z\}\}$,

$$Y^M(u,z) = \sum_{n\in[\epsilon_u]} u_n^M z^{-n-\Delta_u} = \sum_{n\in[\epsilon_u]+[\Delta_u]} u_{(n)}^M z^{-n-1},$$

satisfying the quantum field property. The following axioms are to be satisfied.

- $Y^M(|0\rangle, z) = I_M$.
- For all $u, v \in V$, $x \in M$, $n \in \mathbb{Z}$, $m \in [\epsilon_u]$, and $k \in [\epsilon_v]$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+}\binom{m+\Delta_u-1}{j}(u_{(n+j)}v)^M_{m+k}x\\ =\sum_{j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\left[u^M_{m+n-j}v^M_{k+j-n}-p(u,v)(-1)^n v^M_{k-j}u^M_{m+j}\right]x.\end{aligned} \tag{1.1.5}$$

A *positive energy g-twisted V-module* is a g-twisted V-module M such that

- $M = \oplus_{j\in\mathbb{R}_+} M_j$ is $\mathbb{R}_+$-graded with $M_0 \neq 0$, and each graded piece is finite dimensional.
- $u_n^M M_j \subseteq M_{j-n}$ for all $u \in V$, $n \in [\epsilon_u]$, $j \in \mathbb{R}_+$.

Definition 1.1.3. The *Zhu VOSA structure* is $(V, |0\rangle, \tilde{\omega}, Y[u,z])$ where

$$\tilde{\omega} = (2\pi i)^2(\omega - \tfrac{\mathsf{c}}{24}|0\rangle)$$

and

$$Y[u,z] = e^{2\pi i \Delta_u z} Y(u, e^{2\pi i z} - 1).$$

If we write $L[z] = Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L_{[n]} z^{-n-2}$ then

$$L_{[-2]} = (2\pi i)^2(L_{-2} - c/24), \quad L_{[-1]} = 2\pi i(L_{-1} + L_0),$$
$$\text{and} \quad L_{[0]} = L_0 - \sum_{j\in\mathbb{Z}_{>0}} \frac{(-1)^j}{j(j+1)} L_j.$$

The eigenvalue ∇_u of an eigenvector u with respect to $L_{[0]}$ is called the *Zhu weight* of u. We write $V_{[k]} = \{u \in V | \nabla_u = k\}$ and

$$Y[u,z] = \sum_{n\in\mathbb{Z}} u_{([n])} z^{-n-1} = \sum_{n\in -[\nabla_u]} u_{[n]} z^{-n-\nabla_u}.$$

Explicitly we have

$$\begin{aligned} u_{([n])} v &= \mathrm{Res}_z\, z^n e^{2\pi i \Delta_u z} Y(u, e^{2\pi i z} - 1) v dz \\ &= (2\pi i)^{-n-1} \mathrm{Res}_w [\ln(1+w)]^n (1+w)^{\Delta_u - 1} Y(u,w) v dw, \end{aligned}$$

where $w = e^{2\pi i z} - 1$. An automorphism of the new VOSA structure is the same as an automorphism of the old one. Vectors of homogeneous conformal weight are not generally of homogeneous Zhu weight and vice versa.

We use the following notation below: V is a VOSA, G a finite group of automorphisms of V, and $g, h \in G$ two commuting automorphisms. Unless otherwise stated an element of V is a simultaneous eigenvector of g and h. For such an eigenvector u we write $\mu(u)$ and $\lambda(u)$ for its g- and h-eigenvalues respectively.

We define a right action of $SL_2(\mathbb{Z})$ on $G \times G$ by $(g,h) \cdot A = (g^{\mathsf{a}} h^{\mathsf{c}}, g^{\mathsf{b}} h^{\mathsf{d}})$ where $A = \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{smallmatrix}\right)$. Similarly $(\mu, \lambda) \cdot A = (\mu^{\mathsf{a}} \lambda^{\mathsf{c}}, \mu^{\mathsf{b}} \lambda^{\mathsf{d}})$. We use the standard notation $A\tau$ for $\frac{\mathsf{a}\tau + \mathsf{b}}{\mathsf{c}\tau + \mathsf{d}}$.

Chapter 2

VOSAs with non integer conformal weights

In this chapter we prove Theorem 1.0.5 stated in the introduction. Examples and applications are given in Chapter 3.

Definitions pertaining to VOSAs, modules, etc. have all been given in Section 1.1 of the introduction. We begin in Section 2.1 with further definitions and facts about modular forms needed later in the chapter. In Section 2.2 we define the spaces $\mathcal{C}(g,h)$ of *conformal blocks* associated to VOSA V with automorphisms g, h, and we prove the $SL_2(\mathbb{Z})$ transformations of the $\mathcal{C}(g,h)$ in Theorem 2.2.1.

The C_2-cofiniteness condition on V implies that for $S \in \mathcal{C}(g,h)$ and fixed $u \in V$, the function $S(u,\tau)$ satisfies a Fuchsian differential equation. Moreover there is a *Frobenius expansion* of S in powers of q and $\log q$ whose coefficients are linear maps $V \to \mathbb{C}$. We sketch the proofs in Section 2.3, referring to [8] for details.

In Section 2.4 we analyze the leading coefficients in the Frobenius expansion of a conformal block. These coefficients descend to linear maps $\mathrm{Zhu}_g(V) \to \mathbb{C}$. We establish that these maps are h-supersymmetric functions on $\mathrm{Zhu}_g(V)$ (see Section 2.5 for the definition). In Sections 2.5 and 2.6 we construct a basis of h-supersymmetric functions on $\mathrm{Zhu}_g(V)$ and extend each one to a supertrace function on V, arriving at the definition in equation (1.0.8) above.

We then prove that the $S_M(u,\tau)$ lie in $\mathcal{C}(g,h)$. Finally in Section 2.7 we prove that the $S_M(u,\tau)$ span $\mathcal{C}(g,h)$. The outcome of all this is Theorem 2.7.2 which is stated already as Theorem 1.0.5 in the introduction.

2.1 Modular Forms

In this section we recall some functions that appear in connection with modular forms and elliptic curves. Consider the ill-defined expression

$$2\pi i {\sum_{n\in[\epsilon]}}' \frac{e^{2\pi i n z}}{1-\lambda q^n},$$

where λ is a root of unity and $[\epsilon]$ is a coset of $\mathbb{Q}$ modulo $\mathbb{Z}$ (also fix $\epsilon \in [\epsilon]$ such that $-1 < \epsilon \leq 0$, and let $\mu = e^{2\pi i \epsilon}$). By $\sum'$ we mean the summation over all nonsingular terms, i.e., if $[\epsilon] = \mathbb{Z}$ and $\lambda = 1$ then $n = 0$ is to be excluded from the sum.

To make sense of the sum we first rewrite it as

$$\frac{2\pi i \delta}{1-\lambda} + 2\pi i \sum_{n \in [\epsilon]_{>0}} \frac{e^{2\pi i n z}}{1 - \lambda q^n} - 2\pi i \sum_{n \in [\epsilon]_{<0}} \frac{\lambda^{-1} e^{2\pi i n z} q^{-n}}{1 - \lambda^{-1} q^{-n}},$$

where

$$\delta = \begin{cases} 1 & \text{if } [\epsilon] = \mathbb{Z} \text{ and } \lambda \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we expand in non-negative powers of q to get

$$\frac{2\pi i \delta}{1-\lambda} + 2\pi i \sum_{n \in [\epsilon]_{>0}} e^{2\pi i n z} + 2\pi i \sum_{m \in \mathbb{Z}_{>0}} \left[\sum_{n \in [\epsilon]_{>0}} e^{2\pi i n z} (\lambda q^n)^m - \sum_{n \in [\epsilon]_{<0}} e^{2\pi i n z} (\lambda^{-1} q^{-n})^m \right].$$

This is still not well-defined, because of the second term. Let us re-sum the second term using the geometric series formula. We arrive at the following formula, which we regard as a definition.

$$\begin{aligned} P^{\mu,\lambda}(z,q) = {} & \frac{2\pi i \delta}{1-\lambda} - 2\pi i \frac{e^{2\pi i (1+\epsilon) z}}{e^{2\pi i z} - 1} \\ & + 2\pi i \sum_{m \in \mathbb{Z}_{>0}} \left[\sum_{n \in [\epsilon]_{>0}} e^{2\pi i n z} (\lambda q^n)^m - \sum_{n \in [\epsilon]_{<0}} e^{2\pi i n z} (\lambda^{-1} q^{-n})^m \right]. \end{aligned} \tag{2.1.1}$$

Let us write

$$P^{\mu,\lambda}(z,q) = -z^{-1} + \sum_{k=0}^{\infty} P_k^{\mu,\lambda}(q) z^k.$$

The *Bernoulli polynomials* $B_n(\gamma)$ are defined by

$$\frac{e^{\gamma z}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{n!} B_n(\gamma).$$

For example $B_0(\gamma) = 1$, $B_1(\gamma) = \gamma - 1/2$, $B_2(\gamma) = \gamma^2 - \gamma + 1/6$, etc. The *Bernoulli numbers* are $B_n = B_n(1)$. Using the definition of the Bernoulli polynomials and the series expansion of $e^{2\pi i n z}$ we directly obtain the following.

Lemma 2.1.1. *For $k \in \mathbb{Z}_+$ we have*

$$P_k^{\mu,\lambda}(q) = \delta_{k,0}\frac{2\pi i\delta}{1-\lambda} - \frac{(2\pi i)^{k+1}}{(k+1)!}B_{k+1}(1+\epsilon)$$
$$+ \frac{(2\pi i)^{k+1}}{k!}\sum_{m\in\mathbb{Z}_{>0}}\left[\sum_{n\in[\epsilon]_{>0}} n^k(\lambda q^n)^m - \sum_{n\in[\epsilon]_{<0}} n^k(\lambda^{-1}q^{-n})^m\right].$$

We now record the modular transformation properties of the functions $P_k^{\mu,\lambda}$. For $(\mu,\lambda) \neq (1,1)$, our functions are essentially the same as the Q-functions of [8]. Indeed for all $k \in \mathbb{Z}_+$,

$$P_k^{\mu,\lambda}(q) = (2\pi i)^{k+1}Q_{k+1}(\mu,\lambda,q) \quad \text{when } (\mu,\lambda) \neq (1,1).$$

Section 4 of [8], in particular Theorem 4.6, tells us that $P_k^{\mu,\lambda}$, when summed in order of increasing powers of q, converges to a holomorphic function of $\tau \in \mathcal{H}$. Furthermore

$$P_k^{\mu,\lambda}(A\tau) = (\mathsf{c}\tau+\mathsf{d})^{k+1}P_k^{(\mu,\lambda)\cdot A}(\tau). \tag{2.1.2}$$

Since μ and λ are roots of unity, there exists $N \in \mathbb{Z}_+$ such that $\mu^N = \lambda^N = 1$. Therefore $P_k^{\mu,\lambda}(A\tau) = (\mathsf{c}\tau+\mathsf{d})^{k+1}P_k^{\mu,\lambda}(\tau)$ for all $A = \left(\begin{smallmatrix}\mathsf{a}&\mathsf{b}\\\mathsf{c}&\mathsf{d}\end{smallmatrix}\right) \in \Gamma_0(N)$ where $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ is the subgroup of matrices $\left(\begin{smallmatrix}\mathsf{a}&\mathsf{b}\\\mathsf{c}&\mathsf{d}\end{smallmatrix}\right)$ satisfying $\mathsf{a} \equiv \mathsf{d} \equiv 1 \pmod N$ and $\mathsf{b} \equiv \mathsf{c} \equiv 0 \pmod N$. Hence $P_k^{\mu,\lambda}(\tau)$ is a holomorphic modular form on $\Gamma_0(N)$ of weight $k+1$.

Now we consider the case $(\mu,\lambda) = (1,1)$. Comparing Lemma 2.1.1 with the standard definition of the *Eisenstein series*

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \quad (\text{for } k \geq 2)$$

shows that $P_k^{1,1}(q) = G_{k+1}(\tau)$ for $k \geq 1$. We also have $P_0^{1,1}(q) = -\pi i$. Therefore equation (2.1.2) holds when $(\mu,\lambda) = (1,1)$ and $k \geq 2$. It is well-known that $G_2(q)$ is not a modular form, but instead satisfies

$$G_2(A\tau) = (\mathsf{c}\tau+\mathsf{d})^2G_2(\tau) - 2\pi i\mathsf{c}(\mathsf{c}\tau+\mathsf{d}).$$

The function $P^{1,1}(z,q)$ (which we abbreviate to $P(z,q)$ below) is closely related to the classical Weierstrass zeta function

$$\zeta(z,\tau) = z^{-1} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{z-m\tau-n} + \frac{1}{m\tau+n} + \frac{z}{(m\tau+n)^2}\right]$$
$$= z^{-1} - \sum_{k=4}^{\infty} z^{k-1}G_k(\tau),$$

(the latter is the Laurent expansion about $z = 0$). We have

$$\zeta(z, \tau) = -P(z, q) + zG_2(q) - \pi i.$$

The Weierstrass elliptic function is $\wp(z, \tau) = -\frac{\partial}{\partial z}\zeta(z, \tau)$, so we have

$$\frac{\partial}{\partial z}P(z, q) = \wp(z, q) + G_2(q). \tag{2.1.3}$$

The Dedekind eta function is defined, for $\tau \in \mathcal{H}$, to be

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n). \tag{2.1.4}$$

From [21] p. 253 we have the following.

Proposition 2.1.2.

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau) \quad \textit{and} \quad \eta(\tfrac{-1}{\tau}) = (-i\tau)^{1/2}\eta(\tau).$$

The Jacobi theta function is defined, for $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, to be

$$\theta(z; \tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}. \tag{2.1.5}$$

From [28], p. 475 we have the following.

Proposition 2.1.3.

$$\theta(\tfrac{z}{\tau}; \tfrac{-1}{\tau}) = (-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta(z; \tau) \quad \textit{and} \quad \theta(z; \tau + 1) = \theta(z + \tfrac{1}{2}; \tau).$$

2.2 Conformal Blocks

Let $\mathcal{M}_N$ be the vector space of holomorphic modular forms on $\Gamma_0(N)$, i.e., the vector space of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ such that

- $f(A\tau) = f(\tau)$ for all $A \in \Gamma_0(N)$.
- $f(A\tau)$ is meromorphic at $\tau = i\infty$ for all $A \in SL_2(\mathbb{Z})$.

Definition 2.2.1. Let $\mathcal{V} = \mathcal{M}_{|G|} \otimes_{\mathbb{C}} V$. Define $\mathcal{O}(g, h)$ to be the $\mathcal{M}_{|G|}$-submodule of $\mathcal{V}$ generated by

$$\begin{cases} X_1(u, v) = \mathrm{Res}_z\, Y[u, z]v dz = u_{([0])}v & \text{for } (\mu(u), \lambda(u)) = (1, 1), \\ X_2(u, v) = \mathrm{Res}_z\, \wp(z, q)Y[u, z]v dz & \text{for } (\mu(u), \lambda(u)) = (1, 1), \\ u & \text{for } (\mu(u), \lambda(u)) \neq (1, 1), \\ X_3^{g,h}(u, v) = \mathrm{Res}_z\, P^{\mu(u),\lambda(u)}(z, q)Y[u, z]v dz & \text{for } (\mu(u), \lambda(u)) \neq (1, 1). \end{cases}$$

Definition 2.2.2. The space $\mathcal{C}(g,h)$ of *conformal blocks* is the space of functions $S : \mathcal{V} \times \mathcal{H} \to \mathbb{C}$ satisfying

CB1 $S(x+y,\tau) = S(x,\tau) + S(y,\tau)$ for all $x,y \in \mathcal{V}$, and $S(f(\tau)u,\tau) = f(\tau)S(u,\tau)$ for all $f(\tau) \in \mathcal{M}_{|G|}$, $u \in V$.

CB2 $S(u,\tau)$ is holomorphic in τ for each $u \in V$.

CB3 $S(x,\tau) = 0$ for all $x \in \mathcal{O}(g,h)$.

CB4 For all $u \in V$ such that $(\mu(u),\lambda(u)) = (1,1)$,

$$\left[(2\pi i)^2 q\frac{d}{dq} + \nabla_u G_2(q)\right] S(u,\tau) = S(\operatorname{Res}_z \zeta(z,q)L[z]udz,\tau). \tag{2.2.1}$$

An equivalent form of (2.2.1) is

$$(2\pi i)^2 q\frac{d}{dq} S(u,\tau) = -S(\operatorname{Res}_z P(z,q)L[z]udz,\tau). \tag{2.2.2}$$

2.2.1 Modular transformations of conformal blocks

Let K be a positive integer such that $1/K$ divides the conformal weight of each vector in V (the C_2-cofiniteness condition implies that K exists, see the first paragraph in the proof of Lemma 2.3.1 below). Let $\sqrt[K]{z}$ denote the principal K^{th} root of z i.e., $-\pi/K < \arg(\sqrt[K]{z}) \le \pi/K$. In the following theorem $(\mathsf{c}\tau + \mathsf{d})^{-k}$ is defined as the appropriate integer power of $\sqrt[K]{\mathsf{c}\tau + \mathsf{d}}$.

Theorem 2.2.1. *Let $S \in \mathcal{C}(g,h)$ and $A \in SL_2(\mathbb{Z})$. Define $S \cdot A : \mathcal{V} \times \mathcal{H} \to \mathbb{C}$ by*

$$[S\cdot A](u,\tau) = (\mathsf{c}\tau+\mathsf{d})^{-k} S(a, A\tau) \quad \textit{for } u \in V_{[k]},$$
$$\textit{and} \quad [S\cdot A](f(\tau)u,\tau) = f(\tau)[S\cdot A](u,\tau) \quad \textit{for } u \in V,\ f(\tau) \in \mathcal{M}_{|G|}.$$

Then $S \cdot A \in \mathcal{C}((g,h)\cdot A)$.

Proof. Fix $g,h \in G$, $A \in SL_2(\mathbb{Z})$, and let $S \in \mathcal{C}(g,h)$. It is obvious that $S \cdot A$ satisfies **CB1**. Because $S(u,\tau)$ is holomorphic in τ, $S(u, A\tau)$ is too. Because $\mathsf{c}\mathcal{H}+\mathsf{d}$ is disjoint from the branch cut, $(\mathsf{c}\tau+\mathsf{d})^{-k}$ is holomorphic in τ. Therefore $S \cdot A$ satisfies **CB2**.

Clearly $(\mu(u),\lambda(u)) = (1,1)$ if and only if $(\mu(u),\lambda(u)) \cdot A = (1,1)$. Suppose $(\mu(u),\lambda(u)) = (1,1)$, then we have $S(X_1(u,v),\tau) = 0$. Hence

$$[S\cdot A](X_1(u,v),\tau) = (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u - \nabla_v - 1} S(X_1(u,v), A\tau) = 0.$$

Next $[S \cdot A](X_2(u,v),\tau)$ reduces to

$$
\begin{aligned}
&[S \cdot A](u_{([-2])}v,\tau) + \sum_{k=2}^{\infty}(2k-1)G_{2k}(\tau)[S \cdot A](u_{([2k-2])}v,\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v-1}S(u_{([-2])}v, A\tau) \\
&\quad + \sum_{k=2}^{\infty}(2k-1)G_{2k}(\tau)(\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v+2k-1}S(u_{([2k-2])}v, A\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v-1}\left[S(u_{([-2])}v, A\tau) + \sum_{k=2}^{\infty}(2k-1)G_{2k}(A\tau)S(u_{([2k-2])}v, A\tau)\right] \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v-1}S(X_2(u,v), A\tau) = 0.
\end{aligned}
$$

Now suppose $(\mu,\lambda) = (\mu(u),\lambda(u))_{(g,h)} \neq (1,1)$ (so $(\mu(u),\lambda(u))_{(g,h)\cdot A} \neq (1,1)$ too). Then $[S \cdot A](X_3^{(g,h)\cdot A}(u,v),\tau)$ reduces to

$$
\begin{aligned}
&[S \cdot A](u_{([-1])}v,\tau) + \sum_{k=0}^{\infty}P_k^{(\mu,\lambda)\cdot A}(\tau)[S \cdot A](u_{([k])}v,\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v}S(u_{([-1])}v, A\tau) \\
&\quad - (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v}\sum_{k=0}^{\infty}P_k^{(\mu,\lambda)\cdot A}(\tau)(\mathsf{c}\tau+\mathsf{d})^{k+1}S(u_{([k])}v, A\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v}\left[S(u_{([-1])}v, A\tau) - \sum_{k=0}^{\infty}P_k^{\mu,\lambda}(A\tau)S(u_{([k])}v, A\tau)\right]. \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-\nabla_v}S(X_3^{g,h}(u,v), A\tau) = 0
\end{aligned}
$$

(having used the transformation property (2.1.2) of $P_k^{\mu,\lambda}$). Finally note that $[S \cdot A](u,\tau) = 0$ whenever $(\mu(u),\lambda(u)) \neq (1,1)$ because the same is true for S. Thus $S \cdot A$ satisfies **CB3**.

Let $(\mu(u),\lambda(u)) = (1,1)$ again. By a calculation similar to the one above, we have

$$
[S \cdot A](\mathrm{Res}_z\, \zeta(z,\tau)L[z]udz,\tau) = (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-2}S(\mathrm{Res}_z\, \zeta(z,A\tau)L[z]udz, A\tau).
$$

On the other hand

$$
\begin{aligned}
&\left[2\pi i\frac{d}{d\tau} + \nabla_u G_2(\tau)\right][S\cdot A](u,\tau) \\
&= \left[2\pi i\frac{d}{d\tau} + \nabla_u G_2(\tau)\right](\mathsf{c}\tau+\mathsf{d})^{-\nabla_u} S(u,A\tau) \\
&= \Big[-2\pi i\mathsf{c}\nabla_u(\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-1} + (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u}\frac{d(A\tau)}{d\tau}\frac{d}{d(A\tau)} \\
&\qquad\qquad + \nabla_u G_2(\tau)(\mathsf{c}\tau+\mathsf{d})^{-\nabla_u}\Big] S(u,A\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-2}\left[-2\pi i\mathsf{c}\nabla_u(\mathsf{c}\tau+\mathsf{d}) + \frac{d}{d(A\tau)} + \nabla_u G_2(\tau)(\mathsf{c}\tau+\mathsf{d})^2\right] S(u,A\tau) \\
&= (\mathsf{c}\tau+\mathsf{d})^{-\nabla_u-2}\left[\frac{d}{d(A\tau)} + \nabla_u G_2(A\tau)\right] S(u,A\tau).
\end{aligned}
$$

So $S\cdot A$ satisfies **CB4**. □

2.3 Differential equations satisfied by conformal blocks

We recall the crucial C_2-*cofiniteness* condition introduced in [29]. This condition, together with the conformal block axioms, implies the existence of an ordinary differential equation (ODE) satisfied by the conformal blocks.

Definition 2.3.1. The vertex operator algebra V is said to be C_2-*cofinite* if the subspace

$$C_2(V) = \mathrm{Span}\{u_{(-2)}v | u, v \in V\} \subseteq V$$

has finite codimension in V.

Lemma 2.3.1. *If V is C_2-cofinite then the $\mathcal{M}_{|G|}$-module $\mathcal{V}/\mathcal{O}(g,h)$ is finitely generated, for each $g, h \in G$.*

Proof. Since $C_2(V)$ is a graded subspace of V (under the Δ-grading) there exists $n_0 \in \mathbb{Z}_+$ such that $V_n \subseteq C_2(V)$ for all $n > n_0$. Let $W = \oplus_{k\le n_0} V_k \subseteq V$. Since $\Delta_{u_{(-2)}v} = \Delta_u + \Delta_v + 1$, every vector in V with conformal weight greater than n_0 can be expressed in terms of n^{th} products of vectors in W. Therefore all conformal weights in V are integer multiples of $1/K$ for some positive integer K.

Let $\mathcal{W} = \mathcal{M}_{|G|} W \subseteq \mathcal{V}$. Recall that

$$L_{[0]} = L_0 + \sum_{i\ge1} \alpha_{0i} L_i \quad \text{and} \quad L_0 = L_{[0]} + \sum_{i\ge1} \beta_{0i} L_{[i]},$$

for certain $\alpha_{0i}, \beta_{0i} \in \mathbb{C}$. Suppose $u \in V_n$, i.e., $L_0 u = nu$. Then $L_{[0]}u = nu$ modulo terms with strictly lower Δ. Similarly if $v \in V_{[n]}$ then $L_0 v = nv$ modulo terms with strictly lower ∇. Thus $\oplus_{k\le n} V_k = \oplus_{k\le n} V_{[k]}$ for any $n \in \mathbb{Q}$.

We will prove by induction on conformal weight (which is possible since conformal weights are multiples of $1/K$ and are bounded below) that $V_{[n]} \subseteq \mathcal{W} + \mathcal{O}(g,h)$ for all n. According to the last paragraph this holds for $n \leq n_0$ already.

Let $n > n_0$, and let $x \in V_{[n]}$. Since $V = W + C_2(V)$ we may write x as $w \in W$ plus a sum of vectors of the form

$$u_{(-2)}v = u_{([-2])}v + \sum_{j>-2} \alpha_{-2,j} u_{([j])}v,$$

where we assume u, v are homogeneous in the ∇-grading. It is clear that we can choose all the pairs of vectors u, v so that $\nabla_u + \nabla_v + 1 \leq n$. Therefore all the terms in the j-summation have $\nabla < n$, hence they lie in $\mathcal{W} + \mathcal{O}(g,h)$ by the inductive assumption. It suffices to show that $u_{([-2])}v \in \mathcal{W} + \mathcal{O}(g,h)$.

If $(\mu(u), \lambda(u)) = (1,1)$ then

$$X_2(u,v) = u_{([-2])}v + \sum_{k=2}^{\infty} (2k-1) G_{2k}(\tau) u_{([2k-2])}v \in \mathcal{O}(g,h).$$

The terms in the summation have $\nabla < n$, hence they lie in $\mathcal{W} + \mathcal{O}(g,h)$ by the inductive assumption. Therefore $u_{([-2])}v$ does too.

If $(\mu(u), \lambda(u)) \neq (1,1)$ then

$$X_3(u,v) = -u_{([-1])}v + \sum_{k=0}^{\infty} P_k^{\mu(u),\lambda(u)}(q) u_{([k])}v \in \mathcal{O}(g,h).$$

Substituting $L_{[-1]}u$ in place of u shows that

$$u_{([-2])}v - \sum_{k=0}^{\infty} k P_k^{\mu(u),\lambda(u)}(q) u_{([k-1])}v \in \mathcal{O}(g,h)$$

too. As before, $u_{([-2])}v \in \mathcal{W} + \mathcal{O}(g,h)$. □

Remark 2.3.2. Inspection of the proof of Lemma 2.3.1 reveals that the C_2-cofiniteness condition can be weakened to the following: $V/C^{(g,h)}$ is finite dimensional where $C^{(g,h)}$ is defined to be the span of the vectors

$$u_{(-2)}v \text{ for } (\mu(u), \lambda(u)) = (1,1), \quad \text{and} \quad u_{(-1)}v \text{ for } (\mu(u), \lambda(u)) \neq (1,1).$$

The following lemma is stated in [8].

Lemma 2.3.3. *For any integer $N \geq 1$, $\mathcal{M}_N$ is a Noetherian ring.*

Lemma 2.3.4. *Let V be C_2-cofinite, let $u \in V$, and let $S \in \mathcal{C}(g,h)$. There exists $m \in \mathbb{Z}_+$ and $r_0(\tau), \ldots r_{m-1}(\tau) \in \mathcal{M}_{|G|}$ such that*

$$S(L_{[-2]}^m u, \tau) + \sum_{i=0}^{m-1} r_i(\tau) S(L_{[-2]}^i u, \tau) = 0. \tag{2.3.1}$$

Proof. Let $I_n(u) \subseteq \mathcal{V}/\mathcal{O}(g,h)$ be the $\mathcal{M}_{|G|}$-submodule generated over $\mathcal{M}_{|G|}$ by the images of $u, L_{[-2]}u, \ldots L_{[-2]}^n u$. Because $\mathcal{V}/\mathcal{O}(g,h)$ is a finitely generated $\mathcal{M}_{|G|}$-module and $\mathcal{M}_{|G|}$ is a Noetherian ring, $\mathcal{V}/\mathcal{O}(g,h)$ is a Noetherian $\mathcal{M}_{|G|}$-module: meaning that the ascending chain $I_0(u) \subseteq I_1(u) \subseteq \ldots$ stabilizes. For some $m \in \mathbb{Z}_+$ we have $I_m(u) = I_{m-1}(u)$, which implies

$$L_{[-2]}^m u + \sum_{i=0}^{m-1} r_i(\tau) L_{[-2]}^i u \in \mathcal{O}(g,h)$$

for some $r_i(\tau) \in \mathcal{M}_{|G|}$. Equation (2.3.1) follows from this formula and **CB3**. □

The axiom **CB4** states an equality between $S(L_{[-2]}u,\tau)$ and $(q\frac{d}{dq})S(u,\tau)$ modulo 'terms of lower order', i.e., terms of the form $S(v,\tau)$ with $\nabla_v < \nabla_u$. We use this to convert (2.3.1) into an ODE satisfied by $S(u,\tau)$. More precisely we have

Theorem 2.3.5. *Let $\overline{q} = q^{1/|G|}$, and let $S \in \mathcal{C}(g,h)$. For each $u \in V$, $S(u,\tau)$ satisfies an ODE of the form*

$$(\overline{q}\frac{d}{d\overline{q}})^m S(u,\tau) + \sum_{i=0}^{m-1} g_i(\overline{q})(\overline{q}\frac{d}{d\overline{q}})^i S(u,\tau) + \sum_{j=0}^{m-1}\sum_{k=0}^{\infty} h_{jk}(\overline{q})(\overline{q}\frac{d}{d\overline{q}})^j S(x_{jk},\tau) = 0,$$

where the $x_{jk} \in V$ are of strictly lower conformal weight than u and the functions $g_i(\overline{q})$ and $h_{jk}(\overline{q})$ are polynomials in elements of $\mathcal{M}_{|G|}$ and derivatives of G_2 with respect to $\overline{q}$. In particular these functions are all regular at $\overline{q} = 0$, and so the ODE has a regular singular point there.

We write $\overline{q}$ here instead of q because the elements of $\mathcal{M}_{|G|}$ can be expressed as series in integer powers of $\overline{q}$, rather than q. For the proof of Theorem 2.3.5 see Section 6 of [8].

For $u^{(0)} \in V$ of minimal conformal weight the ODE satisfied by $S(u^{(0)},\tau)$ is homogeneous because there are no nonzero vectors with strictly lower conformal weight. The theory of Frobenius-Fuchs tells us that $S(u^{(0)},\tau)$ may be expressed in a certain form (2.3.2) below. For arbitrary $u \in V$ the same conclusion cannot be drawn directly because of the presence of the inhomogeneous term. However an induction on ∇_u shows that $S(u,\tau)$ does take the form (2.3.2) for all $u \in V$. The form in question is

$$\begin{aligned} S(u,\tau) &= \sum_{i=0}^{R} (\log q)^i S_i(u,\tau), \\ \text{where} \quad S_i(u,\tau) &= \sum_{j=1}^{b(i)} q^{\lambda_{ij}} S_{ij}(u,\tau), \\ \text{where} \quad S_{ij}(u,\tau) &= \sum_{n=0}^{\infty} C_{i,j,n}(u) q^{n/|G|}, \end{aligned} \tag{2.3.2}$$

where $\lambda_{ij_1} - \lambda_{ij_2} \notin \frac{1}{|G|}\mathbb{Z}$ for $1 \leq j_1 \neq j_2 \leq b(i)$. We call (2.3.2) the *Frobenius expansion* of $S(u,\tau)$.

A priori the parameters R, $b(i)$, and $\lambda_{i,j}$ in the Frobenius expansion of $S(u,\tau)$ depend on u. However if $\{u^{(i)}\}$ is a basis of W then the conformal blocks $S(u^{(i)},\tau)$ obviously span $\mathcal{C}(g,h)$. This implies that $\mathcal{C}(g,h)$ is finite dimensional, and that the Frobenius expansion of $S(u,\tau)$ may be written with fixed R, $b(i)$, λ_{i,j_r} independent of u.

2.4 Coefficients of Frobenius expansions

In this section we study the coefficients $C_{R,j,0} : V \to \mathbb{C}$ from (2.3.2). First we recall the definition of the g-twisted Zhu algebra $\mathrm{Zhu}_g(V)$.

Definition 2.4.1. For $u, v \in V$, $n \in \mathbb{Z}$ let

$$\begin{aligned} u \circ_n v &= \mathrm{Res}_w\, w^n (1+w)^{\Delta_u + \epsilon_u} Y(u,w) v\, dw \\ &= 2\pi i\, \mathrm{Res}_z\, e^{2\pi i (1+\epsilon_u) z} (e^{2\pi i z} - 1)^n Y[u,z] v\, dz. \end{aligned}$$

Let $J_g \subseteq V$ be the span of all elements of the form

$$\begin{aligned} &u \quad \text{for } \mu(u) \neq 1 \\ &u \circ_n v \quad \text{for } \mu(u) = \mu(v) = 1 \text{ and } n \leq -2, \\ &u \circ_n v \quad \text{for } \mu(u) = \mu(v)^{-1} \neq 1 \text{ and } n \leq -1, \\ \text{and}\quad &(L_{-1} + L_0) u \quad \text{for } \mu(u) = 1. \end{aligned}$$

The *g-twisted Zhu algebra* is $\mathrm{Zhu}_g(V) = V/J_g$ as a vector superspace, with the product induced by $\circ_{-1}$.

We denote the projection of $u \in V$ to $\mathrm{Zhu}_g(V)$ as $[u]$ or simply u. The following two theorems are proved in [4]; note that $u_{([0])}v$ in our notation is $(2\pi i)^{-1}[u,v]_{\hbar=1}$ in theirs.

Theorem 2.4.1.

- *The product $\circ_{-1}$ is well-defined on $\mathrm{Zhu}_g(V)$ and makes it into an associative superalgebra with unit $[|0\rangle]$. We denote the product by $*$.*
- *The 0^{th} Zhu product $\cdot_{([0])}\cdot$ is well-defined on $\mathrm{Zhu}_g(V)$ and we have*

$$u * v - p(u,v) v * u = 2\pi i u_{([0])} v \quad \text{for all } u, v \in \mathrm{Zhu}_g(V). \tag{2.4.1}$$

- *$[\omega]$ is central in $\mathrm{Zhu}_g(V)$.*

Theorem 2.4.2.

- *There is a restriction functor Ω from the category $\mathsf{PEMod}(g,V)$ of positive energy g-twisted V-modules to the category $\mathrm{Zhu}_g(V)$-mod of $\mathrm{Zhu}_g(V)$-modules. It sends M to M_0 with the action $[u] * x = u_0^M x$ for $u \in V$ and $x \in M_0$.*

- *There is an induction functor L going in the other direction, and we have $\Omega(L(N)) \cong N$ for any $\mathrm{Zhu}_g(V)$-module N.*
- *Ω and L are inverse bijections between the sets of irreducible modules in each category.*

It is shown in Proposition 2.17(c) of [4] that if the VOSA V is C_2-cofinite then $\mathrm{Zhu}_g(V)$ is finite dimensional for each g. The automorphism h descends to an automorphism of $\mathrm{Zhu}_g(V)$, which we also denote h.

Proposition 2.4.3. *Let $S \in \mathcal{C}(g,h)$ with Frobenius expansion (2.3.2). Fix $j \in \{1,2,\dots b(R)\}$, and let $f = C_{R,j,0}$. We have*

- *$f(u) = 0$ for all $u \in J_g(V)$, so f descends to a map $f : \mathrm{Zhu}_g(V) \to \mathbb{C}$.*
- *$f(u * v) = \delta_{\lambda(u)\lambda(v),1} p(u,v)\lambda(u)^{-1} f(v * u)$ for all $u, v \in \mathrm{Zhu}_g(V)$.*

Proof. By definition $S(\cdot,\tau)$ annihilates $\mathcal{O}(g,h)$. Therefore f annihilates the q^0 coefficient of any element of $\mathcal{O}(g,h)$. If $u \in V$ with $\mu(u) \neq 1$ then $u \in \mathcal{O}(g,h)$, so $f(u) = 0$. If $\mu(u) = 1$ then $2\pi i(L_{-1} + L_0)u = \tilde{\omega}_{([0])}u \in \mathcal{O}(g,h)$ is annihilated by f. Now

$$
\begin{aligned}
[q^0] : X_1(u,v) &= X_1(u,v) = u_{([0])}v, \\
[q^0] : X_2(u,v) &= \mathrm{Res}_z \left([q^0] : \partial_z P(z,q) - G_2(q)\right) Y[u,z]v dz \\
&= \mathrm{Res}_z \left[2\pi i \partial_z \frac{e^{2\pi i z}}{e^{2\pi i z} - 1} - 2\zeta(2)\right] Y[u,z]v dz \\
&= -(2\pi i)^2 \mathrm{Res}_z \frac{e^{2\pi i z}}{(e^{2\pi i z} - 1)^2} Y[u,z]v dz - 2\zeta(2) u_{([0])}v \\
&= -2\pi i u \circ_{-2} v - 2\zeta(2) u_{([0])}v, \\
\text{and} \quad [q^0] : X_3(u,v) &= \mathrm{Res}_z \left([q^0] : P^{\mu(u),\lambda(u)}(z,q)\right) Y[u,z]v dz \\
&= \mathrm{Res}_z \left[\frac{2\pi i \delta}{1 - \lambda(u)} - 2\pi i \frac{e^{2\pi i(1+\epsilon_u)z}}{e^{2\pi i z} - 1}\right] Y[u,z]v dz \\
&= \frac{2\pi i \delta}{1 - \lambda(u)} u_{([0])}v - u \circ_{-1} v.
\end{aligned}
$$

Hence f annihilates J_g, and descends to a function on $\mathrm{Zhu}_g(V)$.

Now for the second part. Let $\mu(u) = \mu(v) = 1$. If $\lambda(u)\lambda(v) \neq 1$ then $u * v$, $v * u$ and $u_{([0])}v$ all lie in $\mathcal{O}(g,h)$ and so are annihilated by f. If $\lambda(u) = \lambda(v) = 1$ then f annihilates $u_{([0])}v$, hence $f(u * v) = p(u,v) f(v * u)$. If $\lambda(u)\lambda(v) = 1$ with $\lambda(u) \neq 1$, then f annihilates

$$\frac{2\pi i}{1 - \lambda(u)} u_{([0])}v - u * v.$$

Combining this with (2.4.1) shows that

$$f(u * v) = p(u,v)\lambda(u)^{-1} f(v * u),$$

so we are done. ∎

2.5 h-supersymmetric functions

Let h be an automorphism of an associative superalgebra A, and let $\lambda(a)$ denote the h-eigenvalue of an eigenvector $a \in A$. In Section 2.4 we were led to consider functions $f : A = \mathrm{Zhu}_g(V) \to \mathbb{C}$ satisfying

$$f(a * b) = \delta_{\lambda(a)\lambda(b),1} p(a,b) \lambda(a)^{-1} f(b * a) \tag{2.5.1}$$

for all $a, b \in A$. We refer to such functions as *h-supersymmetric* functions on A, and we denote the space of all such functions $\mathcal{F}_h(A)$. We write $\mathcal{F}(A)$ for $\mathcal{F}_1(A)$ and call these *supersymmetric* functions.

Let A be semisimple now, and let $A = \oplus_{i \in I} A_i$ be its decomposition into simple components. The automorphism h permutes the A_i. The following fact is proved in Lemma 10.7 of [8] (for algebras, but the proof carries over to the present case with trivial modification).

Lemma 2.5.1. *$\mathcal{F}_h(A) = \oplus_{i \in J} \mathcal{F}_h(A_i)$ where the direct sum is over the subset $J \subset I$ of h-invariant simple components.*

For the rest of this section let A be a simple superalgebra. The Wedderburn theorem for superalgebras [14] is as follows.

Theorem 2.5.2. *A finite dimensional simple associative superalgebra over $\mathbb{C}$ is isomorphic to one of the following superalgebras.*

- $\mathrm{End}(\mathbb{C}^{m|k})$, *where* $m + k \geq 1$.
- $A_0[\xi]/(\xi^2 = 1)$, *where* $A_0 = \mathrm{End}(\mathbb{C}^n)$ *for some* $n \geq 1$, *and* ξ *is an odd indeterminate.*

We refer to these as Type I and Type II superalgebras, respectively. Supersymmetric functions on a simple superalgebra are characterized as follows [14].

Lemma 2.5.3. *Let A be a simple superalgebra, and let N be its unique up to isomorphism irreducible module. If A is of* Type I *then $\mathcal{F}(A)$ is spanned by the function $a \mapsto \mathrm{STr}_N\, a$. If A is of* Type II *then $\mathcal{F}(A)$ is spanned by the function $a \mapsto \mathrm{Tr}_N(a\xi)$.*

Now we describe h-supersymmetric functions on A by reducing to the case of supersymmetric functions. We deal with the Type I and Type II cases separately.

Lemma 2.5.4. *If A is of* Type I *then $\mathcal{F}_h(A)$ is spanned by the map $a \mapsto \mathrm{STr}_N(a\gamma)$ where $\gamma \in A$ has the property $h(a) = \gamma^{-1} a \gamma$.*

Proof. Since N is the unique irreducible A-module, the action $(a, x) \mapsto ax$ of A on N is equivalent to the action $(a, x) \mapsto h(a)x$. Hence there exists $\gamma : N \to N$ such that $h(a) = \gamma^{-1} a \gamma$ for all $a \in A$, $x \in N$. Since h is even by definition, we may take γ even too. Now the set of endomorphisms of N is just A itself, so $\gamma \in A_0$ (this is essentially the proof of the Skolem-Noether theorem). Since $h(\gamma) = \gamma^{-1}\gamma\gamma = \gamma$ we have $\lambda(\gamma) = 1$.

Let $f \in \mathcal{F}_h(A)$. If we put $\overline{f}(a) = f(a\gamma^{-1})$ then we have

$$\begin{aligned}\overline{f}(ab) = f(ab\gamma^{-1}) &= \lambda(a)^{-1}\delta_{\lambda(a)\lambda(b),1}p(a,b)f(b\gamma^{-1}a) \\ &= \lambda(a)^{-1}\delta_{\lambda(a)\lambda(b),1}p(a,b)f(b\gamma^{-1}a\gamma\gamma^{-1}) \\ &= \delta_{\lambda(a)\lambda(b),1}p(a,b)f(ba\gamma^{-1}) \\ &= \delta_{\lambda(a)\lambda(b),1}p(a,b)\overline{f}(ba).\end{aligned} \tag{2.5.2}$$

In particular $\overline{f}$ is supersymmetric, and therefore is a multiple of $\mathrm{STr}_N(a)$. Now, $\mathrm{STr}_N(a) = 0$ unless $\lambda(a) = 1$ for otherwise a maps one γ-eigenspace in N to another. Thus STr_N satisfies (2.5.2). The claim follows. □

The Type II case is a little more complicated because $A = \mathrm{End}(N)$ is no longer true (in other words the Skolem-Noether theorem is not true for Type II superalgebras). Let $A = A_0[\xi]/(\xi^2 = 1)$ be of Type II. We have $h(\xi)^2 = h(\xi^2) = h(1) = 1$, and $h(a)h(\xi) = h(a\xi) = h(\xi a) = h(\xi)h(a)$ for all $a \in A$. Hence $h(\xi) = \pm\xi$.

Lemma 2.5.5. *Let $A = A_0[\xi]/(\xi^2 = 1)$ be of* Type II. *Then $\mathcal{F}_h(A)$ is spanned by*

$$\begin{aligned} & a \mapsto \mathrm{Tr}_N(a\xi\gamma_0) \quad \text{if } h(\xi) = \xi, \\ \text{or} \quad & a \mapsto \mathrm{STr}_N(a\sigma_N\gamma_0) \quad \text{if } h(\xi) = -\xi,\end{aligned}$$

where $\gamma_0 \in A_0$ is chosen to satisfy $h(a) = \gamma_0^{-1}a\gamma_0$ for $a \in A_0$.

Proof. For the case of $h(\xi) = \xi$, note that $\gamma_0^{-1}\xi\gamma_0 = \xi = h(\xi)$, so $h(a) = \gamma_0^{-1}a\gamma_0$ for all $a \in A$. The proof of Lemma 2.5.4 shows that $f(a) = \overline{f}(a\gamma_0)$ where $\overline{f}$ is supersymmetric. The result now follows from Lemma 2.5.3.

Now consider the case $h(\xi) = -\xi$. This time $\gamma_0^{-1}a\gamma_0 = \sigma_A(h(a))$. So let $\overline{f}(a) = f(a\gamma_0^{-1})$ and repeat calculation (2.5.2) to obtain

$$\overline{f}(ab) = \delta_{\lambda(a)\lambda(b),1}p(a,b)(-1)^{p(a)}\overline{f}(ba),$$

Let $a, b \in A_0$, we have

$$\overline{f}(\xi a) = -\delta_{\lambda(a),-1}\overline{f}(a\xi)$$

and

$$\overline{f}(ab) = \delta_{\lambda(a)\lambda(b),1}\overline{f}(ba).$$

So $\overline{f}(\xi a) = 0$ and $a \mapsto f(a\gamma_0^{-1})$ is a symmetric function on A_0. Hence f is a scalar multiple of $\mathrm{Tr}_{N_0}(a\gamma_0) = \frac{1}{2}\mathrm{STr}_N(a\gamma_0\sigma_N)$. □

In summary we have (where γ and γ_0 are as in Lemmas 2.5.4 and 2.5.5)

Case	$h(a)$	$\mathcal{F}_h(A)$
A : A is of Type I	$\gamma^{-1}a\gamma$	$\mathbb{C}\,\mathrm{STr}_N(\cdot\gamma)$
B : A is of Type II and $h(\xi) = \xi$	$(\gamma_0\xi)^{-1}a(\gamma_0\xi)$	$\mathbb{C}\,\mathrm{Tr}_N(\cdot\gamma_0\xi)$
C : A is of Type II and $h(\xi) = -\xi$	$(\gamma_0\sigma_N)^{-1}a(\gamma_0\sigma_N)$	$\mathbb{C}\,\mathrm{STr}_N(\cdot\gamma_0\sigma_N)$

In case B (resp. C) let $\gamma = \gamma_0 \xi$ (resp. $\gamma = \gamma_0 \sigma_N$). Then we may write $h(a) = \gamma^{-1} a \gamma$ in all cases. Note that γ is odd in case B.

2.6 Supertrace Functions

Recall that we assume $\mathrm{Zhu}_g(V)$ to be semisimple (and finite dimensional, which follows from C_2-cofiniteness). Let $A \subseteq \mathrm{Zhu}_g(V)$ be a h-invariant simple component, and N its irreducible module. Let $M = L(N)$. The isomorphism $\gamma : N \longrightarrow N$ intertwines the two equivalent actions of $\mathrm{Zhu}_g(V)$ on N,i.e., the usual one $(u, x) \mapsto ux$, and $(u, x) \mapsto h(u)x$. By Theorem 2.4.2 γ lifts to a grade-preserving isomorphism $M \longrightarrow M$ which we also call γ, such that

$$\gamma^{-1} u_n^M \gamma = h(u)_n^M \tag{2.6.1}$$

for all $u \in V$, $n \in [\epsilon_u]$ (below we drop the M superscripts).

Definition 2.6.1. The supertrace function associated to M is

$$S_M(u, \tau) = \begin{cases} \mathrm{STr}_M\, u_0 \gamma q^{L_0 - c/24} & \text{in cases A and C,} \\ \mathrm{Tr}_M\, u_0 \gamma q^{L_0 - c/24} & \text{in case B,} \end{cases}$$

where cases A, B and C refer to the table of Section 2.5 and γ is as defined immediately below that table.

Theorem 2.6.1. *Fix $g, h \in G$. The supertrace function S_M, associated to $M = L(N)$ for some h-invariant irreducible $\mathrm{Zhu}_g(V)$-module N, lies in $\mathcal{C}(g, h)$. Furthermore the S_M are linearly independent.*

The h-supersymmetric functions constructed in Section 2.5 are linearly independent, so the same is true of the S_M. We prove that $S_M \in \mathcal{C}(g, h)$ via the technical Propositions 2.6.2-2.6.7 below.

For the rest of this section we fix $M = L(N) \in \mathsf{PEMod}_h(g, V)$. We use the following notation below. For an endomorphism α of a graded piece M_r of M,

$$T_{M_r}(\alpha) = \begin{cases} \mathrm{STr}_{M_r}(\alpha\gamma) & \text{in cases A and C,} \\ \mathrm{Tr}_{M_r}(\alpha\gamma) & \text{in case B.} \end{cases}$$

If $\alpha : M \longrightarrow M$ is a grade-preserving linear map then we define $T_M(\alpha)$ to be $\sum_r T_{M_r}(\alpha|_{M_r})$.

In Propositions 2.6.2 and 2.6.3 we show that T_M annihilates vectors $u \in V$ such that $(\mu(u), \lambda(u)) \neq (1, 1)$, and $u_{([0])} v$ for $(\mu(u), \lambda(u)) = (1, 1)$. In Proposition 2.6.5 we show that

$$\begin{aligned} &T_M((\mathrm{Res}_z\, P^{\mu(u),\lambda(u)}(z, q) Y[u, z] v dz)_0 q^{L_0}) \\ &\qquad = \begin{cases} -T_M(u_0 v_0 q^{L_0}) & \text{if } (\mu(u), \lambda(u)) = (1, 1), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In Proposition 2.6.6 we use this identity to show that T_M annihilates the remaining elements of $\mathcal{O}(g,h)$. All of these facts hold for S_M immediately, so **CB3** is verified. In Proposition 2.6.7 we use the identity above to show that S_M satisfies **CB4**, this is the single place at which the extra factor of $q^{-c/24}$ is important. **CB2** is automatic.

To verify **CB1**, we must show that the power series S_M converges to a holomorphic function in $|q| < 1$. In the presence of the C_2-cofiniteness condition, this actually follows from the other axioms. We may repeat the calculations of Section 2.3 to show that the power series S_M formally satisfies a Fuchsian ODE, it then follows from general theory that S_M converges to a solution to this ODE.

Since $\omega \in V$ is h-invariant, γ commutes with L_0 and q^{L_0}. Since ω is in the center of $\mathrm{Zhu}_g(V)$ and N is irreducible, L_0 acts on N as a scalar $h(N) \in \mathbb{C}$. Recall that $[L_0, u_k] = -ku_k$. It follows that L_0 acts on M_r as the scalar $h(N) + r$, and that u_0 commutes with L_0 and q^{L_0}.

Proposition 2.6.2. *If* $(\mu(u), \lambda(u)) \neq (1,1)$, *then* $T_{M_r}(u_0) = 0$.

Proof. The operators u_n are defined for $n \in \epsilon_u + \mathbb{Z}$, so if $\mu(u) \neq 1$ then $u_0 = 0$ and $T_{M_r}(u_0) = 0$.

Let $x \in M$ have γ-eigenvalue α. If $\lambda(u) \neq 1$ then

$$\gamma^{-1}(u_0\gamma x) = h(u)_0 x = \lambda u_0 x = \lambda\alpha^{-1}(u_0\gamma x).$$

Thus $u_0\gamma$ permutes γ-eigenspaces leaving none of them fixed. Hence the (super)trace of $u_0\gamma$ is zero. □

Proposition 2.6.3. *If* $\mu(u) = 1$ *then*

$$T_{M_r}(u_{([0])}v)_0 = [1-\lambda(u)](2\pi i)^{-1} T_{M_r} u_0 v_0.$$

Proof. Assume $\mu(v) = 1$ and $\lambda(v) = \lambda(u)^{-1}$, for otherwise both sides of the equation vanish and the result is trivially true.

The commutator formula (1.1.3) with $m = k = 0$ is

$$[u_0, v_0] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_u - 1}{j} (u_{(j)}v)_0 = (\mathrm{Res}_w (1+w)^{\Delta_u - 1} Y(u,w)v dw)_0.$$

Using the substitution $w = e^{2\pi i z} - 1$ gives

$$\begin{aligned}(u_{([0])}v)_0 &= (\mathrm{Res}_z Y[u,z]v dz)_0 = \left(\mathrm{Res}_z e^{2\pi i \Delta_u z} Y(u, e^{2\pi i z} - 1)v dz\right)_0 \\ &= (2\pi i)^{-1}\left(\mathrm{Res}_w (1+w)^{\Delta_u - 1} Y(u,w)v dw\right)_0 \\ &= (2\pi i)^{-1}[u_0, v_0].\end{aligned}$$

In cases A and C γ is even and we have

$$\begin{aligned}\mathrm{STr}_{M_r}[u_0, v_0]\gamma &= \mathrm{STr}_{M_r} u_0 v_0 \gamma - p(u,v)\,\mathrm{STr}_M v_0 u_0 \gamma \\ &= \mathrm{STr}_{M_r} u_0 v_0 \gamma - p(u,v)\lambda(u)\,\mathrm{STr}_M v_0 \gamma u_0 \\ &= [1-\lambda(u)]\,\mathrm{STr}_{M_r} u_0 v_0 \gamma.\end{aligned}$$

In case B γ is odd and we have

$$\begin{aligned}\mathrm{Tr}_{M_r}[u_0, v_0]\gamma &= \mathrm{Tr}_{M_r}\, u_0v_0\gamma - \mathrm{Tr}_{M_r}\, v_0u_0\gamma \\ &= \mathrm{Tr}_{M_r}\, u_0v_0\gamma - \mathrm{Tr}_{M_r}\, u_0\gamma v_0 \\ &= [1 - \lambda(v)^{-1}]\, \mathrm{Tr}_{M_r}\, u_0v_0\gamma.\end{aligned}$$

□

Proposition 2.6.4. *If $\mu(v) = \mu(u)^{-1}$ and $n \in [\epsilon_u]_{>0}$, then*

$$T_{M_r} u_n v_{-n} = \lambda(u)^{-1} p(u, v) T_{M_{r+n}} v_{-n} u_n.$$

Proof. Consider the (super)trace on the space $M_r \oplus M_{r+n}$ and note that, in this direct sum, u_n annihilates M_r and v_{-n} annihilates M_{r+n}. In cases A and C we have

$$\begin{aligned}\mathrm{STr}_{M_r}\, u_nv_{-n}\gamma &= \mathrm{STr}_{M_r\oplus M_{r+n}}\, u_nv_{-n}\gamma \\ &= p(u,v)\, \mathrm{STr}_{M_r\oplus M_{r+n}}\, v_{-n}\gamma u_n \\ &= p(u,v)\lambda(u)^{-1}\, \mathrm{STr}_{M_r\oplus M_{r+n}}\, v_{-n}u_n\gamma \\ &= p(u,v)\lambda(u)^{-1}\, \mathrm{STr}_{M_{r+n}}\, v_{-n}u_n\gamma.\end{aligned}$$

In case B we have

$$\begin{aligned}\mathrm{Tr}_{M_r}\, u_nv_{-n}\gamma &= \mathrm{Tr}_{M_r\oplus M_{r+n}}\, u_nv_{-n}\gamma \\ &= \mathrm{Tr}_{M_r\oplus M_{r+n}}\, v_{-n}\gamma u_n \\ &= \lambda(u)^{-1}\, \mathrm{Tr}_{M_r\oplus M_{r+n}}\, v_{-n}u_n\gamma \\ &= \lambda(u)^{-1}\, \mathrm{Tr}_{M_{r+n}}\, v_{-n}u_n\gamma.\end{aligned}$$

In the case $p(u,v) = 1$ the result holds. If $p(u,v) = -1$ then u_nv_{-n} is even, both sides vanish and the result holds. □

No further calculations require resolution into the cases A, B and C.

Proposition 2.6.5.

$$\begin{aligned}&T_M((\mathrm{Res}_z\, P^{\mu(u),\lambda(u)}(z,q)Y[u,z]vdz)_0 q^{L_0}) \\ &\qquad = \begin{cases} -T_M(u_0v_0q^{L_0}) & \text{if } (\mu(u),\lambda(u)) = (1,1), \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.6.2}$$

Proof. Let $\epsilon = \epsilon_u$, $\mu = \mu(u)$ and $\lambda = \lambda(u)$. Assume that $\mu(v) = \mu^{-1}$ and $\lambda(v) = \lambda^{-1}$, for otherwise both sides of the claimed equality vanish automatically. Assume without loss of generality that $L_0|_{M_0} = 0$.

For any $r \geq 0$, the q^r coefficient of

$$T_M(\mathrm{Res}_z\, P^{\mu,\lambda}(z,q)Y[u,z]vdz)_0 q^{L_0}$$

is $X - Y + Z$, where

$$\begin{aligned} X &= \frac{2\pi i\delta}{1-\lambda} T_{M_r}(u_{([0])}v)_0, \\ Y &= 2\pi i T_{M_r}(\mathrm{Res}_z \frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi i z}-1} Y[u,z]vdz)_0, \\ \text{and}\quad Z &= 2\pi i \sum_{m\in\mathbb{Z}_{>0}} \lambda^m \sum_{n\in[\epsilon]_{>0}} T_{M_{r-mn}}(\mathrm{Res}_z\, e^{2\pi i n z} Y[u,z]vdz)_0 \\ &\quad - 2\pi i \sum_{m\in\mathbb{Z}_{>0}} \lambda^{-m} \sum_{n\in[\epsilon]_{<0}} T_{M_{r+mn}}(\mathrm{Res}_z\, e^{2\pi i n z} Y[u,z]vdz)_0 \end{aligned} \tag{2.6.3}$$

(the sum defining Z is finite since terms with $|mn| > r$ contribute nothing).

Using the change of variable $w = e^{2\pi i z} - 1$ and the commutator formula, we have

$$\begin{aligned} \left(\mathrm{Res}_z\, e^{2\pi i n z} Y[u,z]vdz\right)_0 &= (2\pi i)^{-1} \left(\mathrm{Res}_w (1+w)^{n+\Delta_u-1} Y(u,w)vdw\right)_0 \\ &= (2\pi i)^{-1} \sum_{j\in\mathbb{Z}_+} \binom{n+\Delta_u-1}{j} (u_{(j)}v)_0 = (2\pi i)^{-1}[u_n, v_{-n}] \end{aligned} \tag{2.6.4}$$

for any $n \in [\epsilon_u]$. Now let $n \in [\epsilon_u]_{>0}$. Proposition 2.6.4 implies that

$$\begin{aligned} \sum_{m\in\mathbb{Z}_{>0}} \lambda^m T_{M_{r-mn}} u_n v_{-n} &= p(u,v)\lambda^{-1} \sum_{m\in\mathbb{Z}_{>0}} \lambda^m T_{M_{r-(m-1)n}} v_{-n} u_n \\ &= p(u,v) \sum_{m\in\mathbb{Z}_+} \lambda^m T_{M_{r-mn}} v_{-n} u_n, \end{aligned}$$

from which it follows that

$$\sum_{m\in\mathbb{Z}_{>0}} \lambda^m T_{M_{r-mn}} [u_n, v_{-n}] = p(u,v) T_{M_r} v_{-n} u_n. \tag{2.6.5}$$

If $n \in [\epsilon_u]_{<0}$ then $-n \in [\epsilon_v]_{>0}$, and so

$$\begin{aligned} \sum_{m\in\mathbb{Z}_{>0}} \lambda^{-m} T_{M_{r+mn}} [u_n, v_{-n}] &= -p(u,v) \sum_{m\in\mathbb{Z}_{>0}} \lambda^{-m} T_{M_{r-m(-n)}} [v_{-n}, u_n] \\ &= -T_{M_r} u_n v_{-n}. \end{aligned} \tag{2.6.6}$$

Combining (2.6.5) and (2.6.6) yields

$$\begin{aligned} Z &= \sum_{m\in\mathbb{Z}_{>0}} \lambda^m \sum_{n\in[\epsilon]_{>0}} T_{M_{r-mn}} [u_n, v_{-n}] - \sum_{m\in\mathbb{Z}_{>0}} \lambda^{-m} \sum_{n\in[\epsilon]_{<0}} T_{M_{r+mn}} [u_n, v_{-n}] \\ &= \sum_{n\in[\epsilon]_{<0}} T_{M_r} u_n v_{-n} + p(u,v) \sum_{n\in[\epsilon]_{>0}} T_{M_r} v_{-n} u_n. \end{aligned} \tag{2.6.7}$$

Next we have

$$(\mathrm{Res}_z \frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1} Y[u,z]vdz)_0 = (\mathrm{Res}_z \frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1} e^{2\pi i\Delta_u z} Y(u, e^{2\pi iz}-1)vdz)_0$$
$$= (2\pi i)^{-1}(\mathrm{Res}_w\, w^{-1}(1+w)^{\Delta_u+\epsilon} Y(u,w)vdw)_0$$
$$= (2\pi i)^{-1} \sum_{j\in\mathbb{Z}_+} \binom{\Delta_u+\epsilon}{j} (u_{(j-1)}v)_0.$$

Plugging this into the Borcherds identity with $n=-1$ and $m = 1+\epsilon_u = -k$ yields

$$Y = \sum_{j\in\mathbb{Z}_+} T_{M_r}\left(u_{-j+\epsilon}v_{j-\epsilon} + p(u,v)v_{-j-1-\epsilon}u_{j+1+\epsilon}\right). \tag{2.6.8}$$

Now we see that

$$-Y+Z = \begin{cases} -T_{M_r}u_0v_0 & \text{if } \mu = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mu \neq 1$ then $X = \delta = 0$ and we are done, similarly if $\mu = \lambda = 1$. Suppose $\mu = 1 \neq \lambda$ now. From Proposition 2.6.3 we have

$$X - Y + Z = T_{M_r}u_0v_0 - T_{M_r}u_0v_0 = 0$$

as required. □

In the two following lemmas suppose $(\mu(u), \lambda(u)) = (1,1)$.

Proposition 2.6.6. *We have* $T_M(\mathrm{Res}_z\, \wp(z,q)Y[u,z]v, \tau) = 0$.

Proof. For all $u \in V$ we have $(L_{[-1]}u)_0 = 2\pi i(L_{-1}u + L_0u)_0 = 0$. By Proposition (2.6.5) we have

$$\begin{aligned} 0 &= T_M((L_{[-1]}u)_0 v_0 q^{L_0}) \\ &= T_M((\mathrm{Res}_z\, P(z,q)Y[L_{[-1]}u,z]vdz)_0 q^{L_0}) \\ &= T_M((\mathrm{Res}_z\, P(z,q)\partial_z Y[u,z]vdz)_0 q^{L_0}) \\ &= -T_M((\mathrm{Res}_z\, \partial_z P(z,q)Y[u,z]vdz)_0 q^{L_0}) \\ &= -T_M((\mathrm{Res}_z(\wp(z,q) + G_2(q))Y[u,z]vdz)_0 q^{L_0}). \end{aligned}$$

By Proposition (2.6.3), the $G_2(q)$ term contributes nothing, so the result follows. □

Proposition 2.6.7. *We have*

$$\left[(2\pi i)^2 q\frac{d}{dq} + \nabla_u G_2(q)\right] S_M(u,\tau) = S_M(\mathrm{Res}_z\, \zeta(z,q)L[z]udz, \tau).$$

Proof. We start with equation (2.6.2). Multiply through by $q^{-c/24}$ and substitute

$u = \tilde{\omega} = (2\pi i)^2(\omega - c/24|0\rangle)$, so that $u_0 = (2\pi i)^2(L_0 - c/24)$. The right hand side is

$$-(2\pi i)^2 T_M((L_0 - c/24)v_0 q^{L_0 - c/24}) = -(2\pi i)^2 q\frac{d}{dq}T_M(v_0 q^{L_0-c/24}).$$

The left hand side is

$$\begin{aligned} &T_M((\mathrm{Res}_z\, P(z,q)L[z]vdz)_0 q^{L_0-c/24}) \\ &= T_M((\mathrm{Res}_z[-\zeta(z,q) + zG_2(q) - \pi i]L[z]vdz)_0 q^{L_0-c/24}) \\ &= G_2(q)\nabla_v T_M(v_0 q^{L_0-c/24}) - T_M((\mathrm{Res}_z\, \zeta(z,q)L[z]vdz)_0 q^{L_0-c/24}), \end{aligned}$$

having used $\tilde{\omega}_{([1])}v = \nabla_v b$ and Proposition (2.6.3). □

Thus Theorem 2.6.1 is proved.

2.7 Exhausting a conformal block by supertrace functions

Let $\phi, \psi \in \mathbb{C}$. We will say ϕ is *lower* than ψ (and ψ is *higher* than ϕ) if the real part of ϕ is strictly less than that of ψ.

Let $S(u,\tau) \in \mathcal{C}(g,h)$. In this section we show that S may be written as a linear combination of supertrace functions $S_M(u,\tau)$ for $M \in \mathsf{PEMod}_h(g,V)$. We need the following proposition.

Proposition 2.7.1. *Let $S \in \mathcal{C}(g,h)$ with Frobenius expansion (2.3.2).*

- *Let $j \in \{1,2,\dots b(R)\}$, then $C_{R,j,0}((\omega - \frac{c}{24} - \lambda_{R,j}) * u) = 0$ for all $u \in V^g$.*
- *Let $j \in \{1,2,\dots b(R-1)\}$, then $C_{R-1,j,0}((\omega - \frac{c}{24} - \lambda_{R-1,j})^2 * u) = 0$ for all $u \in V^g$.*

Proof. Recall equation (2.2.2) – the equivalent form of **CB4**. Equating coefficients of $\log^R q$ shows that (2.2.2) holds with $S_{R,j}$ in place of S, that is

$$(2\pi i)^2 q\frac{d}{dq}S_{R,j}(u,\tau) = -S_{R,j}(\mathrm{Res}_z\, P(z,q)L[z]udz, \tau). \tag{2.7.1}$$

Let us equate coefficients of $q^{\lambda_{R,j}}$ in (2.7.1). The left hand side gives $(2\pi i)^2\lambda_{pj}C_{R,j,0}(u)$, while the right hand side gives $C_{R,j,0}$ applied to

$$\begin{aligned} 2\pi i\, \mathrm{Res}_z \frac{e^{2\pi i z}}{e^{2\pi i z}-1}L[z]udz &= \mathrm{Res}_w\, w^{-1}(1+w)^{\Delta_\omega}Y((2\pi i)^2\omega, w)udw \\ &\quad - (c/24)\,\mathrm{Res}_w\, w^{-1}(1+w)^{\Delta_{|0\rangle}}Y((2\pi i)^2|0\rangle, w)udw \\ &= (2\pi i)^2(\omega - c/24) * u. \end{aligned}$$

This proves the first part.

Without loss of generality let $\lambda_{R,j} = \lambda_{R-1,j}$. Equating coefficients of $\log^{R-1} q$ in (2.2.2) yields

$$(2\pi i)^2 q\frac{d}{dq}S_{R-1,j}(u,\tau) = -S_{R-1,j}(\mathrm{Res}_z\, P(z,q)L[z]udz,\tau) - p(2\pi i)^2 S_{R,j}(u,\tau). \tag{2.7.2}$$

Equating coefficients of $q^{\lambda_{R-1,j}}$ yields

$$C_{R-1,j,0}((\omega - c/24 - \lambda_{R-1,j}) * u) = RC_{R,j,0}(u).$$

This, together with the first part of the proposition, implies the second part. □

Let S have the expansion (2.3.2). Following Section 2.5 we have

$$C_{R,j,0}(u) = \sum_N \alpha_N T_N(u)$$

for some constants $\alpha_N \in \mathbb{C}$. The sum runs over the h-invariant irreducible $\mathrm{Zhu}_g(V)$-modules N. Proposition 2.7.1 implies that α_N is nonzero only for N that satisfy $\omega|_N = \lambda_{R,j} + c/24$. Now consider

$$\sum_N \alpha_N S_{L(N)}(u,\tau) \in q^{\lambda_{R,j}}\mathbb{C}[[q^{1/|G|}]].$$

The coefficient of $q^{\lambda_{R,j}}$ is nothing but $C_{R,j,0}$. Therefore the series

$$S'(u,\tau) = S_{R,j}(u,\tau) - \sum_N \alpha_N S_{L(N)}(u,\tau)$$

has lowest power of q whose exponent is higher than $\lambda_{R,j}$. The coefficient of the lowest power of q in $S'(u,\tau)$ is h-supersymmetric, so we may write it as a linear combination of T_N as above. The modules N that occur must be different than the ones used in the first iteration because ω acts on these modules by some constant higher than $\lambda_{R,j} + c/24$. We subtract the corresponding $S_{L(N)}(u,\tau)$ as before and repeat. The process terminates because there are only finitely many irreducible $\mathrm{Zhu}_g(V)$-modules. We obtain $S_{R,j}(u,\tau)$ as a linear combination of $S_M(u,\tau)$. It follows that $S_{R,j}(u,\tau) \in \mathcal{C}(g,h)$.

We may repeat the argument above, using the second part of Proposition 2.7.1, to conclude that $S_{R-1,j}(u,\tau) \in \mathcal{C}(g,h)$ also. Hence $S_{R-1,j}$ satisfies (2.7.1) in addition to (2.7.2). Together these equations imply $R = 0$. Thus $S = \sum_j S_{R,j}$ is a linear combination of supertrace functions.

In summary we have proved the following theorem.

Theorem 2.7.2. *Let V be a C_2-cofinite VOSA with rational conformal weights. Let G be a finite group of automorphisms of V. Suppose $\mathrm{Zhu}_g(V)$ is finite dimensional and semisimple for each $g \in G$. Now fix commuting $g, h \in G$. Let $A^1, \ldots A^s$ be the h-invariant simple components of $\mathrm{Zhu}_g(V)$, let N^i be the irreducible A^i-module, and* *let $M^i = L(N^i)$. The space $\mathcal{C}(g,h)$ of conformal blocks is spanned by the supertrace function $S_{M^i}(u,\tau)$ defined as follows*

$$\begin{cases} \mathrm{STr}_{M^i}\, u_0 \gamma q^{L_0 - c/24} & \text{if } A^i \text{ is of } \mathsf{Type\ I}, \text{ or is of } \mathsf{Type\ II} \text{ and } h(\xi) = -\xi, \\ \mathrm{Tr}_{M^i}\, u_0 \gamma q^{L_0 - c/24} & \text{if } A^i \text{ is of } \mathsf{Type\ II} \text{ and } h(\xi) = \xi, \end{cases}$$

where γ is as defined in Section 2.5 on N^i and then extended to M^i using equation (2.6.1).

Chapter 3

Some examples and applications

In this chapter we work out the consequences of Theorem 2.7.2 on five examples.

3.1 The neutral free fermion VOSA

As a vector superspace, the neutral free fermion VOSA $V = F(\varphi)$ ([17], pg. 98) is the span of the monomials

$$\varphi_{n_1} \cdots \varphi_{n_s}|0\rangle,$$

where $n_i \in 1/2 + \mathbb{Z}$, $n_1 < \ldots < n_s < 0$, and the monomial has parity $s \bmod 2$. The VOSA structure is generated by the single odd field

$$Y(\varphi, z) = \sum_{n\in\mathbb{Z}} \varphi_{(n)} z^{-n-1} = \sum_{n\in 1/2+\mathbb{Z}} \varphi_n z^{-n-1/2}.$$

The action of the modes on V is by left multiplication, subject to the relations $\varphi_n|0\rangle = 0$ for $n > 0$ and the commutation relation

$$\varphi_m\varphi_n + \varphi_n\varphi_m = \delta_{m,-n} \quad \Longleftrightarrow \quad [Y(\varphi, z), Y(\varphi, w)] = \delta(z, w).$$

The Virasoro vector is

$$\omega = \frac{1}{2}\varphi_{-3/2}\varphi_{-1/2}|0\rangle.$$

The element $\varphi = \varphi_{-1/2}|0\rangle$ has conformal weight $1/2$, and the central charge of V is $c = 1/2$. This VOSA is C_2-cofinite.

Let $G = \{1, \sigma_V\} \cong \mathbb{Z}/2\mathbb{Z}$. In this section we explicitly compute the conformal blocks $\mathcal{C}(g, h; u)$ for $u = |0\rangle$ and $u = \varphi$. To do so we determine the g-twisted Zhu algebra and its modules, look up the corresponding h-supersymmetric functions in the table of Section 2.5, and extend these to supertrace functions using equation (2.6.1).

Let $g = \sigma_V$. We have $\epsilon_\varphi = -1/2$ and

$$\varphi \circ_n v = \mathrm{Res}_w\, w^n (1+w)^{1/2-1/2} Y(\varphi, w) v dw = \varphi_{(n)} v$$

(Definitions 1.1.2 and 2.4.1). Therefore J_{σ_V} contains all monomials except $|0\rangle$, hence $\mathrm{Zhu}_{\sigma_V}(V)$ is either $\mathbb{C}|0\rangle$ or 0. Note that the adjoint module V is a positive energy σ_V-twisted V-module, therefore $\mathrm{Zhu}_{\sigma_V}(V) = \mathbb{C}|0\rangle$. The unique irreducible $\mathrm{Zhu}_{\sigma_V}(V)$-module is $N = \mathbb{C}$ and the corresponding σ_V-twisted V-module is $L(N) = V$.

Let $h = 1$. Then we can take $\gamma = 1$ and $\mathcal{C}(\sigma_V, 1; u)$ is spanned by

$$\mathrm{STr}_V\, u_0 q^{L_0 - c/24}.$$

Let $h = \sigma_V$ now, which acts on $\mathrm{Zhu}_{\sigma_V}(V)$ as the identity. Again $\gamma = 1$, but its extension to V according to equation (2.6.1) is now σ_V rather than the identity. $\mathcal{C}(\sigma_V, \sigma_V; u)$ is spanned by

$$\mathrm{STr}_V\, u_0 \sigma_V q^{L_0 - c/24} = \mathrm{Tr}_V\, u_0 q^{L_0 - c/24}.$$

Let $g = 1$. We have

$$\varphi \circ_n v = \mathrm{Res}_w\, w^n (1+w)^{1/2} Y(\varphi, w) v dw,$$

so in $\mathrm{Zhu}_1(V)$ any mode $\varphi_{(n)}$ for $n \leq -2$ is a linear combination of modes $\varphi_{(k)}$ for $k > n$. Hence $\mathrm{Zhu}_1(V)$ is a quotient of $\mathbb{C}|0\rangle + \mathbb{C}\varphi$. In fact $\mathrm{Zhu}_1(V) = \mathbb{C}|0\rangle + \mathbb{C}\varphi$, we prove it below by exhibiting an irreducible positive energy 1-twisted V-module M such that M_0 is 2 dimensional. The unit element of $\mathrm{Zhu}_1(V)$ is $|0\rangle$ and we readily compute that $\varphi * \varphi = \frac{1}{2}|0\rangle$. Therefore $\mathrm{Zhu}_1(V) \cong \mathbb{C}[\xi]/(u^2 = 1)$, where 1 is the image of $|0\rangle$ and ξ is the image of $\sqrt{2}\varphi$.

We construct M as follows: it has basis

$$\varphi^M_{n_1} \cdots \varphi^M_{n_s} 1$$

where $n_i \in \mathbb{Z}$, $n_1 < \ldots < n_s \leq 0$ and the parity of this monomial is s mod 2. The modes of the field

$$Y^M(\varphi, z) = \sum_{n \in \mathbb{Z}} \varphi^M_n z^{-n-1}$$

satisfy $\varphi^M_n |0\rangle = 0$ for $n \geq 1$ and $\varphi^M_m \varphi^M_n + \varphi^M_n \varphi^M_m = \delta_{m,-n}$. Note that $M_0 = \mathbb{C}1 + \mathbb{C}\varphi^M_0 1$. The unique irreducible $\mathrm{Zhu}_1(V)$-module is $N = M_0$ and the corresponding 1-twisted V-module is $L(N) = M$.

Let $h = 1$. According to the table of Section 2.5 we have $\gamma = \xi$ and the space of h-supersymmetric functions is spanned by $u \mapsto \mathrm{Tr}_N\, u\xi$. The extension of γ to M is

$$\gamma : \varphi^M_{n_1} \cdots \varphi^M_{n_s} 1 \mapsto \sqrt{2}\varphi^M_{n_1} \cdots \varphi^M_{n_s} \varphi^M_0 1$$

and $\mathcal{C}(1,1;u)$ is spanned by

$$\mathrm{Tr}_M\, u_0^M \gamma q^{L_0 - c/24}.$$

Let $h = \sigma_V$. In this case $h|_{N_0} = \mathrm{id}_{N_0}$ so $\gamma = \sigma_N$. When extended to M we have $\gamma = \sigma_M$. Therefore $\mathcal{C}(1, \sigma_V; u)$ is spanned by

$$\mathrm{STr}_M\, u_0^M \gamma q^{L_0 - c/24} = \mathrm{Tr}_M\, u_0^M q^{L_0 - c/24}.$$

3.1.1 Conformal blocks in weights 0 and 1/2

Recall the Dedekind eta function defined by (2.1.4). Each of the spaces of conformal blocks in the following table is spanned by the accompanying function, which is determined from the formulas of the last section and some basic combinatorics.

Conformal block	Character	Infinite product	η-quotient
$\mathcal{C}(\sigma_V, 1; \|0\rangle)$	$q^{-1/48}\,\mathrm{SCh}(V)$	$q^{-1/48}\prod_{n\geq 0}(1 - q^{n+1/2})$	$\frac{\eta(\tau/2)}{\eta(\tau)}$
$\mathcal{C}(\sigma_V, \sigma_V; \|0\rangle)$	$q^{-1/48}\,\mathrm{Ch}(V)$	$q^{-1/48}\prod_{n\geq 0}(1 + q^{n+1/2})$	$\frac{\eta(\tau)^2}{\eta(2\tau)\eta(\tau/2)}$
$\mathcal{C}(1, 1; \|0\rangle)$	0	0	0
$\mathcal{C}(1, \sigma_V; \|0\rangle)$	$q^{-1/48} q^{1/16}\,\mathrm{Ch}(M)$	$q^{-1/24}\prod_{n\geq 0}(1 + q^n)$	$2\frac{\eta(\tau)}{\eta(2\tau)}$

In the fourth line we use $L_0|_{M_0} = 1/16$ which follows a direct computation with the Borcherds identity.

Theorem 1.0.5 implies that if $f(\tau) \in \mathcal{C}(g, h; |0\rangle)$ then $f(A\tau) \in \mathcal{C}((g,h)\cdot A; |0\rangle)$ for all $A \in SL_2(\mathbb{Z})$. This may be verified directly for the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ using the explicit forms above together with Proposition 2.1.2.

Since $L_{[0]}\varphi = L_0\varphi$, we have $\nabla_\varphi = \Delta_\varphi = 1/2$. Note that if $(g,h) \neq (1,1)$ then $\mathcal{C}(g,h;\varphi) = 0$ because it is spanned by the (super)trace of an odd linear operator. For $g = h = 1$ we have

$$[\varphi_0^M \gamma] : \varphi_{n_1}^M \cdots \varphi_{n_s}^M 1 \mapsto \sqrt{2}\varphi_0^M \varphi_{n_1}^M \cdots \varphi_{n_s}^M \varphi_0^M 1 = \frac{\epsilon}{\sqrt{2}} \varphi_{n_1}^M \cdots \varphi_{n_s}^M 1$$

where $\epsilon = (-1)^{s-1}$ if $n_s = 0$ and $(-1)^s$ otherwise. Therefore

$$\mathcal{C}(1,1;\varphi) \text{ is spanned by } q^{-1/48} q^{1/16} \prod_{n=1}^{\infty}(1 - q^n) = \eta(\tau),$$

which is a weight 1/2 modular form in accordance with Theorem 1.0.5.

3.2 The charged free fermions VOSA with real conformal weights

As a vector superspace, the charged free fermions VOSA $V = F^a_{\text{ch}}(\psi, \psi^*)$ ([17], pg. 98) is the span of the monomials

$$\psi_{(i_1)}\psi_{(i_2)}\cdots\psi_{(i_m)}\psi^*_{(j_1)}\psi^*_{(j_2)}\cdots\psi^*_{(j_n)}|0\rangle, \tag{3.2.1}$$

where $i_r, j_s \in \mathbb{Z}$, $i_1 < \ldots < i_m \leq -1$, $j_1 < \ldots < j_n \leq -1$, and the parity of the monomial is $(m+n)$ mod 2. The VOSA structure is generated by the two odd fields

$$Y(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_{(n)} z^{-n-1} \quad \text{and} \quad Y(\psi^*, z) = \sum_{n\in\mathbb{Z}} \psi^*_{(n)} z^{-n-1}.$$

The action of the modes on V is by left multiplication, subject to the relations $\psi_{(n)}|0\rangle = \psi^*_{(n)}|0\rangle = 0$ for $n \geq 0$, and the commutation relations

$$[\psi_{(m)}, \psi^*_{(n)}] = \psi_{(m)}\psi^*_{(n)} + \psi^*_{(n)}\psi_{(m)} = \delta_{m+n,-1} \Longleftrightarrow [Y(\psi,z), Y(\psi^*, w)] = \delta(z,w). \tag{3.2.2}$$

All other commutators vanish.

Let $a \in \mathbb{R}$. We define a Virasoro vector ([17], pg. 102)

$$\omega^a = a\left(\psi_{(-2)}\psi^*_{(-1)}|0\rangle\right) + (1-a)\left(\psi^*_{(-2)}\psi_{(-1)}|0\rangle\right). \tag{3.2.3}$$

We write $L^a(z) = Y(\omega^a, z)$. With respect to this choice of Virasoro vector we have

$$\Delta_\psi = 1 - a \quad \text{and} \quad \Delta_{\psi^*} = a.$$

The central charge of V is $c = -2(6a^2 - 6a + 1)$. This VOSA is C_2-cofinite.

3.2.1 Twisted modules

Fix $\mu, \lambda \in \mathbb{C}$ of unit modulus and let the automorphisms g and h of V be defined by

$$\begin{aligned} g(\psi) &= \mu^{-1}\psi, \quad g(\psi^*) = \mu\psi^*, \\ h(\psi) &= \lambda^{-1}\psi, \quad h(\psi^*) = \lambda\psi^*, \end{aligned}$$

extended to all n^{th} products (note that the Virasoro element is indeed fixed by g and h). Let $\mu = e^{2\pi i\delta}$ and $\lambda = e^{2\pi i\rho}$ where $\delta, \rho \in [0,1)$, we do not require these to be rational numbers. Hence g and h need not have finite order.

A g-twisted V-module M will have fields

$$Y^M(\psi, z) = \sum_{n\in-[\delta]} \psi^M_n z^{-n-(1-a)} \quad \text{and} \quad Y^M(\psi^*, z) = \sum_{n\in[\delta]} \psi^{*M}_n z^{-n-a},$$

and the modes must satisfy

$$[\psi_m^M, \psi_n^{*M}] = \psi_m^M \psi_n^{*M} + \psi_n^{*M}\psi_m^M = \delta_{m+n,0}. \tag{3.2.4}$$

Let us put $M = V$, $\psi^M(z) = z^{-x}\psi(z)$ and $\psi^{*M}(z) = z^x\psi^*(z)$ where $x \in -[\delta] - [a]$. We may easily confirm that

$$\psi_n^M = \psi_{(n-a-x)} \quad \text{and} \quad \psi_n^{*M} = \psi^*_{(n-1+a+x)}$$

have the correct commutation relations (3.2.4). Therefore $M = M^{(g)} = V$ is given the structure of a positive energy g-twisted V-module, as long as we choose x so that ψ_n^M and ψ_n^{*M} annihilate $|0\rangle$ for $n > 0$. This leads to the requirement $0 \leq a + x \leq 1$.

For $g \neq 1$ this requirement fixes x uniquely and so $M \in \mathsf{PEMod}(g, V)$ (it is clear that M is irreducible). Notice that $a + x = 1 - \delta$.

If $g = 1$ then $\delta = 0$, we have the choice of putting $x = 1 - a$ or $x = -a$ and we get a module in $\mathsf{PEMod}(1, V)$ either way. In the first case $\psi_n^M = \psi_{(n-1)}$ and $\psi_n^{*M} = \psi^*_{(n)}$, so $M_0 = \mathbb{C}|0\rangle + \mathbb{C}\psi$. In the second case $M_0 = \mathbb{C}|0\rangle + \mathbb{C}\psi^*$ and $f : M_0 \to M_0$ defined by $f(|0\rangle) = \psi^*$ and $f(\psi) = |0\rangle$ lifts to an equivalence from the first to the second module. It is convenient to take the first model of the 1-twisted module, since then in all cases we have $a + x = 1 - \delta$.

3.2.2 Zhu algebras

Let $g \neq 1$. If $\epsilon_v + \epsilon_\psi = -1$ then

$$\psi \circ_n v = \text{Res}_w w^n (1 + w)^{\Delta_\psi + \epsilon_\psi} Y(\psi, z) v dw \in J_g.$$

It is possible to write $\psi_{(n)}v$ as a linear combination of $\psi_{(k)}v$ for $k > n$ whenever $n \leq -1$. The same goes for ψ^*. Iterating this procedure reveals that $\text{Zhu}_g(V)$ is a quotient of $\mathbb{C}|0\rangle$. The existence of the positive energy g-twisted V-module exhibited above shows that $\text{Zhu}_g(V) = \mathbb{C}|0\rangle$.

Let $g = 1$. The same argument as above holds but with $n \leq -2$. Thus $\text{Zhu}_1(V)$ is a quotient of $\mathbb{C}|0\rangle + \mathbb{C}\psi + \mathbb{C}\psi^* + \mathbb{C}\psi_{(-1)}\psi^*$. Calculating the products of these four elements reveals that their span is isomorphic to $\text{End}(\mathbb{C}^{1|1})$ via $\psi \mapsto E_{21}$, $\psi^* \mapsto E_{12}$ and $\psi_{(-1)}\psi^* \mapsto (a-1)E_{11} + aE_{22}$. Since we have constructed an irreducible positive energy 1-twisted V-module we have $\text{Zhu}_1(V) \cong \text{End}(\mathbb{C}^{1|1})$.

Note that Theorem 1.0.5 has nothing to say unless g, h have finite order and $a \in \mathbb{R}$ is chosen to lie in $\mathbb{Q}$. Even so, we can press on and write down supertrace functions. We have seen that the Zhu algebra is always of $\mathsf{Type\ I}$, so there is a single supertrace function associated to (g, h), which is

$$\text{STr}_M\, u_0^M \gamma q^{L_0 - c/24},$$

where M is the unique module from $P(g, V)$. But what is γ? For $g \neq 1$ the automorphism h restricts to the identity on $\text{Zhu}_g(V)$, therefore equation (2.6.1) gives $\gamma = h^{-1}$. From the construction of $M^{(1)}$ we see that $\gamma = h^{-1}$ in the $g = 1$ case too.

Next we restrict attention to $u = |0\rangle$ and write

$$\chi_{\mu,\lambda}(\tau) := \mathrm{STr}_M\, h^{-1} q^{L_0 - c/24}$$

for brevity. Next we will express the twisted supercharacters $\chi_{\mu,\lambda}(\tau)$ in terms of Jacobi theta functions and derive modular transformations.

3.2.3 Twisted Supercharacters

The *conformal weight* $h = h(M)$ of an irreducible V-module M is defined to be the eigenvalue of L_0^M on the lowest graded piece of M. We use the Borcherds identity to compute $h(M)$ for the twisted V-modules M defined above.

Put $u = \psi$, $v = \psi^*$, so that $[\epsilon_u] = -[\delta]$ and $[\epsilon_v] = [\delta]$ in the Borcherds identity, (1.1.2). Let $m = -\delta$, $k = \delta$ and denote by LHS(n) the left hand side of (1.1.2) with these choices of u, v, m, and k. We have

$$\mathrm{LHS}(-1) = (\psi_{(-1)}\psi^*)_0^M - (\delta + a)(\psi_{(0)}\psi^*)_0^M$$

and
$$\mathrm{LHS}(-2) = (\psi_{(-2)}\psi^*)_0^M - (\delta + a)(\psi_{(-1)}\psi^*)_0^M + \tfrac{1}{2}(\delta + a)(\delta + a + 1)(\psi_{(0)}\psi^*)_0^M.$$

Rearranging and using $\psi_{(0)}\psi^* = |0\rangle$ yields

$$(\psi_{(-2)}\psi^*)_0 = \mathrm{LHS}(-2) + (\delta + a)\,\mathrm{LHS}(-1) + \tfrac{1}{2}(\delta + a)(\delta + a - 1).$$

The corresponding right hand side of (1.1.2) is

$$\mathrm{RHS}(n) = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} \left[\psi_{-\delta+n-j}\psi^*_{\delta+j-n} + (-1)^n \psi^*_{\delta-j}\psi_{-\delta+j}\right].$$

If we apply this to $|0\rangle$, then the first term vanishes and the second term equals $(-1)^n$. Therefore, when applied to $|0\rangle$,

$$(\psi_{(-2)}\psi^*)_0 = 1 - (\delta + a) + \tfrac{1}{2}(\delta + a)(\delta + a - 1).$$

A similar calculation shows that, when applied to $|0\rangle$,

$$(\psi^*_{(-2)}\psi)_0 = -(\delta + a) + \tfrac{1}{2}(\delta + a)(\delta + a + 1).$$

Combining these with (3.2.3) yields

$$h(M) = \tfrac{1}{2}(\delta - a)(\delta + a - 1) = \tfrac{1}{2}\left[x(x-1) + 2ax\right] \tag{3.2.5}$$

(having also used $a + x = 1 - \delta$).

Let $\phi(\tau) = \prod_{n=1}^{\infty}(1 - q^n)$, so that $\eta(\tau) = q^{1/24}\phi(\tau)$.

Applying $\psi_{(-n)} = \psi^M_{-n+a+x}$ to a monomial in V raises its L_0-eigenvalue by $n - a -$

$x = n - (1-\delta)$. Similarly $\psi^*_{(-n)} = \psi^{*M}_{-n+1-a-x}$ raises the eigenvalue by $n-\delta$. We have

$$\chi_{\mu,\lambda}(\tau) = q^{h(M)-c/24} \prod_{n=1}^{\infty}(1-\lambda q^{n-(1-\delta)}) \prod_{n=1}^{\infty}(1-\lambda^{-1}q^{n-\delta}).$$

The first product is the contribution of the ψ terms, the second is that of the ψ^* terms. Note that when $g = h = 1$ the supercharacter vanishes.

Recall the Jacobi triple product identity:

$$\prod_{m=1}^{\infty}(1-z^{2m})(1+z^{2m-1}y^2)(1+z^{2m-1}y^{-2}) = \sum_{n\in\mathbb{Z}} z^{n^2}y^{2n}.$$

Set $z = q^{1/2}$, and $y^2 = -\lambda q^{\delta-1/2}$. We obtain

$$\begin{aligned}\chi_{\mu,\lambda}(\tau) &= \frac{q^{h(M)-c/24}}{\phi(q)} \sum_{n\in\mathbb{Z}}(-\lambda)^n q^{n^2/2+(\delta-1/2)n} \\ &= \frac{e^{2\pi i[h(M)-(c-1)/24]\tau}}{\eta(\tau)}\theta((\delta-\tfrac{1}{2})\tau + (\rho-\tfrac{1}{2});\tau)\end{aligned}$$

where $\theta(z;\tau)$ is the Jacobi theta function defined by (2.1.5).

3.2.4 Modular transformations

Let $A = \delta - \frac{1}{2}$ and $B = \rho - \frac{1}{2}$. Using Proposition 2.1.3 we have

$$\begin{aligned}\chi_{\mu,\lambda}(\tau+1) &= \frac{e^{2\pi i[h-(c-1)/24](\tau+1)}}{\eta(\tau+1)}\theta(A(\tau+1)+B;\tau+1) \\ &= e^{2\pi i[h-(c-1)/24]}\frac{e^{2\pi i[h-(c-1)/24]\tau}}{e^{\pi i/12}\eta(\tau)}\theta(A\tau+(B+A+\tfrac{1}{2});\tau) \\ &= e^{2\pi i[h-c/24]}\frac{e^{2\pi i[h-(c-1)/24]\tau}}{\eta(\tau)}\theta((\delta-\tfrac{1}{2})\tau+(\delta+\rho-\tfrac{1}{2});\tau).\end{aligned}$$

This is proportional to $\chi_{\mu,\lambda\mu}(\tau)$ the $\lambda\mu$-twisted supercharacter of the irreducible μ-twisted V-module.

Using Proposition 2.1.3 again we have

$$\begin{aligned}\chi_{\mu,\lambda}(-1/\tau) &= \frac{e^{-2\pi i[h-(c-1)/24]/\tau}}{\eta(-1/\tau)}\theta((B\tau-A)/\tau;-1/\tau) \\ &= \frac{e^{-2\pi i[h-(c-1)/24]/\tau}}{(-i\tau)^{1/2}\eta(\tau)}(-i\tau)^{1/2}e^{\pi i(B\tau-A)^2/\tau}\theta(B\tau-A;\tau) \\ &= e^{-2\pi i AB}e^{-2\pi i[h-(c-1)/24-A^2/2]/\tau}\frac{e^{2\pi i[B^2/2]\tau}}{\eta(\tau)}\theta(B\tau-A;\tau).\end{aligned}$$

Recall the formula for c in terms of a, we use it to obtain

$$\begin{aligned} \tfrac{1}{24}(c-1)+\tfrac{1}{2}A^2 &= \tfrac{1}{8}(-4a^2+4a-1)+\tfrac{1}{2}(\delta-\tfrac{1}{2})^2 \\ &= \tfrac{1}{2}(\delta-\tfrac{1}{2})^2-\tfrac{1}{2}(a-\tfrac{1}{2})^2 \\ &= \tfrac{1}{2}(\delta-a)(\delta+a-1)=h(M). \end{aligned} \tag{3.2.6}$$

Hence

$$\chi_{\mu,\lambda}(-1/\tau)=e^{-2\pi iAB}\frac{e^{2\pi i[B^2/2]\tau}}{\eta(\tau)}\theta(B\tau-A;\tau).$$

But by calculation (3.2.6) again, $\frac{1}{2}B^2=h(M')-\frac{1}{24}(c-1)$ where M' is the irreducible λ-twisted V-module. Therefore $\chi_{\mu,\lambda}(-1/\tau)$ is proportional to $\chi_{\lambda,\mu^{-1}}(\tau)$ the μ^{-1}-twisted supercharacter of the irreducible positive energy λ-twisted V-module.

In summary $\chi_{1,1}=0$ and $\chi_{\mu,\lambda}(A\tau)\propto\chi_{(\mu,\lambda)\cdot A}(\tau)$ for $(g,h)\neq(1,1)$. If $a\in\mathbb{Q}$ our VOSA has rational conformal weights, if μ and λ are roots of unity then g,h lie in some finite group of automorphisms of V. In this case we may apply Theorem 1.0.5 and it confirms the modular transformations of the $\chi_{\mu,\lambda}(\tau)$ computed above. Note that the direct computation holds for all $a\in\mathbb{R}$ and g,h of unit modulus.

3.3 VOSAs associated to integral lattices

Let $(Q,\langle\cdot,\cdot\rangle)$ be a rank r integral lattice with positive definite bilinear form $\langle\cdot,\cdot\rangle$. Let $\mathfrak{h}=Q\otimes_{\mathbb{Z}}\mathbb{C}$ with the induced positive definite bilinear form $\langle\cdot,\cdot\rangle$. The *loop algebra* $\tilde{\mathfrak{h}}=\mathfrak{h}[t^{\pm1}]$ is equipped with a Lie bracket as follows:

$$[ht^m,h't^n]=m\,\langle h,h'\rangle\,\delta_{m,-n}.$$

Let $S_-(\mathfrak{h})=U(\tilde{\mathfrak{h}})/U(\tilde{\mathfrak{h}})\mathfrak{h}[t]$. We write h_m for ht^m. Explicitly $S_-(\mathfrak{h})$ has a basis of monomials

$$h^1_{n_1}\cdots h^s_{n_s}1$$

where the h^i range over a basis of $\mathfrak{h}$, and $n_1\leq\ldots n_s\leq-1$ are integers.

The *twisted group algebra* $\mathbb{C}_\epsilon[Q]$ of Q is a unital associative algebra with basis $\{e^\alpha|\alpha\in Q\}$, unit element $1=e^0$, and multiplication $e^\alpha e^\beta=\epsilon(\alpha,\beta)e^{\alpha+\beta}$ where the function $\epsilon:Q\times Q\to\{\pm1\}$ has been chosen to satisfy

- $\epsilon(0,a)=\epsilon(a,0)=1$ for all $a\in Q$,
- $\epsilon(a,b)\epsilon(a+b,c)=\epsilon(a,b+c)\epsilon(b,c)$ for all $a,b,c\in Q$,
- $\epsilon(a,b)=\epsilon(b,a)(-1)^{\langle a,b\rangle+\langle a,a\rangle\langle b,b\rangle}$ for all $a,b\in Q$.

It may be shown that such ϵ exists.

Associated to this data there is a VOSA ([17], pg. 148).

Definition 3.3.1. The lattice VOSA $(V_Q, |0\rangle, Y, \omega)$ associated to Q is defined to be $V_Q = S_-(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[Q]$ as a vector superspace, where the parity of $s \otimes e^\alpha$ is $\langle \alpha, \alpha \rangle$ mod 2. The vacuum vector is $|0\rangle = 1 \otimes 1$. Let $h \in \mathfrak{h}$, $\alpha \in Q$, and $n \in \mathbb{Z}$. Define $h_n : V_Q \to V_Q$ by

$$\begin{aligned} h_n(s \otimes e^\alpha) &= (h_n s) \otimes e^\alpha \quad \text{for } n < 0, \\ h_n(1 \otimes \mathbb{C}_\epsilon[Q]) &= 0 \quad \text{for } n > 0, \\ h_0(1 \otimes e^\alpha) &= \langle h, \alpha \rangle\, 1 \otimes e^\alpha, \\ \text{and} \quad [h_m, h'_n] &= m\, \langle h, h' \rangle\, \delta_{m,-n}. \end{aligned}$$

Put $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}$ and

$$\Gamma_\alpha(z) = e^\alpha z^{\alpha_0} \exp(-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j) \exp(-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j),$$

where by definition $e^\alpha(s \otimes e^\beta) = \epsilon(\alpha, \beta)(s \otimes e^{\alpha+\beta})$. The state-field correspondence is given by $Y(h \otimes 1, z) = h(z)$, $Y(1 \otimes e^\alpha, z) = \Gamma_\alpha(z)$, and is extended to all of V_Q by normally ordered products, i.e.,

$$Y(h_n a, w) = \mathrm{Res}_z \left[h(z) Y(u, w) i_{z,w}(z-w)^n - Y(u, w) h(z) i_{w,z}(z-w)^n\right] dz.$$

Finally, the Virasoro vector is

$$\omega = \frac{1}{2} \sum_{i=1}^{r} a^i_{(-1)} b^i_{(-1)} |0\rangle$$

where $\{a^i\}$ and $\{b^i\}$ are bases of $\mathfrak{h}$ dual under $\langle \cdot, \cdot \rangle$, i.e., $\langle a^i, b^j \rangle = \delta_{ij}$.

Some commutators between the generating fields are

$$\begin{aligned} [h(z), h'(w)] &= \langle h, h' \rangle\, \partial_w \delta(z, w), \\ [h(z), \Gamma_\alpha(w)] &= \langle \alpha, h \rangle\, \Gamma_\alpha(w) \delta(z, w). \end{aligned}$$

There is an explicit expression for $[\Gamma_\alpha(z), \Gamma_\beta(w)]$ which we do not require. The conformal weight of $h = h_{-1}|0\rangle \in S_-(\mathfrak{h}) \otimes 1$ is 1, the conformal weight of $1 \otimes e^\alpha$ is $\langle \alpha, \alpha \rangle / 2$. The central charge of V equals the rank r of Q. The lattice VOSAs are known to be C_2-cofinite.

3.3.1 Irreducible modules and their (super)characters

Let $G = \{1, \sigma_V\}$. It is explained in [17] that V_Q is σ_V-rational. If Q is an even lattice, i.e., $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in Q$, then V_Q is purely even and so $\sigma_V = 1$. We focus on the case Q is not even. The same proof as in [17] shows that V_Q is also 1-rational.

Let $Q^\circ \subseteq \mathfrak{h}$ be the lattice dual to Q and let $\delta \in Q^\circ$. We define

$$\begin{aligned} Y^\delta(h,z) &= h(z) + \langle \delta, h \rangle z^{-1}, \\ Y^\delta(e^\alpha, z) &= z^{\langle \delta, \alpha \rangle} \Gamma_\alpha(z). \end{aligned} \tag{3.3.1}$$

Under this modified state-field correspondence V acquires the structure of a positive energy σ_V-twisted V-module (see [23] Section 5). The V-module structure depends on δ only through $\delta + Q$. Via this construction the cosets of Q° modulo Q are in bijection with the irreducible positive energy (σ_V-twisted) V_Q-modules [17]. The Virasoro field acts on (V, Y^δ) as

$$L^\delta(z) = L(z) + \frac{1}{2} z^{-1} \sum_{i=1}^{r} \left[\langle \delta, b^i \rangle a^i(z) + \langle \delta, a^i \rangle b^i(z) \right] + \frac{\langle \delta, \delta \rangle}{2} z^{-2}.$$

From this we see that the L_0^δ-eigenvalue of $1 \otimes e^\alpha$ is $\langle \alpha + \delta, \alpha + \delta \rangle / 2$. The L_0^δ-eigenvalue of $h \in \mathfrak{h}$ is 1 as before.

In a similar way the irreducible positive energy 1-twisted V-modules are exactly (V, Y^ρ) (defined as in (3.3.1)) but for $\rho \in \mathfrak{h}$ satisfying

$$\langle 2\rho, \alpha \rangle \equiv \langle \alpha, \alpha \rangle \pmod 2 \tag{3.3.2}$$

for every $\alpha \in Q$.

Let $\{a^i\}$ be a basis of Q and let $p(a^i)$ denote the parity of $\langle a^i, a^i \rangle$. Let $\{b^i\}$ be the basis of Q° dual to $\{a^i\}$, and let

$$\rho = \frac{1}{2} \sum_{p(a^i)=1} b^i + \sum_{p(a^i)=0} b^i.$$

Clearly ρ satisfies equation (3.3.2) for $\alpha \in \{a^i\}$. Now let $\alpha = \sum_i k_i a^i$ where $k_i \in \mathbb{Z}$. Then

$$\langle 2\rho, \alpha \rangle = \sum_i k_i \langle 2\rho, a^i \rangle \quad \text{and} \quad \langle \alpha, \alpha \rangle = \sum_{i,j} k_i k_j \langle a^i, a^j \rangle$$

but

$$\sum_{i,j} k_i k_j \langle a^i, a^j \rangle \equiv \sum_i k_i^2 \langle a^i, a^i \rangle \equiv \sum_i k_i \langle a^i, a^i \rangle \equiv \sum_i k_i \langle 2\rho, a^i \rangle \pmod 2,$$

so ρ satisfies (3.3.2) for all $\alpha \in Q$. Let $Q^\bullet$ be the set of all elements of $\mathfrak{h}$ satisfying (3.3.2) for all $\alpha \in Q$. If Q is an even lattice then V_Q is a VOA rather than a VOSA, $\sigma_V = 1$, and $Q^\bullet = Q^\circ$. If Q is integral but not even then $Q^\circ \cup Q^\bullet$ is a lattice containing Q° as an index 2 sublattice.

As noted in the introduction, all components of $\text{Zhu}_g(V)$ are h-invariant for each $h \in G$ and $\gamma = h$. The space $\mathcal{C}(g, h; |0\rangle)$ is spanned by $\text{STr}_M \, h q^{L_0^\delta - c/24}$ where M ranges over $P(g, V)$.

The bosonic part $S_-(\mathfrak{h})$ of the tensor product $S_-(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[Q]$ is purely even and

$$q^{-c/24} \operatorname{Tr}_{S_-(\mathfrak{h})} q^{L_0} = \frac{q^{-r/24}}{\phi(\tau)^r} = \frac{1}{\eta(\tau)^r}.$$

To determine the contribution of $\mathbb{C}_\epsilon[Q]$ we introduce

$$\Theta_{\delta,Q}(q) = \operatorname{Tr}_{\mathbb{C}_\epsilon[Q]} q^{L_0^\delta} = \sum_{\alpha\in Q} e^{\pi i\tau\langle\alpha+\delta,\alpha+\delta\rangle}$$
$$\text{and} \quad \Pi_{\delta,Q}(q) = \operatorname{STr}_{\mathbb{C}_\epsilon[Q]} q^{L_0^\delta} = \sum_{\alpha\in Q} e^{\pi i\tau\langle\alpha+\delta,\alpha+\delta\rangle} e^{\pi i\langle\alpha,\alpha\rangle}.$$

We see that

$$\mathcal{C}(g,\sigma_V;|0\rangle) \text{ is spanned by } \frac{\Theta_{\delta,Q}(\tau)}{\eta(\tau)^r}, \text{ and } \mathcal{C}(g,1;|0\rangle) \text{ is spanned by } \frac{\Pi_{\delta,Q}(\tau)}{\eta(\tau)^r},$$

where in each case δ ranges over Q°/Q (resp. $Q^\bullet/Q$) if $g=\sigma_V$ (resp. $g=1$).

The transformation

$$\sum_{\alpha\in Q} e^{-i\pi\langle\alpha,\alpha\rangle/\tau} = (\operatorname{disc} Q)^{-1/2}(-i\tau)^{r/2} \sum_{\beta\in Q^\circ} e^{i\pi\tau\langle\beta,\beta\rangle}$$

of the usual lattice theta function under $\tau \mapsto -1/\tau$ is proved using Poisson summation [13] (here $\operatorname{disc} Q$ is the discriminant, defined to be the determinant of the Gram matrix of an integral basis of Q). Using the same method, the $SL_2(\mathbb{Z})$ transformations of Θ and Π may be deduced. The results confirm the conclusion from Theorem 1.0.5, that $\mathcal{C}(1,1;|0\rangle)$ is a modular invariant family of weight 0, as is the direct sum of the other three spaces $\mathcal{C}(1,\sigma_V;|0\rangle) \oplus \mathcal{C}(\sigma_V,1;|0\rangle) \oplus \mathcal{C}(\sigma_V,\sigma_V;|0\rangle)$.

3.4 The $N = 1$ superconformal VOSA

The $N=1$ superconformal VOSAs, also known as Neveu-Schwarz VOSAs, are a sort of minimal supersymmetric generalization of the class of Virasoro VOAs. Recall the Neveu-Schwarz (NS) Lie superalgebra $\mathrm{NS} = (\oplus_{n\in\mathbb{Z}}\mathbb{C}L_n) \oplus (\oplus_{n\in 1/2+\mathbb{Z}}\mathbb{C}G_n) \oplus \mathbb{C}C$ as a vector superspace, here C and the L_n are even vectors and the G_n are odd. The commutation relations are

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C, \\ [G_m, L_n] &= (m-\frac{n}{2})G_{m+n}, \\ [G_m, G_n] &= 2L_{m+n} + \frac{1}{3}(m^2+\frac{1}{4})\delta_{m,-n}C, \end{aligned} \tag{3.4.1}$$

and C is central.

A nice summary reference for highest weight representations of NS is [24]. Let

NS_+ denote the span of vectors with strictly positive subscript, along with C and L_0. As usual we are interested in NS-modules of the form

$$M(c,h) = U(\mathrm{NS}) \otimes_{U(\mathrm{NS}_+)} S_{c,h}$$

where $S_{c,h} = \mathbb{C}v_{c,h}$ is the 1-dimensional NS_+-module on which C acts by c, L_0 acts by h and all positive 'modes' act trivially. We also denote the irreducible quotient of $M(c,h)$ by $L(c,h)$.

Inside $L(c,0)$ we identify the elements $\omega = L_{-2}v_{c,0}$ and $\tau = G_{-3/2}v_{c,0}$. The Neveu-Schwarz VOSA NS_c is defined to be $L(c,0)$ as a vector superspace, the vacuum vector is $|0\rangle = v_{c,0}$, Virasoro vector is ω, and the translation operator is L_{-1}. The state-field correspondence is defined by

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad Y(\tau,z) = \sum_{n\in 1/2+\mathbb{Z}} G_n z^{-n-3/2},$$

extended to all of V by the usual normal-ordering prescription. As suggested by the notation, L_m (resp. G_n) lowers conformal weight by m (resp. n).

The representation theory is quite similar to that of the Virasoro algebra; generically the Verma module $M(c,0)$ is irreducible and so all $L(c,h)$ acquire the structure of positive energy modules over NS_c. For certain values of c though, $L(c,0)$ is a nontrivial quotient of $M(c,0)$ and the irreducible positive energy NS_c-modules are finite in number and are all of the form $L(c,h)$. In fact, NS_c is a rational VOSA when

$$c = c_{p,p'} = \frac{3}{2}\left(1 - \frac{2(p'-p)^2}{pp'}\right)$$

for $p,p' \in \mathbb{Z}_{>0}$ with $p<p'$, $p'-p \in 2\mathbb{Z}$ and $\gcd(\frac{p'-p}{2}, p) = 1$. From now on when we write NS_c we implicitly mean NS_c for one of the special values of c above.

The irreducible positive energy NS_c-modules are $L(c,h)$ for the following values of h:

$$h = h_{r,s} = \frac{(rp'-sp)^2 - (p'-p)^2}{8pp'}$$

for $1 \leq r \leq p-1$ and $1 \leq s \leq p'-1$ with $r-s$ even. Since NS_c has vectors of half-integer conformal weight (precisely the odd vectors) the modules described above are actually, in our notation, σ-twisted where we write $\sigma = \sigma_{\mathrm{NS}_c}$. So it is more accurate to say that NS_c is σ-rational. In fact NS_c is also 1-rational and the 1-twisted modules, which are often referred to as 'Ramond twisted' modules, are easily described in terms of a variant on the Neveu-Schwarz Lie superalgebra called the Ramond superalgebra.

The Ramond Lie superalgebra is $\mathrm{R} = (\oplus_{n\in\mathbb{Z}}\mathbb{C}L_n) \oplus (\oplus_{n\in\mathbb{Z}}\mathbb{C}G_n) \oplus \mathbb{C}C$, with commutation relations formally identical to (3.4.1). We set R_+ to be the sum of the strictly positive modes along with C, L_0 and G_0. We define $M(c,h)$ and $L(c,h)$ as above, but with a slightly different definition of $S_{c,h}$. Notice that in R we have

$L_0 = G_0^2 - c/24$, so we should take

$$S_{c,h} = \begin{cases} \mathbb{C}v_{c,h} \oplus \mathbb{C}\theta_{c,h} \text{ with } G_0 v_{c,h} = \theta_{c,h} & \text{if } h \neq c/24 \\ \mathbb{C}v_{c,h} \text{ with } G_0 v_{c,h} = 0 & \text{if } h = c/24. \end{cases}$$

The irreducible 1-twisted positive energy NS_c-modules are precisely $L(c,h)$ where

$$h = h_{r,s} = \frac{(rp' - sp)^2 - (p' - p)^2}{8pp'} + \frac{1}{16}$$

for $1 \leq r \leq p-1$ and $1 \leq s \leq p'-1$ with $r - s$ odd.

The supercharacter of any of these 1-twisted modules is trivial, being either 0 or 1. The characters of the 1-twisted modules, along with the characters and supercharacters of the σ-twisted modules, are known for any given $c_{p,p'}$ and together form an $SL_2(\mathbb{Z})$-invariant space of weight 0.

In this section we consider the untwisted conformal blocks $\mathcal{C}_\tau(1,1)$ evaluated on the element τ of weight 3/2. As in Section 3.1, the other spaces of conformal blocks $\mathcal{C}_\tau(g,h)$ are zero, and so $\mathcal{C}_\tau(1,1)$ is $SL_2(\mathbb{Z})$-invariant of weight 3/2. As usual the first step is to compute the Zhu algebra $\mathrm{Zhu}_1(\mathrm{NS}_c)$.

The 1-twisted Zhu algebra of the corresponding universal Neveu-Schwarz VOSA is a supercommutative polymonial algebra,

$$\mathrm{Zhu}_1(M(c,0)) \cong \mathbb{C}[x,\xi]/(\xi^2 = x - c/24).$$

Here x is the image of ω and acts on a module as L_0 while ξ is the image of τ and acts as G_0.

Since NS_c is 1-rational, its Zhu algebra is a direct sum of simple supercommutative superalgebras. This forces it to be a direct sum of copies of $\mathbb{C}$ and the queer superalgebra $Q_1 \cong \mathbb{C}[\xi]/(\xi^2 = h - c/24)$. In fact the irreducible 1-twisted positive energy NS_c-module $L(c,h)$ corresponds to a copy of $\mathbb{C}$ in the Zhu algebra if $h = c/24$, and a copy of Q_1 if $h \neq c/24$. Only modules of the second type have nontrivial supertrace functions at half-integer level.

Let $M = L(c,h)$ be a 1-twisted module where $h \neq c/24$. We have $M_0 = S_{c,h} = \mathbb{C}v + \mathbb{C}\theta$, which carries an action of Q_1 where ξ sends 1 to θ and θ to $h - c/24$. The unique, up to a scalar, supersymmetric function on Q_1 is $x \mapsto \mathrm{Tr}_{S_{c,h}} u\xi$. The corresponding supertrace function on $L(c,h)$ is now

$$S_{L(c,h)}(u,\tau) = \mathrm{Tr}_{L(c,h)}\, u_0 \gamma q^{L_0 - c/24}$$

where γ is defined on a typical vector $\mathbf{m}x$ (for $\mathbf{m}$ some monomial in L_m and G_n, and $x = v$ or θ) by

$$\gamma : \mathbf{m}x \mapsto \mathbf{m}G_0 x.$$

Since we are dealing with the Neveu-Schwarz element $u = \tau$, the supertrace function is $\mathrm{Tr}_{L(c,h)}\, \varphi q^{L_0 - c/24}$ where

$$\varphi : \mathbf{m}x \mapsto G_0 \mathbf{m} G_0 x.$$

A case of particular interest is when $p = 2$, $p' = 8$ so that $c = c_{p,p'} = -21/4$. Then there are two 1-twisted modules $L(c, h)$, where $h = -3/32$ and where $h = -7/32$, the second of these satisfies $h = c/24$ and so has trivial supertrace function associated to τ. Consequently the supertrace function

$$F(\tau) = \mathrm{Tr}_{L(-21/4,-3/32)}\, \varphi q^{L_0 + 7/32}$$

is a modular form on $SL_2(\mathbb{Z})$ of weight $3/2$.

3.4.1 Computing this function explicitly

On the top level of $L(c, h)$, φ acts as $h - c/24$, so the top level contribution to $F(\tau)$ is $\frac{1}{4}q^{1/8}$. This is already enough information to determine $F(\tau)$ completely. The cube of the Dedekind eta function is $q^{1/8}$ times an ordinary power series in q, so the quotient $f(\tau) = F(\tau)/\eta(\tau)^3$ is a holomorphic modular form of weight 0 for $SL_2(\mathbb{Z})$, possibly with a multiplier system. Since the q-series of f has integer powers of q we have $f(T\tau) = f(\tau)$, and since $S^2 = 1$ we have only the possibilities $f(S\tau) = \pm f(\tau)$. But in $SL_2(\mathbb{Z})$ we have the relation $(ST)^3 = 1$, so if S acted by -1 on f we would have $f(\tau) = f(-T^3\tau) = -f(\tau)$. Hence $f(S\tau) = f(\tau)$ and, since it is a genuine holomorphic modular form on $SL_2(\mathbb{Z})$, we have $f(\tau) = 1$. Thus

$$F(\tau) = \frac{1}{4}\eta(\tau)^3.$$

It is interesting to try to compute $F(\tau)$ in a more direct fashion as well. As a first step let us study the action of φ on a general Verma module $M = M(c, h)$. On the top level $M_0 = \mathbb{C}1 + \mathbb{C}\theta$ we find φ acts by the scalar $h - c/24$, so the trace on this level is $2(h - c/24)$. We claim that on all other levels of M, the trace of φ vanishes.

Consider the action of φ on the monomial

$$\mathbf{m}x = L_{m_1} \cdots L_{m_s} G_{n_1} \cdots G_{n_t} x$$

where $m_1 \leq \ldots \leq m_s \leq -1$ and $n_1 < \ldots < n_t \leq -1$. We place a G_0 to the immediate left of x, and another at the left of the whole term which we move through $\mathbf{m}$ using the commutation relations of R. This process generates a complicated set of terms, but most of these monomials are *shorter* than the one we start with (which has length $s + t$) and so do not contribute to the trace of φ.

The first term to consider is the one where G_0 moves all the way to the end without interacting, so we get the starting monomial multiplied by $(-1)^t(h - c/24)$. We wish to compare how many monomials (of fixed weight $N = \sum_{i=1}^{s} m_i + \sum_{j=1}^{t} n_j$) have t even versus how many have t odd. Let us introduce a generating function $G(t) = 1 + \sum_{i=1}^{\infty} G_i t^i$ where G_N counts (with parity) the number of partitions of N

into pieces which may have even or odd parity. This generating function is clearly

$$G(t) = \prod_{i=1}^{\infty} \frac{1}{1-t^i} \prod_{j=1}^{\infty} (1-t^j) = 1.$$

So equally many monomials have even t as have odd t. The next simplest terms that arise in the course of PBW ordering $\varphi(\mathbf{m}x)$ are ones in which $G_0 L_n$ is replaced by G_n or $G_0 G_n$ is replaced by L_n. These terms cannot equal a multiple of $\mathbf{m}x$ because the number of Ls and Gs has changed, so these terms do not contribute to the trace. All terms of lower order are shorter than $\mathbf{m}x$ itself and do not contribute to the trace either. Thus we have

$$\mathrm{Tr}_{M(c,h)}\, \varphi q^{L_0 - c/24} = 2(h - c/24) q^{h-c/24}.$$

Since only the top level of a Verma module contributes to $\mathrm{Tr}\,\varphi$, the trace of φ over $L = L(c,h)$ has nonzero q-coefficients only in levels of L in which there are singular vectors.

Now suppose again that $c = c_{p,p'}$ and h is one of the special values above. It is known [19] that the maximal R-submodule of $M(c,h)$ is generated by two top pieces S_{c,h_1} and S_{c,h_2} with known levels h_1 and h_2. These give rise to maps from $M(c,h_1)$ and $M(c,h_2)$ into $M(c,h)$ and the intersection of their images contains another two top pieces. This continues ad infinitum. This structural knowledge allows one to write the (super)character of $L(c,h)$ as an alternating sum of Verma module characters (see [24] for explicit formulas). Using a similar trick to directly evaluate the trace of φ seems to require more information about singular vectors than just their levels, and the details become very complicated. On the other hand, it was known to Jacobi that

$$\eta(\tau)^3 = q^{1/8} \sum_{n=-\infty}^{\infty} (4n+1) q^{n(2n+1)}$$

(this can be derived from the Jacobi triple product identity). The powers of q appearing here with nonzero coefficient match the levels of singular vectors in the Verma module $M(-21/4, -3/32)$, as they should.

3.5 Affine Kac-Moody algebras at admissible level

Recall the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}, (\cdot,\cdot)$ in the introduction. Let $\Lambda = k\Lambda_0$ where $\Lambda_0 \in \mathfrak{h}^*$ is defined by $\Lambda_0(\alpha_i^\vee) = \delta_{i,0}$. It is well known that the Verma module $M(k\Lambda_0)$ can each be given the structure of a vertex algebra [12] [17], which we denote $V^k(\mathfrak{g})$. Further, $V^k(\mathfrak{g})$ has the structure of a VOA if the level $k \neq -h^\vee$. The vacuum vector $|0\rangle$ is the highest weight vector. The state-field correspondence is defined by

$$Y(a,w) = \sum_{n\in\mathbb{Z}} a_n w^{-n-1}$$

for $a \in \mathfrak{g}$, and is extended to all $v \in V$ via the n^{th} products

$$\begin{aligned} Y(a_{(n)}v, w) &= Y(a,w)_{(n)} Y(v,w) \\ &= \mathrm{Res}_z \left[Y(a,z)Y(v,w) i_{z,w}(z-w)^n - p(u,v) Y(v,w) Y(a,z) i_{w,z}(z-w)^n \right]. \end{aligned}$$

The translation operator T is uniquely defined by $T|0\rangle = 0$ together with $[T, u_{(n)}] = -nu_{(n-1)}$. Finally the Virasoro vector is

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} a^i_{(-1)} b^i_{(-1)} |0\rangle,$$

where $\{a^i\}$ and $\{b^i\}$ are a pair of bases of $\mathfrak{g}$ dual with respect to $(\cdot,\cdot)$. The irreducible quotient $L(k\Lambda_0)$ is also a VOA, denoted $V_k(\mathfrak{g})$. Now $\mathfrak{g}$ naturally embeds into $V_k(\mathfrak{g})$ via the mapping $a \mapsto a_{(-1)}|0\rangle$. The elements of $\mathfrak{g}$ in $V_k(\mathfrak{g})$ are all of conformal weight 1. The central charge is

$$c = \frac{k \dim \mathfrak{g}}{k+h^\vee}.$$

For $V = V^k(\mathfrak{g})$ the quotient $V/C_2(V)$ has basis given by monomials

$$a^1_{(-1)} \cdots a^s_{(-1)} |0\rangle$$

for $a^i \in \mathfrak{g}$, so V is not C_2-cofinite. In [29] it is noted that if $k \in \mathbb{Z}_+$ then the irreducible quotient $V_k(\mathfrak{g})$ is C_2-cofinite. It follows from standard structure theory of integrable $\hat{\mathfrak{g}}$-modules [16] that if $k \in \mathbb{Z}_+$ then $L(k\Lambda_0)$ is the quotient of $M(k\Lambda_0)$ by the $\hat{\mathfrak{g}}$-submodule generated by the singular vector $(e_\theta)^{k+1}_{(-1)}|0\rangle$. All vectors $a^{k+1}_{(-1)}|0\rangle$ for $a \in \mathfrak{g}$ are seen to lie in the maximal $\hat{\mathfrak{g}}$-submodule, and C_2-cofiniteness follows.

If $k \in \mathbb{Z}_+$ then $V_k(\mathfrak{g})$ is also rational [12]. The finitely many irreducible positive energy $V_k(\mathfrak{g})$-modules are precisely the integrable highest weight $\hat{\mathfrak{g}}$-modules of level k, i.e., $L(\Lambda)$ for $\Lambda \in \mathfrak{h}^*$ dominant integral with level k. The action of

$$u = a^1_{(n_1)} \cdots a^s_{(n_s)} |0\rangle$$

on $M = L(\Lambda)$ is through the product of generating fields

$$Y^M(a^1, z)_{(n_1)} \left(Y^M(a^2, z)_{(n_2)} \cdots Y^M(a^s, z) \right),$$

where $Y^M(a,z) = \sum_n at^n|_M z^{-n-1}$.

Recall the element $\delta \in \hat{\mathfrak{h}}$ satisfying $[\delta, at^m] = -mat^m$. Since $[L_0, at^m] = -mat^m$ too the actions of L_0 and δ on $L(\Lambda)$ agree up to an overall additive scalar. Interestingly though, the action of L_0 is fixed unambiguously while that of δ could be adjusted by an additive scalar. So L_0 provides a sort of intrinsically defined replacement for δ with a preferred normalization. In particular it may be verified (using the Borcherds identity for example) that L_0 acts on the singular vector of $L(\Lambda)$ as the scalar

$$\frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}$$

(where λ is the projection along δ of Λ to $\mathfrak{h}$). Thus

$$q^{s_\Lambda} \widetilde{\mathrm{Ch}}_{L(\Lambda)}(q) = \mathrm{Tr}_{L(\Lambda)}\, q^{L_0 - c/24},$$

and Zhu's theorem implies the $SL_2(\mathbb{Z})$-invariance of normalized $\hat{\mathfrak{g}}$-module characters first proved by Kac and Peterson. It also (partially) explains the strange factor s_Λ required for modularity.

3.5.1 The case of non integer level k

The structure of $V_k(\mathfrak{g})$ depends a lot on the level k. For $k \in \mathbb{Z}_+$ as we have seen $V_k(\mathfrak{g})$ is much smaller that $V^k(\mathfrak{g})$, having only finitely many irreducible modules. For irrational k we have $V_k(\mathfrak{g}) = V^k(\mathfrak{g})$. In between there is the case of 'admissible level' for which the structure of $V_k(\mathfrak{g})$ becomes rather complicated.

A weight $\Lambda \in \hat{\mathfrak{h}}^*$ is said to be admissible if

- $\langle \Lambda + \rho, \alpha_i^\vee \rangle \notin \{0, -1, -2, \ldots\}$ for $i = 0, 1, \ldots r$.
- The set $\Delta_\Lambda^\vee$ of positive real coroots $\alpha^\vee$ satisfying $\langle \Lambda, \alpha^\vee \rangle \in \mathbb{Z}$, spans $\mathbb{Q}\Delta^\vee$.

The number $k \in \mathbb{Q}$ is said to be admissible if $k\Lambda_0$ is an admissible weight. There are finitely many admissible weights of a given level. Kac and Wakimoto [20] studied highest weight modules with admissible highest weight (more generally over an arbitrary symmetrizable Kac-Moody algebra). They extended many results from the integrable case, in particular the Weyl-Kac character formula.

The connection with Zhu's theory is a little obscure because for non integer k, $V_k(\mathfrak{g})$ is neither C_2-cofinite nor rational. We consider the following modification though: For $x \in \mathfrak{h}$ let $\omega^x = \omega + h_{(-2)}|0\rangle$. It may be verified that ω^x is a Virasoro vector, but with central charge

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee} - 12k(x, x).$$

Denote the corresponding VOA $V_k(x, \mathfrak{g})$. In $V_k(x, \mathfrak{g})$ the energy operator is $L_0^x = L_0 - x_0$, so the conformal weights of elements of $\mathfrak{g}$ are no longer all 1 and may be very much non integer. Suppose x lies in the positive cone of $\mathfrak{h}^*$. Since the operators e_0 for $e \in \mathfrak{n}_+$ are grade-lowering, the positive energy condition on a $V_k(x, \mathfrak{g})$-module M implies that it lies in the category $\mathcal{O}$ when regarded as a $\hat{\mathfrak{g}}$-module. This drastically reduces the number of positive energy modules relative to the $x = 0$ case.

Indeed it is conjectured [1] that for admissible level k, the irreducible $V_k(\mathfrak{g})$-modules which lie in category $\mathcal{O}$ are precisely $L(\Lambda)$ for Λ admissible of level k. This conjecture has been verified in the following cases:

- $\mathfrak{g} = \mathfrak{sl}_2$ for all admissible levels [1] [5]. Furthermore both sets of authors showed that $V_k(\mathfrak{g})$ is 'rational in the category $\mathcal{O}$'.
- $\mathfrak{g} = \mathfrak{sl}_{2m+1}$ for $m \in \mathbb{Z}_{>0}$ and $k = -m - 1$ [26].
- $\mathfrak{g} = \mathfrak{sl}_3$ and $k = -1/2$, $-5/3$, $-12/5$ or $-18/7$ by the present author.

3.5.2 Digression on Lie conformal superalgebras

To go any further we need to pause to introduce the theory of Lie conformal superalgebras. This material is used in other parts of the thesis also. A complete introduction can be found in the book [17]. A good concise overview is given in Section 1.7 of [4].

Definition 3.5.1. A *Lie conformal superalgebra* $(R, T, [\cdot_\lambda \cdot])$ is a $\mathbb{C}[T]$-module R, endowed with a λ-*bracket*, $R \otimes R \to R \otimes \mathbb{C}[\lambda]$ (denoted $u \otimes v \mapsto [u_\lambda v]$) for which the following axioms hold:

- Sesquilinearity axiom, $[(Tu)_\lambda v] = -\lambda[u_\lambda v]$ and $[u_\lambda (Tv)] = (T + \lambda)[u_\lambda v]$.
- Skew-commutativity axiom, $[v_\lambda u] = -p(u, v)[u_{-T-\lambda} v]$.
- Jacobi Identity, $[u_\lambda [v_\mu x]] = [[u_\lambda v]_{\lambda+\mu} x] + p(u, v)[v_\mu [u_\lambda x]]$.

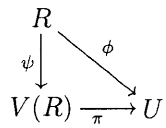
We write $[u_\lambda v] = \sum_{j \in \mathbb{Z}_+} \lambda^{(j)} u_{(j)} v$ where $u_{(j)} v \in R$. A homomorphism of Lie conformal algebras is a homomorphism of $\mathbb{C}[T]$-modules that sends λ-brackets to λ-brackets.

Example 3.5.1. The *Virasoro* Lie conformal algebra is $\text{Vir} = \mathbb{C}[T]L \oplus \mathbb{C}C$ with $TC = 0$. It suffices to define the λ-brackets between L and C and then all further λ-brackets are determined by the sesquilinearity axiom; $[L_\lambda L] = (T + 2\lambda)L + \lambda^3 C/12$ and C is central. We have $L_{(0)} L = TL$, $L_{(1)} L = 2L$, $L_{(3)} L = C/2$ and all other $L_{(n)} L$ vanish.

Example 3.5.2. Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra with invariant supersymmetric bilinear form $(\cdot, \cdot)$. The *Current* Lie conformal algebra $\text{Cur}(\mathfrak{g})$ is $\text{Cur}(\mathfrak{g}) = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}K$ with $TK = 0$ (the elements $a \in \mathfrak{g} \subseteq \text{Cur}(\mathfrak{g})$ are referred to as currents). The λ-brackets between currents are $[a_\lambda b] = [a, b] + \lambda(a, b)K$, the element K is central. We have $a_{(0)} b = [a, b]$ and $a_{(1)} b = (a, b)K$.

Any vertex algebra can be given the structure of a Lie conformal algebra by defining the λ-bracket $[u_\lambda v] = \sum_{j \in \mathbb{Z}_+} \lambda^{(j)} u_{(j)} v$. One should keep in mind the following analogy: Vertex algebras are to Lie conformal algebras as associative algebras are to Lie algebras.

For a Lie conformal algebra $(R, T, [\cdot_\lambda \cdot])$, an enveloping vertex algebra is a pair (U, ϕ) where U is a vertex algebra and $\phi : R \to U$ is a homomorphism of Lie conformal algebras. The universal enveloping vertex algebra $(V(R), \psi)$ is an enveloping vertex algebra such that for any enveloping vertex algebra (U, ϕ) there is a unique vertex algebra homomorphism $\pi : V(R) \to U$ such that the following diagram commutes



As a vector space $V(R) = U(\mathrm{Lie}\, R)/U(\mathrm{Lie}\, R)(\mathrm{Lie}\, R)_-$, where

$$\begin{aligned} (\mathrm{Lie}\, R)_- &= \langle u_{(n)} | u \in R, n \geq 0 \rangle \\ \text{and} \quad (\mathrm{Lie}\, R)_+ &= \langle u_{(n)} | u \in R, n < 0 \rangle, \end{aligned}$$

and the map $R \to V(R)$ is $u \mapsto u_{(-1)}$.

Recall the definition of $\mathrm{Lie}\, V$ (recall $g = 1$ in this section) following Lemma 5.0.3. It turns out that it uses only that V is a Lie conformal algebra with a diagonal operator H. Hence we may define $\mathrm{Lie}\, R$ for an arbitrary H-graded Lie conformal algebra by replacing all occurrences of V in the definition with R.

We have that $V(\mathrm{Vir})/(C = c|0\rangle) = \mathrm{Vir}^c$ is the Virasoro vertex algebra of level c, and that $\mathrm{Lie}\,\mathrm{Vir}$ is the Virasoro Lie algebra (which is also given the symbol Vir). Similarly $V(\mathrm{Cur}\, \mathfrak{g})/(K = k|0\rangle) = V^k(\mathfrak{g})$, and $\mathrm{Lie}\,\mathrm{Cur}\, \mathfrak{g} = \hat{\mathfrak{g}}$, the affine Kac-Moody algebra.

Recall that a *restricted* $\mathrm{Lie}\, R$-module is a $\mathrm{Lie}\, R$-module M such that for each $u \in R$ and $v \in M$ we have $u_{(n)}v = 0$ for $n \gg 0$.

Definition 3.5.2. A *homomorphism* of Lie conformal superalgebras is a morphism ϕ of $\mathbb{C}[T]$-modules such that

$$\phi([a_\lambda b]) = [\phi(a)_\lambda \phi(b)].$$

Isomorphism and automorphism are defined as expected.

Let R be an LCSA with a finite order automorphism g. Then R splits into g-eigenspaces R^ϵ where the ϵ are roots of unity. Associated to (R, g) is the Lie algebra $\mathrm{Lie}_g(R)$ defined as follows:

Definition 3.5.3. Let $(R, T, [\cdot_\lambda \cdot])$ be a Lie conformal algebra with a finite order automorphism g. Let

$$Q = \bigoplus_{\epsilon \in \mathbb{R}/\mathbb{Z}} t^\epsilon R^{[\epsilon]} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}],$$

(where ϵ is any representative of the coset $[\epsilon]$ and $R^{[\epsilon]} \subseteq R$ is the $e^{2\pi i \epsilon}$-eigenspace of g). Define $\mathrm{Lie}^g(R)$ to be the quotient

$$Q/(T + H + \partial_t)Q$$

with the Lie bracket

$$[u_m, v_k] = \sum_{j\in\mathbb{Z}_+} \binom{m+\Delta_u - 1}{j} (u_{(j)}v)_{m+k}, \tag{3.5.1}$$

where u_n denotes the image of ut^n in the quotient. We also use the notation $u_{(n)} = u_{n-\Delta_u+1}$ for $n \in [\gamma_u] = [\Delta_u] + [\epsilon_u]$. In terms of this indexing the commutation relation is

$$[u_{(m)}, v_{(k)}] = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (u_{(j)}v)_{(m+k-j)}. \tag{3.5.2}$$

Theorem 3.5.1. *There is a natural bijection between the set of all g-twisted (not necessarily positive energy) $V(R)$-modules and the set of restricted* $\mathrm{Lie}_g(R)$*-module.*

Proof. Let (M, Y^M) be a g-twisted $V(R)$-module. Recall the Borcherds identity: for all $u, v \in V$, $x \in M$, $n \in \mathbb{Z}$, $m \in [\epsilon_u]$, and $k \in [\epsilon_v]$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+} &\binom{m+\Delta_u-1}{j} (u_{(n+j)}v)^M_{m+k}x \\ &= \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \left[u^M_{m+n-j}v^M_{k+j-n} - p(u,v)(-1)^n v^M_{k-j}u^M_{m+j}\right] x.\end{aligned}$$

The special case of the commutator formula

$$[u^M_m, v^M_k] = \sum_{j\in\mathbb{Z}_+} \binom{m+\Delta_u-1}{j} (u_{(j)}v)^M_{m+k}$$

for $u, v \in R$ is just the Lie bracket in $\mathrm{Lie}^g(R)$. Since $(T + H + \partial_t)ut^m$ acts trivially on M as well, M is a $\mathrm{Lie}_g(V)$-module. Because each $Y^M(u,z)$ is a quantum field, M is a *restricted* $\mathrm{Lie}_g(R)$-module.

For the converse, let M be a restricted $\mathrm{Lie}^g(R)$-module. For $u \in R$ let

$$u^M(z) = \sum_{n\in[\epsilon_u]} u^M_n z^{-n-\Delta_u} = \sum_{n\in[\gamma_u]} u^M_{(n)} z^{-n-1}$$

which is an $\mathrm{End}\, M$-valued quantum field. For a coset $[\gamma] \in \mathbb{R}/\mathbb{Z}$ let us define a γ-shifted delta function

$$\delta_{[\gamma]}(z,w) = (w/z)^\gamma \delta(z,w),$$

which does not depend on the choice γ of representative of $[\gamma]$. Then we may write the commutation relation (3.5.2) as

$$[u^M(z), v^M(w)] = \sum_{j\in\mathbb{Z}_+} [u_{(j)}v]^M(w) \partial_w^{(j)} \delta_{[\gamma_u]}(z,w).$$

As for the usual delta function we have

$$(z-w)\partial_w^{(j)}\delta_{[\gamma]}(z,w) = \partial_w^{(j-1)}\delta_{[\gamma]}(z,w)$$

and so the fields we just introduced are local in the usual sense

$$(z-w)^N[u^M(z), v^M(w)] = 0$$

for some $N \gg 0$.

We now define a vertex operation $X(\cdot, z)$ taking vectors in $V(R)$ to quantum fields on M. We start with $X(u,z) = u^M(z)$ for $u \in R$, which is a quantum field by restrictedness of the $\mathrm{Lie}^g(R)$ action on M. Now let $u \in R$ and $v \in V(R)$, we associate to $u_{(n)}v$ the *twisted n^{th} product* field $X(u,w)_{(n,g)}X(v,w)$ defined to be

$$\mathrm{Res}_z(z/w)^{\gamma_u}\left[X(u,z)X(v,w)i_{z,w} - X(v,w)X(u,z)i_{w,z}\right](z-w)^n.$$

where $\gamma_u \in [\gamma_u]$ has been selected arbitrarily. By Proposition 3.9 of [23] the collection $\mathcal{F}$ of fields so defined consists of pairwise local quantum fields. Furthermore (Theorem 2.5, [23].) $\mathcal{F}$ is a vertex superalgebra when equipped with the following state-field correspondence:

$$X(v,z) \mapsto Y(X(v,z),w) = \sum_{n\in\mathbb{Z}} w^{-n-1}X(v,z)_{(n,g)},$$

and (Theorem 3.14, [23]) if we define $Y^M(X(v,z),w)$ to be $v(w)$ then (M, Y^M) is a g-twisted $\mathcal{F}$-module.

Since $R \to \mathcal{F}$ defined by $u \mapsto u^M(z)$ is a homomorphism of LCSAs we obtain a vertex superalgebra homomorphism $V(R) \to \mathcal{F}$, and M becomes a g-twisted $V(R)$-module. $\square$

3.5.3 Admissible level module characters

We now fix a positive rational multiple x of $h \in \mathfrak{h}$ and introduce the automorphism $g_0 = e^{-2\pi i L_0^x}$ which acts as

$$g_0 \cdot e = \alpha_0 e, \quad g_0 \cdot f = \alpha_0^{-1} f \quad \text{and} \quad g_0 \cdot h = h,$$

where $\alpha_0 = e^{2\pi i(x,e)}$. Let G be the (finite) group of automorphisms generated by g_0. In the same way we regard G as acting on the associated LCA $R = \mathrm{Cur}\,\mathfrak{g}$.

Let $g \in G$ and write δ for the smallest nonnegative number such that $e^{2\pi i\delta}e = ge$. The Lie algebra $\mathrm{Lie}^g(R)$ is spanned by K along with the modes $e_{r+\delta}$, $f_{r-\delta}$ and h_r for all $r \in \mathbb{Z}$. The nontrivial commutation relations are

$$[e_{m+\delta)}, f_{n-\delta}] = h_{m+n} + (m + \delta - (x,e))\delta_{m,-n}K.$$

In fact $\hat{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Lie}^g(R)$ via the isomorphism

$$et^m \mapsto e_{m+\delta}, \quad ft^m \mapsto f_{m-\delta}, \quad K \mapsto K,$$
$$ht^m \mapsto h_m \quad \text{for } m \neq 0, \quad h \mapsto h_0 + [\delta - (x,e)]K.$$

So by Theorem 3.5.1, the set of g-twisted $V^k(x, \mathfrak{g})$-modules is naturally identified with the set of restricted $\hat{\mathfrak{g}}$-modules. By our earlier remarks $\mathsf{PEMod}(g, V^k(x, \mathfrak{g}))$ (for $g \neq 1$) is identified with the set of $\hat{\mathfrak{g}}$-modules in category $\mathcal{O}$. Finally the set of irreducible modules in $\mathsf{PEMod}(g, V_k(x, \mathfrak{g}))$ (for $g \neq 1$) is identified with the set of highest weight modules $L(\Lambda)$ for Λ an admissible weight of level k. Since $V_k(\mathfrak{g})$ is rational in the category $\mathcal{O}$, $V_k(x, \mathfrak{g})$ is g-rational for each $g \neq 1$.

Before we can apply Theorem 1.0.5 we need to establish C_2-cofiniteness. $V_k(x, \mathfrak{g})$ is not C_2-cofinite, but it was established in [5] that it is weakly C_2-cofinite in the sense of Remark 2.3.2. The idea is straightforward. The subspace $C^{(g,h)}$ of the universal VOA $V^k(x, \mathfrak{g})$ is spanned by the following elements:

$$\begin{aligned} &u_{(-2)}v \quad \text{for } u \text{ fixed by } g \text{ and } h, \\ &u_{(-1)}v \quad \text{for } u \text{ not fixed by either } g \text{ or } h. \end{aligned}$$

$V^k(x, \mathfrak{g})$ has a basis consisting of monomials

$$e_{(i_1)} \cdots e_{(i_\alpha)} f_{(j_1)} \cdots f_{(j_\beta)} h_{(k_1)} \cdots h_{(k_\gamma)} |0\rangle, \tag{3.5.3}$$

where $i_1 \leq \ldots i_\alpha \leq -1$, $j_1 \leq \ldots j_\beta \leq -1$ and $k_1 \leq \ldots k_\gamma \leq -1$. If any $e_{(n)}$ or $f_{(n)}$ terms appear, or $h_{(n)}$ with $n \leq -2$ then the monomial lies in $C^{(g,h)}$. This means the quotient $V^k(x, \mathfrak{g})/C^{(g,h)}$ is spanned by polynomials in $h_{(-1)}$. Therefore the simple quotient $V_k(x, \mathfrak{g})$ is weakly C_2-cofinite whenever it can be established that there exists a vector $v \in V^k(x, \mathfrak{g}) = M(k\Lambda_0)$ lying in the maximal $\hat{\mathfrak{g}}$-submodule and with nonzero projection to $\mathbb{C}[h_{(-1)}] \subseteq V^k(\mathfrak{g})$. This can be verified explicitly given information about singular vectors.

Corollary 3.5.2 (To Theorem 1.0.5). *Let k be an admissible level for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}_2}$. Fix x a positive rational multiple of $H \in \mathfrak{sl}_2$ and define G as above. For $g \neq 1$ and $h \in G$, the space $\mathcal{C}(g, h, |0\rangle)$ of conformal blocks associated to $V_k(x, \mathfrak{g})$ is spanned by the trace functions*

$$\mathrm{Tr}_{L(\Lambda)} \gamma q^{L_0 - c/24},$$

for the admissible level k weights Λ.

We can write down γ explicitly as

$$\gamma : a^1_{n_1} \cdots a^s_{n_s} v_\Lambda \mapsto h^{-1}(a^1)_{n_1} \cdots h^{-1}(a^s)_{n_s} v_\Lambda$$

which we see coincides with $e^{2\pi i x^*}$ for some element $x^* \in \mathfrak{h}^*$ depending on g. Therefore the characters considered in the corollary are normalized specializations of the characters considered by Kac and Wakimoto.

Chapter 4

A generalization to non rational VOSAs

The key idea in Theorem 1.0.5 (as in Zhu's theorem) is to link the spaces $\mathcal{C}(g,h)$ of conformal blocks with supertrace functions on V-modules. Invariance under $SL_2(\mathbb{Z})$ of the latter then follows from invariance of the former, which is straightforward to prove. Ultimately the connection is established by showing that any element of $\mathcal{C}(g,h)$ gives rise to a linear function on $\mathrm{Zhu}_g(V)$ that is h-supersymmetric (Section 2.5). These functions on $\mathrm{Zhu}_g(V)$ can be constructed explicitly in terms of supertraces on $\mathrm{Zhu}_g(V)$-modules because the superalgebra is semisimple. Finally Zhu's induction functor lifts these supertraces to supertrace functions on the associated V-modules, which are conformal blocks.

Besides semisimplicity of the Zhu algebras, the other essential condition is C_2-cofiniteness. In particular this implies the spaces $\mathcal{C}(g,h)$ of conformal blocks are finite dimensional. Note that semisimplicity of the Zhu algebras is not required to prove finite dimensionality of $\mathcal{C}(g,h)$, nor modular invariance. Therefore it is natural to ask for a description of conformal blocks in the general non rational case. This issue was first investigated by Miyamoto [25]; he introduced 'pseudotrace functions' associated to (possibly reducible) positive energy V-modules, which span $\mathcal{C}$. Interestingly, these functions do not have q-series, they involve the logarithm τ as well. Miyamoto's construction is quite involved, and utilizes the higher level Zhu algebras which we have not yet introduced.

In this chapter we describe a different approach, based on the same ideas, requiring only the ordinary Zhu algebra. The drawback is that it only works for a fairly restricted class of examples (or more accurately it is difficult to verify the conditions required in general). Nevertheless we present the nontrivial example of the free superboson VOSA, in which we see the appearance of logarithmic modular forms.

4.1 The abstract supertrace construction

Let A be an associative unital superalgebra and let P be a finitely generated projective left A-module. There is a well known 'abstract trace' map

$$\mathrm{Tr}_P : \mathrm{End}_A P \longrightarrow A/[A, A],$$

where $[A, A]$ is the span of all commutators $ab - ba$ in A. The trace has the usual property $\mathrm{Tr}_P(fg) = \mathrm{Tr}_P(gf)$ for all $f, g \in \mathrm{End}_A P$.

Suppose more generally that A is a superalgebra with a finitely generated projective module P, and that A carries an automorphism h. Denote by $P_{(h)}$ the module P with a 'twisted' action of A defined by

$$(a, x) \mapsto h^{-1}(a)x.$$

In this section we define a supertrace operation

$$\mathrm{STr}_P : \mathrm{Hom}_A(P, P_{(h)}) \longrightarrow A/[A, A]_h,$$

where $[A, A]_h$ which is the span in A of all elements of the form $ab - p(a, b)h^{-1}(b)a$ for $a, b \in A$. Elements of $\mathrm{Hom}_A(P, P_{(h)})$ are just maps $\gamma : P \longrightarrow P$ satisfying $\gamma^{-1}a\gamma x = h(a)x$ for all $x \in P$, $a \in A$. The supertrace has the property $\mathrm{STr}_P(fg) = p(f, g)\,\mathrm{STr}_P(gf)$ for all $f \in \mathrm{End}_A P$ $g \in \mathrm{Hom}_A(P, P_{(h)})$.

Before proceeding let us settle some notation and conventions about superalgebras. A homomorphism $f : M \longrightarrow N$ of left A-modules is a homogeneous (i.e., either even or odd) linear function satisfying

$$f(ax) = (-1)^{p(a,f)} af(x).$$

Similarly a map of right A-modules is homogeneous $f : M \longrightarrow N$ such that

$$f(xa) = (-1)^{p(a,f)} f(x)a.$$

The opposite superalgebra A^{op} is defined to be A as a vector superspace endowed with the product $a * b := (-1)^{p(a,b)}ba$. A supercommutative algebra is its own opposite. Given a left A-module M we may define $m*a = (-1)^{p(a,m)}am$, whereupon M becomes a right A^{op}-module.

If M is a left A-module then its dual

$$M^* = \mathrm{Hom}_A(M, A) = \{\phi : M \longrightarrow A | \phi(ax) = (-1)^{p(a,\phi)} a\phi(x)\}$$

is a right A-module with the action

$$[\phi a](x) = (-1)^{p(a,x)} \phi(x)a.$$

If M is a left A-module, and N a left A^{op}-module (i.e., a right A-module), then

$N \otimes_A M$ is simply $N \otimes_{\mathbb{C}} M$ modulo terms of the form

$$yb \otimes x - y \otimes bx.$$

The natural 'swap operation' on tensor products of vector superspaces M, N is

$$\begin{aligned} \mathrm{swap} : M \otimes N &\to N \otimes M, \\ x \otimes y &\mapsto p(x,y) y \otimes x. \end{aligned}$$

Lemma 4.1.1. *Suppose the superalgebra A carries an automorphism h and let M be a left A-module. There is a well-defined map $S : M^* \otimes_A M_{(h)} \to A/[A,A]_h$, where by definition*

$$[A,A]_h = \{ab - p(a,b)h^{-1}(b)a | a, b \in A\}.$$

The map is given by $\phi \otimes x \mapsto \phi(x)$, using the natural identification of $M_{(h)}$ with M itself.

Proof. As a map from $M^* \otimes_{\mathbb{C}} M_{(h)}$ to A, S is well-defined. But when tensored over A we have $\phi \otimes ax = \phi a \otimes x$ and the images of these two in A are $\phi(h^{-1}(a)x) = (-1)^{p(a,\phi)} h^{-1}(a)\phi(x)$, and $[\phi a](x) = (-1)^{p(a,x)}\phi(x)a$. But these are equal in the quotient $A/[A,A]_h$, so the map is well-defined. □

Lemma 4.1.2 (Coordinate systems). *Let P be a finitely generated projective left A-module. Then there exist subsets $\{e_i\}_{i=1}^n \subset P$ and $\{e_i^*\}_{i=1}^n \subset P^*$ such that for all $x \in P$,*

$$x = \sum_{i=1}^{n} (-1)^{p(e_i^*(x), e_i)} e_i^*(x) e_i.$$

Proof. We can realize P as a direct summand of $F = A^n$ for some n because P is projective and finitely generated. The injection $\iota : P \to F$ and projection $\pi : F \to P$ are even A-module maps.

Let e_i be the image of the generator $1_i \in A^{(i)} \subset F$ under π. Some of the 1_i may be odd, and e_i has the same parity as 1_i.

Let $1_i^* : F \to A$ be the A-module map sending 1_i to 1 and all other 1_j to 0. Clearly 1_i^* has the same parity as 1_i. Compose with the inclusion ι to obtain $e_i^* : P \to A$ (which has the same parity as e_i).

Now $1_i^*(a1_i) = (-1)^{p(1_i,a)}a$. So under the natural identification of $A1_i$ with A, the map 1_i^* is the identity or σ_A according to whether 1_i is even or odd. Let $x = \sum_i a_i 1_i$, then

$$\sum_i (-1)^{p(1_i^*(x),1_i)} 1_i^*(x) 1_i = \sum_i (-1)^{p(1_i^*(x),1_i)} (-1)^{p(1_i,a_i)} a_i 1_i = \sum_i a_i 1_i = x.$$

Hence for $x \in P$

$$(-1)^{p(e_i^*(x),e_i)}e_i^*(x)e_i = \sum_i (-1)^{p(e_i^*(x),e_i)}1_i^*(\iota x)e_i$$

$$= \pi\left[\sum_i (-1)^{p(1_i^*(x),1_i)}1_i^*(\iota x)1_i\right] = \pi(\iota(x)) = x.$$

□

Lemma 4.1.3. *Let P, Q be two finitely generated projective left A-modules. There is a vector superspace isomorphism $\theta : P^* \otimes_A Q \to \mathrm{Hom}_A(P,Q)$ defined by*

$$[\theta(\phi \otimes x)](y) = (-1)^{p(x,y)}\phi(y)x.$$

Proof. The map θ exists for arbitrary P, Q. In the present case we can choose a coordinate system for P as above and use it to construct an inverse to θ. Define α sending $f \in \mathrm{Hom}_A(P,Q)$ to $\sum_i (-1)^{p(e_i^*,f(e_i))}e_i^* \otimes f(e_i) \in P^* \otimes_A Q$. Then $\theta \circ \alpha = \mathrm{id}$ because

$$[\theta(\alpha(f))](y) = \sum_i (-1)^{p(e_i^*,f(e_i))}\theta(e_i^* \otimes f(e_i))(y)$$

$$= \sum_i (-1)^{p(e_i^*,f(e_i))}(-1)^{p(f(e_i),y)}e_i^*(y)f(e_i)$$

$$= \sum_i (-1)^{p(e_i^*,f(e_i))}(-1)^{p(f(e_i),y)}(-1)^{p(f,e_i^*(y))}f(e_i^*(y)e_i)$$

$$= \sum_i (-1)^{p(e_i^*(y),e_i)}f(e_i^*(y)e_i) = f(y).$$

We just need to check α is surjective. It is because the images of the basic morphisms $e_i \mapsto e_j$ are $e_i \otimes e_j^*$ which clearly span. □

Definition 4.1.1. Let P a finitely generated projective left A-module and let h be an automorphism of A. The abstract supertrace $\mathrm{STr}_P : \mathrm{Hom}_A(P, P_{(h)}) \to A/[A,A]_h$ is defined as the composition

$$\mathrm{STr}_P = S \circ \theta^{-1}.$$

Unwinding the definition yields

$$\mathrm{STr}_P : f \mapsto \sum_i (-1)^{p(e_i^*,f(e_i))}e_i^* \otimes f(e_i) \mapsto \sum_i (-1)^{p(e_i^*,f(e_i))}e_i^*(f(e_i)).$$

Lemma 4.1.4. *The supertrace is supersymmetric in the sense that if $f \in \mathrm{End}_A P$ and $g \in \mathrm{Hom}_A(P, P_{(h)})$ then $\mathrm{STr}_P(fg) - p(f,g)\,\mathrm{STr}_P(gf) \in A/[A,A]_h$.*

The endomorphism f here appears both as an element of $\mathrm{End}_A P$ and as an element of $\mathrm{End}_A(P_{(h)})$; the identification between these sets is just the identity.

Proof. Pick a coordinate system $\{e_i\}$, $\{e_i^*\}$ for P. Put $f(e_i) = \sum_j f_{ij}e_j$ where $f_{ij} = (-1)^{p(f_{ij},e_j)}e_j^*(f(e_i))$, and similarly for g. By definition we have

$$\begin{aligned}
\mathrm{STr}_P(fg) &= \sum_i (-1)^{p(f(g(e_i)),e_i)} e_i^*(f(g(e_i))) \\
&= \sum_{i,j} (-1)^{p(f(g(e_i)),e_i)} e_i^*(f(g_{ij}e_j)) \\
&= \sum_{i,j} (-1)^{p(f(g(e_i)),e_i)} (-1)^{p(f,g_{ij})} e_i^*(g_{ij}f(e_j)) \\
&= \sum_{i,j,k} (-1)^{p(f(g(e_i)),e_i)} (-1)^{p(f,g_{ij})} e_i^*(g_{ij}f_{jk}e_k) \\
&= \sum_{i,j,k} (-1)^{p(f(g(e_i)),e_i)} (-1)^{p(f,g_{ij})} (-1)^{p(g_{ij}f_{jk},e_i^*)} g_{ij}f_{jk}e_i^*(e_k).
\end{aligned}$$

Because $e_i^*(e_j) = \delta_{ij}1$ the expression above reduces to

$$\sum_{j,k} (-1)^{p(f(g(e_k)),e_k)} (-1)^{p(f,g_{kj})} (-1)^{p(g_{kj}f_{jk},e_k^*)} g_{kj}f_{jk}.$$

For the j, k term in this summation, the exponent of -1 is

$$\begin{aligned}
&[p(f) + p(g) + p(e_k)]p(e_k) + p(f)[p(g) + p(e_k) + p(e_j)] \\
&\quad + [p(g) + p(e_k) + p(e_j) + p(f) + p(e_k) + p(e_j)]p(e_k) \\
&= [p(f) + p(g) + p(e_k)]p(e_k) + p(f)[p(g) + p(e_k) + p(e_j)] + [p(g) + p(f)]p(e_k) \\
&= p(e_k) + p(f)[p(g) + p(e_k) + p(e_j)].
\end{aligned} \tag{4.1.1}$$

We may expand $\mathrm{STr}_P(gf)$ in a similar manner. Because $g(ax) = h^{-1}(a)gx$ we find that

$$\mathrm{STr}_P(gf) = \sum_{j,k} (-1)^{p(g(f(e_j)),e_j)} (-1)^{p(g,f_{jk})} (-1)^{p(f_{jk}g_{kj},e_j^*)} h^{-1}(f_{jk})g_{kj},$$

and the exponent of -1 in the j, k term is the same as the last line of (4.1.1) with f and g exchanged, and j and k exchanged.

The difference between the two supertraces is of the right shape to be an element of $[A, A]_h$. We just need to confirm the parities are correct, which reduces to showing

$$\begin{aligned}
&p(e_k) + p(f)[p(g) + p(e_k) + p(e_j)] + p(e_j) + p(g)[p(f) + p(e_k) + p(e_j)] \\
&= p(f)p(g) + [p(f) + p(e_j) + p(e_k)][p(g) + p(e_j) + p(e_k)].
\end{aligned}$$

In fact this is true so we are done. □

In our application there is a finite group G of automorphisms of A and a natural choice of even element $\gamma = \gamma(h) \in \mathrm{Hom}_A(P, P_{(h)})$ for each automorphism $h \in G$. We are most interested in the function $\mathrm{STr}_P^{(h)} : \mathrm{End}_A P \to A/[A,A]_h$ defined by

$$\mathrm{STr}_P^{(h)}(f) = \mathrm{STr}_P(f \circ \gamma).$$

As notes above $\mathrm{End}_A P_{(h)} = \mathrm{End}_A P$, so it makes sense to speak of the action of γ on $\mathrm{End}_A P$ by conjugation, i.e., $f \mapsto \gamma^{-1} f \gamma$. Since we assume h (and therefore γ) has finite order, $\mathrm{End}_A P$ splits into eigenspaces for the action of γ. For the rest of this section elements of $\mathrm{End}_A P$ are implicitly assumed to be eigenvectors for the action of γ, and $\lambda(f)$ denotes the eigenvalue of $f \in \mathrm{End}_A P$.

Lemma 4.1.5. *The map $\mathrm{STr}_P^{(h)}$ is h-supersymmetric, meaning*

$$\mathrm{STr}_P^{(h)}(fg) = \delta_{\lambda(f)\lambda(g),1} p(f,g) \lambda(f)^{-1} \mathrm{STr}_P^{(h)}(gf).$$

Proof. Using lemma 4.1.4 we compute

$$\begin{aligned} \mathrm{STr}_P^{(h)}(fg) &= \mathrm{STr}_P(fg\gamma) = \lambda(g)\,\mathrm{STr}_P(f\gamma g) \\ &= p(f,g)\lambda(g)\,\mathrm{STr}_P(gf\gamma) = p(f,g)\lambda(g)\,\mathrm{STr}_P^{(h)}(gf) \end{aligned}$$

(note γ is even). We also have

$$\begin{aligned} \mathrm{STr}_P^{(h)}(fg) &= \mathrm{STr}_P(fg\gamma) = \mathrm{STr}_P(\gamma fg) \\ &= \lambda(f)^{-1}\,\mathrm{STr}_P(f\gamma g) = p(f,g)\lambda(f)^{-1}\,\mathrm{STr}_P(gf\gamma) \\ &= p(f,g)\lambda(f)^{-1}\,\mathrm{STr}_P^{(h)}(gf). \end{aligned}$$

The lemma follows. □

The following lemma is proved simply by unwinding definitions.

Lemma 4.1.6. *Let A be an arbitrary finite dimensional unital associative algebra, let Z be the free rank 1 left A-module, and let $B = (\mathrm{End}_A Z)^{op}$. We have $B \cong A$ and the right action of B on Z coincides with the natural right action of A. The map*

$$\begin{aligned} (B/[B,B]_h)^* &\to (A/[A,A]_h)^* \\ \phi &\mapsto \phi \circ \mathrm{STr}_Z^{(h)} \end{aligned}$$

naturally identifies the spaces of h-supersymmetric functions on A and B.

4.2 Supertrace functions and conformal blocks

Lemma 4.1.6 suggests a replacement for the use of supertraces of irreducible V-modules in the proof of Theorem 2.6.1. Let $A = \mathrm{Zhu}_g(V)$, which we assume to be finite dimensional but not necessarily semisimple. Let Z denote the free rank one left A-module and let $B = (\mathrm{End}_A(Z))^{\mathrm{op}} \cong A$ which naturally acts on Z from the right.

Let $M = L(Z) = \oplus_{j \in \mathbb{Z}_+} M_j$ be the V-module corresponding to Z under the usual Zhu induction functor. We extend the action of B from Z to M by setting

$$(u_n x) \cdot b = u_n(x \cdot b) \quad \text{for } x \in Z,\, b \in B,\, u \in V.$$

From its construction, this right action of B commutes with the action of V.

The following basic lemma is proved in a similar manner to Lemma 2.3.1.

Lemma 4.2.1. *If V is C_2-cofinite and N is a finite dimensional* $\mathrm{Zhu}_g(V)$*-module, then every graded piece of the induced module $L(N)$ is finite dimensional.*

Certainly M is not finitely generated as a B-module, but the action of B preserves the graded pieces M_n and these are finitely generated because they are finite dimensional.

Assumption 4.2.2. The graded pieces of M are projective as right B-modules.

In Section 4.4 we study an interesting example for which this assumption holds. Therefore the theory developed here is not vacuous! We do not know if the assumption holds in general.

If M is an irreducible positive energy V-module then by Schur's lemma L_0 acts as a scalar $h(M) \in \mathbb{C}$ on the lowest graded piece of M. By the Virasoro commutation relations L_0 acts as $h(M) + j$ on M_j. If V is rational then all positive energy V-modules are direct sums of irreducible ones and so L_0 acts semisimply on them all. In the present case V is not rational so there may be V-modules on which L_0 does not act semisimply. Since the graded pieces are finite dimensional they are direct sums of generalized eigenspaces for L_0.

Since $Z = B$ as a vector space we may put $\gamma = h^{-1} : Z \to Z$ which satisfies $x\gamma b \gamma^{-1} = xh(b)$ for all $b \in B$, $x \in Z$. The induction functor lifts this to a map $\gamma : M \to M$ intertwining the B-action with the h-twisted B-action. We now write a $B/[B,B]_h$-valued supertrace function

$$S_M^{(h)}(u,q) = \mathrm{STr}_M^{(h)}\, u_0 q^{L_0 - c/24} = q^{-c/24} \sum_{j \geq 0} \mathrm{STr}_{M_j}^{(h)}(u_0 q^{L_0}). \tag{4.2.1}$$

On a graded piece M_j we can write the Jordan decomposition $L_0 = L_0^{\mathrm{ss}} + L_0^{\mathrm{nilp}}$ into commuting semisimple and nilpotent parts. We write $q = e^{2\pi i \tau}$ formally, and define q^{L_0} as $q^{L_0^{\mathrm{ss}}} e^{2\pi i \tau L_0^{\mathrm{nilp}}}$. Thus $S_M^{(h)}$ is a series in powers of q, whose coefficients are elements of

$$(B/[B,B]_h) \otimes_{\mathbb{C}} \mathbb{C}[\tau].$$

In this definition we perform the abstract supertrace construction on Z regarded as a *right* B-module, opposite to the convention adopted in Section 4.1, but it is obviously equivalent since a right B-module is simply a left B^{op}-module. For $\phi \in (B/[B,B]_h)^*$ it is sometimes convenient to write $S_M^{(h),\phi} = \phi \circ S_M^{(h)}$. The main result of this chapter is:

Theorem 4.2.3. *Let V be a C_2-cofinite VOSA, G a finite group of automorphisms of V, $g, h \in G$ commuting, and $\mathcal{C}(g,h)$ the space of conformal blocks as in Definition 2.2.2. Suppose Assumption 4.2.2 holds. Then there is a surjection $(B/[B,B]_h)^* \twoheadrightarrow \mathcal{C}(g,h)$, given by*

$$\phi \mapsto \phi \circ S_M^{(h)},$$

where $M = L(Z)$ is the V-module corresponding to Z the free module of rank 1 over $A = \mathrm{Zhu}_g(V)$.

Recall again that C_2-cofiniteness of V implies finite dimensionality of $\mathrm{Zhu}_g(V)$ (Proposition 2.17(c) of [4]). We need to show that $S_M^{(h)}$ is a $B/[B,B]_h$-valued conformal block and that the $S_M^{(h),\phi}$ span $\mathcal{C}((g,h))$. The proof that $S_M^{(h)}$ is a conformal block is almost identical to the proof of Theorem 2.6.1 because all that is used in the proof is the Borcherds identity and the h-supersymmetric property of $S_M^{(h)}$. The only difference is the nonsemisimplicty of L_0, which requires us to pepper the calculations of Propositions 2.6.2-2.6.7 with the factor $q^{L_0^{\mathrm{nilp}}}$. Since $[L_0, u_n] = -nu_n$ and u_n lowers the level in M by n we have that L_0^{nilp} commutes with u_n. Also L_0^{nilp} commutes with γ. Therefore $q^{L_0^{\mathrm{nilp}}}$ is a spectator and these propositions carry over to the present case. What remains is to repeat the exhaustion of a conformal block by supertrace functions.

4.3 Exhaustion procedure

The procedure to exhaust a conformal block by supertrace functions is the same as in Section 2.7 with two complications. One is that the Frobenius expansion of a conformal block can have nontrivial terms in $\log q$. The other is the appearance of $e^{2\pi i\tau L_0^{\mathrm{nilp}}}$ in the definition of supertrace functions. To tackle these two related things we adapt some lemmas from Miyamoto [25].

Let B be an arbitrary finite dimensional unital associative superalgebra with finite order automorphism h. Suppose B contains a central element ω fixed by h. Let $K \subseteq B$ be the 2-sided ideal of eigenvectors of ω, i.e.,

$$K = \{b \in B | (\omega - \omega^{\mathrm{ss}}) * b = 0\}.$$

Of course K is the union of finitely many $K_\lambda = \{b \in B | (\omega - \lambda) * b = 0\}$ which lie in separate direct summands of B.

Let ϕ be a h-supersymmetric function on B, in general it does not descend to a supersymmetric function on B/K. However ϕ', defined by

$$\phi'(b) = \phi(b(\omega - \omega^{\mathrm{ss}})),$$

is a h-supersymmetric function on B/K.

The following lemma is a generalization to the super case of Lemma 4.1 of [2]. The proof is essentially identical to the one in that paper, but we include it for the sake of completeness.

Lemma 4.3.1. *Let Z be a finitely generated projective right B-module and define $Z' = Z/ZK$ which is a finitely generated projective right B'-module. For any $f \in \operatorname{End}_B Z$ we have*

$$\mathrm{STr}_Z^{(h),\phi}(f \circ (\omega - \omega^{ss})) = \mathrm{STr}_{Z'}^{(h),\phi'}(f'),$$

where $f' \in \operatorname{End}_{B'}(Z')$ is the image of f under the canonical projection $\operatorname{End}_B(Z) \twoheadrightarrow \operatorname{End}_{B'}(Z')$.

Proof. Let $\{e_i\} \subseteq Z$ and $\{e_i^*\} \subseteq \operatorname{Hom}_B(Z, B)$ be a coordinate system for Z over B, i.e.,

$$x = \sum_i (-1)^{p(e_i^*(x), e_i)} e_i e_i^*(x) \quad \text{for all } x \in Z.$$

Let $\overline{e}_i = e_i + ZK \in Z'$ and let $\overline{e}_i^* \in \operatorname{Hom}_{B'}(Z', B')$ be defined by $\overline{e}_i^*(x + ZK) = e_i^*(x) + K$ which is well-defined because $e_i^*(xb) = e_i^*(x)b$. These sets form a B'-coordinate system for Z' because for any $x + ZK$ we have

$$\begin{aligned}\sum_i (-1)^{p(\overline{e}_i^*(x), \overline{e}_i)} \overline{e}_i \overline{e}_i^*(x + ZK) &= \sum_i (-1)^{p(e_i^*(x), e_i)} \overline{e}_i (e_i^*(x) + K) \\ &= \sum_i (-1)^{p(e_i^*(x), e_i)} e_i e_i^*(x) + ZK.\end{aligned}$$

By definition

$$\mathrm{STr}_Z^{(h),\phi} f = \phi \left(\sum_i (-1)^{p(e_i^*, f(e_i))} e_i^*(f(\gamma(e_i))) \right).$$

Hence

$$\begin{aligned}\mathrm{STr}_{Z'}^{(h),\phi'} f &= \phi'(\sum_i (-1)^{p(\overline{e}_i^*(x), \overline{e}_i)} \overline{e}_i^*(f(\gamma(\overline{e}_i)))) \\ &= \phi((\omega - \omega^{\mathrm{ss}}) * \sum_i (-1)^{p(e_i^*(x), e_i)} e_i^*(f(\gamma(e_i) + ZK)) + ZK) \\ &= \phi(\sum_i (-1)^{p(e_i^*(x), e_i)} e_i^*(f(\gamma(e_i) * (\omega - \omega^{\mathrm{ss}})))) \\ &= \mathrm{STr}_Z^{(h),\phi}(f \circ (\omega - \omega^{\mathrm{ss}})),\end{aligned}$$

where we used that ω is central in B and fixed by h, and that e_i^* and f commute with the B-action. □

Corollary 4.3.2. *If we define Z' as above, and let $M = L(Z)$ and $M' = L(Z')$, then*

$$\mathrm{STr}_Z^{(h),\phi} u_0 q^{L_0 - c/24} L_0^{nilp} = \mathrm{STr}_{Z'}^{(h),\phi'} u_0 q^{L_0 - c/24}.$$

Proof. The action of L_0 on M_0 agrees with that of $\omega \in B$. By the definition of the right action of B on M we have that L_0 acts by $\omega + n$ on M_n. We have

$$\begin{aligned}
\mathrm{STr}_M^{(h),\phi} u_0 q^{L_0 - c/24} L_0^{\mathrm{nilp}} &= \sum_{n \geq 0} \mathrm{STr}_{M_n}^{(h),\phi} u_0 q^{L_0 - c/24} (L_0 - L_0^{\mathrm{ss}}) \\
&= \sum_{n \geq 0} \mathrm{STr}_{M_n}^{(h),\phi} u_0 q^{L_0 - c/24} (\omega - \omega^{\mathrm{ss}}) \\
&= \sum_{n \geq 0} \mathrm{STr}_{(M_n)'}^{(h),\phi'} u_0 q^{L_0 - c/24},
\end{aligned}$$

where in the last step we used Lemma 4.3.1. It remains to show that $(L(Z)_m)' = L(Z')_m$, which is true by construction. □

Consider a nonzero conformal block $S(u, \tau)$, with Frobenius expansion (2.3.2) and coefficients $C_{i,j,n}$. We are most interested in the coefficients $C_{0,j,0}$. As special cases of Propositions 6.0.5 and 6.0.6 in Chapter 6 we have

- $C_{0,j,0} : V \to \mathbb{C}$ descends to a map $\mathrm{Zhu}_g(V) \to \mathbb{C}$.
- $C_{0,j,0}$ is h-supersymmetric.
- For all $u, v \in \mathrm{Zhu}_g(V)$ we have

  $$C_{0,j,0}((\omega - c/24 - \lambda_j)^{R+1} * u) = 0,$$

  where R is the highest power of $\log q$ appearing in the Frobenius expansion (2.3.2).

From the last of these we see that either $C_{0,j,0}(u) = 0$ or else u lies in the generalized eigenspace for ω of eigenvalue $c/24 + \lambda_j$. Hence, in particular,

$$C_{0,j,0}((\omega - \omega^{\mathrm{ss}})^{R+1} * u) = 0.$$

Each $C_{0,j,0}$ acts nontrivially on a different direct summand of the superalgebra B. Let $C_{0,0} = \sum_j C_{0,j,0}$. Since it is a h-supersymmetric function on $\mathrm{Zhu}_g(V)$, we may write $C_{0,0} = \phi \circ \mathrm{STr}_Z^{(h)}$ as per Lemma 4.1.6. Consider the conformal block

$$S'(u, \tau) = S(u, \tau) - \mathrm{STr}_M^{(h),\phi} u_0 q^{L_0 - c/24},$$

this also has a Frobenius expansion whose coefficients we denote $C'_{i,j,n}$. We wish to show that $C'_{0,j,0}$ is identically zero for each j.

Let

$$K^{(j)} = \{b \in B | (\omega - \omega^{\mathrm{ss}})^j * b = 0\},$$

which is a 2-sided ideal in B. We put $B^{(k)} = B/K^{(j)}$ and $Z^{(k)} = Z/ZK^{(j)}$ the free rank 1 module over $B^{(k)}$. Finally we define $M^{(j)} = L^0(Z^{(j)})$. For a h-supersymmetric function ϕ on B, we define the h-supersymmetric function $\phi^{(j)}$ on $B^{(j)}$ by $\phi^{(j)}(b) = \phi((\omega - \omega^{\text{ss}})^j * b)$.

Lemma 4.3.3. *There exist (fixed) constants $b_1, b_2, \ldots \in \mathbb{C}$ such that*

$$\operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{ss}-c/24} = S_M^{(h),\phi}(u,\tau) - \sum_{n=1}^{R} b_n (\log q)^n S_{M^{(n)}}^{(h),\phi^{(n)}}(u,\tau).$$

We have stated this lemma for $M = L(Z)$ since this is the only case we use, but it is true if Z is replaced by any finitely generated projective module.

Proof. Let $b_n \in \mathbb{C}$ be defined by $(e^x - 1)/e^x = \sum_{n=1}^{\infty} b_n x^n$. Then

$$\begin{aligned}
\operatorname{STr}_M^{(h),\phi}(u,\tau) &= \operatorname{STr}_M^{(h),\phi} u_0 q^{L_0 - c/24} = \operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24} q^{L_0^{\text{nilp}}} \\
&= \operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24} \left[1 + e^{2\pi i \tau L_0^{\text{nilp}}} \sum_{n=1}^{\infty} b_n (2\pi i \tau L_0^{\text{nilp}})^n \right] \\
&= \operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24} + \sum_{n=1}^{\infty} b_n (\log q)^n \operatorname{STr}_M^{(h),\phi} u_0 q^{L_0 - c/24} (L_0^{\text{nilp}})^n.
\end{aligned}$$

Now we invoke Corollary 4.3.2 to rewrite the final line above as

$$\operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24} + \sum_{n=1}^{R} b_n (\log q)^n \operatorname{STr}_{M^{(n)}}^{(h),\phi^{(n)}} u_0 q^{L_0 - c/24},$$

so we are done. □

The function $\operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24}$ contains no terms in $\log q$ because L_0^{ss} is a scalar on each graded piece of M. Therefore we have

$$\operatorname{STr}_M^{(h),\phi} u_0 q^{L_0^{\text{ss}}-c/24} = S_0(u,\tau) = \sum_j \sum_{n \geq 0} q^{\lambda_j + n} C_{0,j,n}(u)$$

as a direct consequence of Lemma 4.3.3. Therefore $C_{0,0}(u) = \operatorname{STr}_Z^{(h),\phi} u$ and so $C'_{0,0} = 0$, indeed each $C_{0,j,0} = 0$. It follows directly from conformal block axiom (**CB4**) that:

Lemma 4.3.4. *If $C_{i,j,0} = 0$ then $C_{i+1,j,0} = 0$ too.*

Therefore $C'_{i,j,0} = 0$ for $i = 0, 1, \ldots R$.

Either $S' = 0$ in which case we are done, or else we repeat the arguments of this section on $S'(a,\tau)$. Because all the initial coefficients are 0, the supersymmetric function ϕ that we obtain this time acts nontrivially on direct summands of B with higher ω eigenvalues. The argument may only repeat finitely many times before ϕ becomes zero and we exhaust S by supertrace functions.

4.4 Application to the superbosons VOSA

Recall the charged free fermions VOSA $F_{\mathrm{ch}}^{a}(\psi,\psi^*)$ of Section 3.2. We now take $a=0$ and consider the sub-VOSA V generated by $\chi=\psi$ and $\chi^*=T\psi^*$. These are both of conformal weight 1. The Virasoro element is

$$\omega=\chi^*_{(-1)}\chi_{(-1)}|0\rangle,$$

and the central charge is -2. and we have the explicit commutator formula

$$[\chi(z),\chi^*(w)]=\partial_w\delta(z,w),$$

so that

$$[\chi_m,\chi^*_n]=m\delta_{m,-n} \tag{4.4.1}$$

(self commutators are all zero). This VOSA V can also be obtained by the standard construction (see [17]) of a VOSA from a finite dimensional Lie superalgebra with supersymmetric invariant bilinear form $(\mathfrak{g},(\cdot,\cdot))$. In this case applied to $\mathfrak{g}=\mathbb{C}\chi+\mathbb{C}\chi^*$ purely odd with trivial bracket and $(\chi,\chi^*)=-(\chi^*,\chi)=1$. For this reason we call V the VOSA of superbosons.

4.4.1 The Zhu algebra

We compute the untwisted Zhu algebra. We have $\chi\circ_n=\chi_{(n)}+\chi_{(n+1)}$, and the same for χ^*. So any monomial can be reduced modulo the Zhu relations to one with all subscripts -1. Because χ and χ^* are odd there can be at most one of each in a non vanishing monomial. So the Zhu algebra is a quotient of the span of $|0\rangle$, χ, χ^* and $\chi^*_{(-1)}\chi=\omega$. Both χ and χ^* square to 0 and we have

$$\chi^* * \chi=-\chi * \chi^*=\omega.$$

The products involving ω are deduced from associativity now, all products are 0 except with $|0\rangle$. Therefore Zhu(V) is a quotient of $U(\mathfrak{g})$ where g is the two dimensional odd Lie superalgebra above.

In fact Zhu$(V)\cong U(\mathfrak{g})$, which follows from the existence of a positive energy V-module with $\Lambda(\mathbb{C}^2)$ as its lowest graded piece. In Section 3.5.2 we define *Lie conformal superalgebras* (LCSAs) and two objects associated to an LCSA R: the Lie superalgebra Lie R, and the *universal enveloping vertex superalgebra* (UEVSA) $V(R)$ of R. Theorem 3.5.1 states that positive energy $V(R)$-modules are the same thing as 'restricted' Lie R-modules.

Let $\mathfrak{g},(\cdot,\cdot)$ be as above. We associate an LCSA $R=\operatorname{Cur}\mathfrak{g}$ to this data as in Example 3.5.2. Then $V(R)$ is just our superboson VOSA. On the other hand Lie $R=\hat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]+\mathbb{C}1$ and if we let $a_n=at^n$ then the Lie bracket is just (4.4.1). Now the generalized Verma module $M=U(\hat{\mathfrak{g}})\otimes_{U(\mathfrak{g}[t]+\mathbb{C}1)}U(\mathfrak{g})$ (which is a restricted $\hat{\mathfrak{g}}$-module) is the V-module we seek. We conclude that Zhu$(V)=U(\mathfrak{g})$ as a superalgebra.

Notice that Zhu(V) is not semisimple. The action of L_0 on Zhu(V) has a nontrivial Jordan block since L_0 annihilates χ, χ^* and ω, but sends 1 to ω. Also Zhu(V) is supercommutative, so its supersymmetric functions are simply all elements of the dual space.

4.4.2 Abstract supercharacters

The supercharacter of V itself is

$$\mathrm{STr}_V\, q^{L_0 - c/24} = q^{1/12} \prod_{n=1}^{\infty} (1-q^n)^2 = \eta(\tau)^2.$$

This is initially puzzling since $\eta(\tau)^2$ is a weight 1 modular form, rather than weight 0 as we would expect! Computing some abstract supertraces will make sense of this.

Let $A = B = Z = \mathrm{Zhu}(V)$ as usual. The V-module $L(Z)$ is defined (see Chapter 5) as the quotient of M by a maximal submodule intersecting the top level trivially. By a standard argument such a submodule is 0 in this case so $M = L(Z)$. Now M is a free B-module so the abstract supertrace makes sense. To compute it we select the following generators of M as a B-module; the vectors $\mathbf{m}|0\rangle$ for all monomials $\mathbf{m}$ in the lowering operators. There is an obvious dual basis for $Z^* = \mathrm{Hom}_B(Z, B)$.

For a monomial

$$\mathbf{m} = \chi_{-m_1} \cdots \chi_{-m_s} \chi^*_{-n_1} \cdots \chi^*_{-n_t}$$

with level $w = \sum m_i + \sum n_j$ (and parity $s+t$) and $x \in \mathrm{Zhu}(V)$ we have $L_0(\mathbf{m}x) = w(\mathbf{m}x) + \mathbf{m}(L_0 x)$. Since L_0 is one-step nilpotent on Zhu(V) we have

$$q^{L_0} \mathbf{m} x = q^w \mathbf{m}(1 + 2\pi i \tau L_0)x.$$

Recall the definition$\mathrm{STr}_P\, f = \sum_i (-1)^{p(f(e_i), e_i^*)} e_i^*(f(e_i)) \in B$. The contribution to $\mathrm{STr}_M\, q^{L_0}$ of the summand attached to the monomial $\mathbf{m}$ is

$$(-1)^{s+t} q^w (1 + 2\pi i \tau \omega).$$

Summing over all monomials gives the B-valued supertrace function

$$\begin{aligned} \mathrm{STr}_M\, q^{L_0 - c/24} &= (1 + 2\pi i \tau \omega) q^{1/12} \prod_{n=1}^{\infty} (1-q^n)^2 \\ &= \eta(\tau)^2 + 2\pi i \tau \eta(\tau)^2 \omega. \end{aligned}$$

From this we may decant the two functions $\eta(\tau)^2$ and $\tau\eta(\tau)^2$. These do indeed span an $SL_2(\mathbb{Z})$-invariant vector space of weight 0.

4.4.3 Higher weight pseudosupertrace functions

Now let us evaluate $\mathrm{STr}_M\, \chi_0^* q^{L_0-c/24}$. Now we have

$$\chi_0^* q^{L_0}\mathbf{m}x = (-1)^{s+t} q^w \mathbf{m}\chi_0^*(1+2\pi i\tau L_0)x.$$

The parities of x and $\chi_0^* x$ are always opposite, so no further $(-1)^{s+t}$ factor appears from taking the supertrace. Now $\chi_0^*(1+2\pi i\tau L_0)1 = \chi^*$, so we end up with the B-valued supertrace function

$$\eta(\tau)^2\chi^*.$$

This is modular invariant of weight 1 as expected.

Chapter 5

Higher level Zhu algebras

We have already encountered the g-twisted Zhu algebra $\mathrm{Zhu}_g(V)$ in Section 2.4. It plays a crucial role in establishing $SL_2(\mathbb{Z})$-invariance of VOSA characters. Analogues of the (untwisted) Zhu algebra, called higher level Zhu algebras, were introduced in [7] and these were used by Miyamoto [25] to prove a generalization of Zhu's theorem to non rational VOAs.

In this chapter we construct a unital associative superalgebra $\mathrm{Zhu}_{P,g}(V)$ (associated to a VOSA V with not necessarily integer conformal weights, an automorphism g of V, and $P \in \mathbb{R}_+$) suitable for studying g-twisted V-modules. Indeed we construct a restriction functor Ω_P taking $M \in \mathsf{PEMod}(g, V)$ to its P^{th} graded piece M_P, which is a $\mathrm{Zhu}_{P,g}(V)$-module. We also construct an induction functor going in the other direction. The following theorem summarizes the main properties of these functors.

Theorem 5.0.1.

- *For any* $\mathrm{Zhu}_{P,g}(V)$*-module* N *we have*

$$\Omega_P(L^P(N)) \cong N.$$

- Ω_P *and* L^P *are inverse bijections between the sets of irreducible modules in the category* $\mathsf{PEMod}(g, V)$ *of positive energy* g*-twisted* V*-modules, and the category* $\mathrm{Zhu}_{P,g}(V)$-mod *of* $\mathrm{Zhu}_{P,g}(V)$*-modules.*

Our approach closely follows De Sole and Kac [4], incorporating much from Dong, Li and Mason [7]. We prove some other results too. In particular

Theorem 5.0.2. *Suppose* $\mathrm{Zhu}_{P,g}(V)$ *is a finite dimensional semisimple superalgebra for each* $P \in \mathbb{R}_+$*, and suppose the central charge* c *of* V *is nonzero. Then* V *is* g*-rational.*

We also compute the higher level Zhu algebras of some particular VOAs, namely the universal Virasoro VOA Vir^c and the universal affine Kac-Moody VOA $V^k(\mathfrak{g})$ at non-critical level $k \neq -h^\vee$. There are compatible surjections $\mathrm{Zhu}_{P,g}(V) \twoheadrightarrow \mathrm{Zhu}_{Q,g}(V)$ whenever $P \geq Q$ so the g-twisted higher Zhu algebras form a directed system. In the two cases above we compute the inverse limit of the directed system.

Throughout most of the chapter we take the level P to be an integer, for clarity's sake. Zhu algebras of non integer level are of interest as well (they are a necessary part of Theorem 5.0.2) because twisted modules typically have nonzero graded pieces in non integer levels. In Section 5.4 we detail the changes necessary for the $P \notin \mathbb{Z}$ case (in the rest of the chapter we reserve lower case p to denote an integer level).

5.0.4 Notation

We use Definition 1.1.1 of VOSA from the introduction, though in this chapter there is no need to restrict conformal weights of V to be rational, we allow arbitrary real conformal weights. We denote the energy operator L_0 of V by H. In our paper [27] (from which the material of this chapter is drawn) we consider the more general setup of a vertex superalgebra (without the data of a Virasoro vector) carrying a diagonalizable (with real eigenvalues) operator $H : V \to V$ satisfying

$$[H, Y(u,z)] = z\partial_z Y(u,z) + Y(Hu,z) \tag{5.0.1}$$

for all $u \in V$. This is actually enough to establish most of the results of this chapter.

Definition 5.0.1. Let V be a VOSA with automorphism g and let M be a g-twisted V-module. For $u \in V$ the number ϵ_u was defined in the introduction. We now also define $\gamma_u = \Delta_u + \epsilon_u$.

If $M \in \mathsf{PEMod}(g, V)$ and we put

$$Y^M(u,z) = \sum_n u^M_{(n)} z^{-n-1}$$

then of course the sum is over $n \in [\gamma_u]$.

The following simple but important lemma follows directly from the defining relation (1.1.5) of a g-twisted V-module.

Lemma 5.0.3. *Let $M \in \mathsf{PEMod}(g,V)$, let $u \in V$, and let $s \in [\epsilon_u]$. Then*

$$[(T + H + s)u]_s M = 0. \tag{5.0.2}$$

In particular $[(T+H)u]_0 M = 0$ for all u such that $\epsilon_u = 0$.

Proof. In (1.1.5) put $b = |0\rangle$, $m = s+1$, $k = -1$, and $n = -2$, then note that $|0\rangle^M_{(n)} = \delta_{n,-1} I_M$. □

We now introduce an important Lie superalgebra associated to V.

Definition 5.0.2. Let V be a VOSA with automorphism g. Put

$$Q = \bigoplus_{[\epsilon] \in \mathbb{R}/\mathbb{Z}} t^{\epsilon} V^{[\epsilon]} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

(where ϵ is any representative of the coset $[\epsilon]$ and $V^{[\epsilon]} \subseteq V$ is the $e^{2\pi i\epsilon}$-eigenspace of g). Define $\mathrm{Lie}^g V$ to be the quotient

$$Q/(T+H+\partial_t)Q,$$

with the Lie bracket

$$[u_m, v_k] = \sum_{j\in\mathbb{Z}_+} \binom{m+\Delta_u-1}{j} (u_{(j)}v)_{m+k}, \tag{5.0.3}$$

where u_n denotes the image of ut^n. With the grading $\deg u_n = n$, $\mathrm{Lie}^g V$ is a graded Lie superalgebra. We also sometimes use the notation $u_{(n)} = u_{n-\Delta_u+1}$ for $n \in [\gamma_u]$, in terms of which (5.0.3) becomes

$$[u_{(m)}, v_{(k)}] = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (u_{(j)}v)_{(m+k-j)}. \tag{5.0.4}$$

It is immediate from the definition that every g-twisted positive energy V-module is automatically a $\mathrm{Lie}^g V$-module. Observe that the weight 0 subspace $(\mathrm{Lie}^g V)_0 = \mathbb{C}\{u_n \in \mathrm{Lie}^g V | [\epsilon_u] = [0], n \in \mathbb{Z}\} \subseteq \mathrm{Lie}^g V$ is a Lie subalgebra.

Finally, we record some properties of the numbers ϵ_u and γ_u that we shall use repeatedly. T preserves , $[\epsilon]$, ϵ, $[\Delta]$ and $[\gamma]$. The $[\epsilon]$ function is additive in the n^{th} products, but ϵ is not, for example if $-1 < \epsilon_u, \epsilon_v \leq -\frac{1}{2}$ then $\epsilon_{u_{(n)}b} = \epsilon_u + \epsilon_v + 1$.

Definition 5.0.3.

$$\chi(u,v) = \begin{cases} 1 & \text{if } \epsilon_u + \epsilon_v \leq -1, \\ 0 & \text{if } \epsilon_u + \epsilon_v > -1. \end{cases}$$

We have

Lemma 5.0.4.

$$\epsilon_{|0\rangle} = 0, \quad \epsilon_{Tu} = \epsilon_u, \quad \textit{and} \quad \epsilon_{u_{(n)}v} = \epsilon_u + \epsilon_v + \chi(u,v),$$

and

$$\gamma_{|0\rangle} = 0, \quad \gamma_{Tu} = \gamma_u + 1, \quad \textit{and} \quad \gamma_{u_{(n)}v} = \gamma_u + \gamma_v - n - 1 + \chi(u,v).$$

One more piece of notation: We denote by $\Gamma \subseteq \mathbb{R}$ the union, ranging over all homogeneous $u \in V$, of $\epsilon \in [\epsilon_u]$. Clearly $\mathrm{Lie}^g V$ is a Γ-graded Lie superalgebra.

5.0.5 Motivation to introduce Zhu algebras

In this section we provide some motivation for the definition of the Zhu algebra and its higher analogues. The zero modes u_0 of an action of the VOSA V preserve graded pieces of V-modules, and can be composed. So the basic idea behind the Zhu algebra is to try to formalize this notion of 'algebra of zero modes'.

More precisely, let V be a g-twisted VOSA and let $M \in \mathsf{PEMod}(g, V)$. Define

$$V_g = \{u \in V | [\epsilon_u] = [0]\},$$

If $u \in V_g$ then each graded piece M_P of M is stable under a_0^M. We would like to intrinsically define a product $*_P : V_g \otimes V_g \to V_g$ such that $(u *_P v)_0^M = u_0^M v_0^M$ for all $M \in \mathsf{PEMod}(g, V)$.

For simplicity let $P = p \in \mathbb{Z}_+$ from now on (in Section 5.4 we will describe the changes that must be made for the general case of $P \in \mathbb{R}_+$) and let $x \in M_p$. Let $u, v \in V$ with $[\epsilon_u] + [\epsilon_v] = [0]$. Put $m = p+1+\epsilon_u$, $k = -(p+1+\epsilon_u)$, $n \in \mathbb{Z}$ in (1.1.5). Because $u_s^M M_p = 0$ for $s > p$, we obtain

$$\sum_{j \in \mathbb{Z}_+} \binom{\gamma_u + p}{j} (u_{(n+j)}v)_0^M x = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} u^M_{p+1+n-j+\epsilon_u} v^M_{-p-1-n+j-\epsilon_u} x. \tag{5.0.5}$$

The right hand side vanishes when $n \leq -2p-2+\chi(u,v)$ (recall $\epsilon_u + \epsilon_v + \chi(u,v) = 0$), in other words

$$(u_{[n]}v)_0^M M_p = 0 \quad \text{whenever} \quad n \leq -2p - 2 + \chi(u, v), \tag{5.0.6}$$

where

$$u_{[n]}v = \sum_{j \in \mathbb{Z}_+} \binom{\gamma_u + p}{j} u_{(n+j)}v.$$

In the notation of Definition 2.4.1 we have for $p = 0$ that $u_{[n]}v = u \circ_n v$. The product $u_{[n]}v$ defined here is not the same as the Zhu product $u_{[n]}v$ used elsewhere in the thesis. There should be no confusion because we never use the latter product in this chapter.

Now let $u, v \in V_g$, so that $\epsilon_u = \epsilon_v = \chi(u, v) = 0$; equation (5.0.5) becomes

$$(u_{[n]}v)_0^M x = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} u^M_{p+1+n-j} v^M_{-p-1-n+j} x.$$

Since we are aiming for $(a *_p v)_0^M = u_0^M v_0^M$, we are led to define $*_p$ by

$$u *_p v = \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-1-m]}v. \tag{5.0.7}$$

Indeed

$$(u *_p v)_0^M x = \sum_{j \in \mathbb{Z}_+} \sum_{m=0}^{p} (-1)^j \binom{-p-1}{m} \binom{-p-1-m}{j} u^M_{-m-j} v^M_{m+j} x.$$

For an integer α, such that $0 \leq \alpha \leq p$, the coefficient of $u^M_{-\alpha} v^M_\alpha x$ here is

$$\begin{aligned}\sum_{j=0}^{\alpha}(-1)^j \binom{-p-1}{\alpha-j}\binom{-p-1-\alpha+j}{j} &= \sum_{j=0}^{\alpha}\binom{-p-1}{\alpha-j}\binom{p+\alpha}{j} \\ &= [\xi^\alpha] : (1+\xi)^{-p-1}(1+\xi)^{p+\alpha} \\ &= [\xi^\alpha] : (1+\xi)^{\alpha-1}\end{aligned}$$

which is 1 if $\alpha = 0$ and 0 if $\alpha > 0$. So $(u *_p v)^M_0 = u^M_0 v^M_0$ and M_p naturally acquires the structure of a module over $(V_g, *_p)$.

The algebra $(V_g, *_p)$ is not associative. But there are elements of V_g that act trivially on all M_p, namely those presented in equations (5.0.2) and (5.0.6), we might as well quotient these out. Indeed let

$$J_{p,g} = \mathbb{C}\{(T+H)u | u \in V_g\} + \mathbb{C}\{u_{[-2p-2+\chi(u,v)]}v | [\epsilon_u] + [\epsilon_v] = [0]\} \subseteq V_g, \tag{5.0.8}$$

so that we automatically have

Lemma 5.0.5.

$$(J_{p,g})^M_0 M_p = 0.$$

Later we shall see that $J_{p,g}$ is a 2-sided ideal of $(V_g, *_p)$, so the following definition is sound.

Definition 5.0.4. Let V be a VOSA and g an automorphism of V. Then the g-twisted level p Zhu algebra is

$$\mathrm{Zhu}_{p,g}(V) = V_g / J_{p,g}.$$

It turns out $\mathrm{Zhu}_{p,g}(V)$ is an associative superalgebra with unit element $[|0\rangle]$. Because of the lemma above the action of V_g on M_p factors to an action of $\mathrm{Zhu}_{p,g}(V)$.

5.1 First properties of the Zhu algebras

5.1.1 A modified state-field correspondence

Motivated by the discussion of the last section we introduce a modified state-field correspondence

$$\begin{aligned} Z(u,z) &= (1+z)^{\gamma_u + p} Y(u,z) = \sum_{n\in\mathbb{Z}} u_{[n]} z^{-n-1}, \\ \text{so that} \quad a_{[n]} &= \sum_{j\in\mathbb{Z}_+} \binom{\gamma_u + p}{j} u_{(n+j)}. \end{aligned} \tag{5.1.1}$$

In section 5.1.2 we develop an analog of the Borcherds identity for the modified fields $Z(u,z)$ and use it to show that $J_{p,g}$ is a right ideal. In section 5.1.3 we prove a skew-symmetry formula

$$u *_p v - p(u,v) v *_p u = [u, b] \pmod{J_{p,g}} \tag{5.1.2}$$

(where $[u,v]$ is defined in that section). Then we use the skew-symmetry formula, in section 5.1.4, to prove that $J_{p,g}$ is also a left ideal. In sections 5.1.5 and 5.1.6 we show that, modulo $J_{p,g}$, the product $*_p$ is associative and $|0\rangle$ is a unit, respectively.

Lemma 5.1.1.

- *For all $u, v \in V$, $n \in \mathbb{Z}$,*

$$(Tu)_{[n]}v + (\gamma_u + p + n + 1)u_{[n]}v = -nu_{[n-1]}v. \tag{5.1.3}$$

- $V_{[j]}V \subseteq V_{[k]}V$ *whenever* $j \leq k \leq -1$.

Proof. For the first part we have

$$\begin{aligned} Z(Tu, z) &= (1+z)^{\gamma_u + 1 + p} Y(Tu, z) = (1+z)^{\gamma_u+1+p} \partial_z Y(u,z) \\ &= (1+z)\partial_z[(1+z)^{\gamma_u+p}Y(u,z)] - (\gamma_u + p)(1+z)^{\gamma_u+p}Y(u,z) \\ &= [(1+z)\partial_z - (\gamma_u + p)]Z(u,z), \end{aligned}$$

equate coefficients of z^{-n-1} to get (5.1.3). The second part follows immediately. □

Now suppose $u \in V_g$, so that $\epsilon_u = 0$ and $\Delta_u = \gamma_u$. From Lemma 5.1.1 we have

$$\begin{aligned} [(T+H)u] *_p v &= \sum_{m=0}^{p} \binom{-p-1}{m} [(T+H)u]_{[-p-1-m]}v \\ &= \sum_{m=0}^{p} \binom{-p-1}{m} \left[(p+m+1)u_{[-p-2-m]}v + mu_{[-p-1-m]}v\right]. \end{aligned}$$

If we put $m = n+1$ in the second summand and manipulate it slightly we can combine the sums to obtain

$$[(T+H)u] *_p v = (2p+1)\binom{-p-1}{p} u_{[-2p-2]}v. \tag{5.1.4}$$

If $v \in V_g$ too then $[\epsilon_u] + [\epsilon_v] = [0]$ and so $((T+H)V_g) *_p V_g \subseteq J_{p,g}$.

Remark 5.1.2. This calculation shows that in the untwisted case, where $\epsilon_u = 0$ for all u, the set J_p may be defined simply as the ideal in $(V, *_p)$ generated by $(T+H)V$ rather than the more complicated equation (5.0.8).

5.1.2 The Borcherds identity

Let $(V, Y, |0\rangle)$ be a vertex algebra, recall [17] the n^{th} product of quantum fields:

$$\begin{aligned} Y(u,w)_{(n)}Y(v,w) = \ & \mathrm{Res}_z \left[Y(u,z)Y(v,w)i_{z,w}(z-w)^n \right. \\ & \left. -p(u,v)Y(v,w)Y(u,z)i_{w,z}(z-w)^n \right]. \end{aligned} \tag{5.1.5}$$

Taking the residue Res_z of the Borcherds identity shows that

$$Y(u,w)_{(n)}Y(v,w) = Y(u_{(n)}v,w). \tag{5.1.6}$$

This is called the n^{th} product identity. In this section we recast the n^{th} product identity and the Borcherds identity, which are both in terms of $Y(u,w)$, in terms of $Z(u,w)$.

Theorem 5.1.3.

$$(1+w)^{n+p+1-\chi(u,v)} Z(u_{[n]}v,w) = Z(u,w)_{(n)}Z(v,w). \tag{5.1.7}$$

Here the n^{th} product of quantum fields is defined as above, i.e., equation (5.1.5) but with Z in place of Y.

Proof. The left hand side is

$$\begin{aligned} (1+w)^{n+p+1-\chi(u,v)} & \sum_{j\in\mathbb{Z}_+} \binom{\gamma_u+p}{j} Z(u_{(n+j)}v,w) \\ &= \sum_{j\in\mathbb{Z}_+} \binom{\gamma_u+p}{j} (1+w)^{\gamma_u+\gamma_v+2p-j} Y(u_{(n+j)}v,w) \end{aligned}$$

(using $\gamma_{u_{(k)}v} = \gamma_u + \gamma_v + \chi(u,v) - k - 1$). Now we use equation (5.1.6) to rewrite this as

$$\begin{aligned} \mathrm{Res}_z & \left[\sum_{j\in\mathbb{Z}_+} \binom{\gamma_u+p}{j} (z-w)^j (1+w)^{\gamma_u+\gamma_v+2p-j} \times \right. \\ & \left. \times \left\{ Y(u,z)Y(v,w)i_{z,w}(z-w)^n - p(u,v)Y(v,w)Y(u,z)i_{w,z}(z-w)^n \right\} \right]. \end{aligned}$$

But

$$\sum_{j\in\mathbb{Z}_+} \binom{\gamma_u+p}{j} (z-w)^j (1+w)^{\gamma_u+\gamma_v+2p-j} = (1+z)^{\gamma_u+p}(1+w)^{\gamma_v+p},$$

so the left hand side of equation (5.1.7) becomes

$$\mathrm{Res}_z \left[Z(u,z)Z(v,w)i_{z,w}(z-w)^n - p(u,v)Z(v,w)Z(u,z)i_{w,z}(z-w)^n \right],$$

which is the right hand side. □

We use Theorem 5.1.3 to write down the modified analog of the Borcherds identity.

Theorem 5.1.4. *For all $u, v, x \in V$,*

$$\sum_{j\in\mathbb{Z}_+}(1+w)^{n+j+p+1-\chi(u,v)}Z(u_{[n+j]}v,w)x\,\partial_w^{(j)}\delta(z,w) \\ = Z(u,z)Z(v,w)x\,i_{z,w}(z-w)^n - p(u,v)Z(v,w)Z(u,z)x\,i_{w,z}(z-w)^n. \tag{5.1.8}$$

Proof. The theorem and its proof are essentially the same as Theorem 2.3 of [4] and its proof. □

We use Theorem 5.1.4 to obtain an expression for $(u_{[n]}v)_{[k]}x$. We begin by extracting the z^{-m-1} coefficient, obtaining

$$\sum_{j\in\mathbb{Z}_+}\binom{m}{j}(1+w)^{n+j+p+1-\chi(u,v)}Z(u_{[n+j]}v,w)xw^{m-j} \\ = \sum_{j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\left[u_{[m+n-j]}\left(Z(v,w)x\right)w^j - p(u,v)(-1)^nZ(v,w)(u_{[m+j]}x)w^{n-j}\right].$$

Next we multiply through by $(1+w)^{-n-p-1+\chi(u,v)}$, expand $(1+w)^{-n-p-1+\chi(u,v)}$ in positive powers of w, and extract the w^{-k-1} coefficient, obtaining

$$\sum_{i,j\in\mathbb{Z}_+}\binom{m}{j}\binom{j}{i}(u_{[n+j]}v)_{[i+k+m-j]}x \\ = \sum_{i,j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\binom{-n-p-1+\chi(u,v)}{i}\times \\ \times\left[u_{[m+n-j]}\left(v_{[i+j+k]}x\right) - p(u,v)(-1)^nv_{[i-j+n+k]}(u_{[m+j]}x)\right].$$

Finally we put $m=0$, to obtain

$$(u_{[n]}v)_{[k]}x = \sum_{i,j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\binom{-n-p-1+\chi(u,v)}{i}\times \\ \times\left[u_{[n-j]}\left(v_{[i+j+k]}x\right) - p(u,v)(-1)^nv_{[i-j+n+k]}(u_{[j]}x)\right]. \tag{5.1.9}$$

If $n, k \le -p-1$, then $-n-p-1+\chi(u,v) \ge 0$ and the sum on the right hand side of (5.1.9) runs over $0 \le i \le -n-p-1+\chi(u,v)$. Therefore $i-j+n+k \le -2p-2+\chi(u,v)$ and the $v_{[\cdot]}(u_{[\cdot]}x)$ terms all lie in $J_{p,g}$. If we put $n = -2p-2+\chi(u,v)$ and $k \le -p-1$

we see that

$$
\begin{aligned}
&(u_{[-2p-2+\chi(u,v)]}v)_{[k]}x \\
&\equiv \sum_{i,j\in\mathbb{Z}_+}(-1)^j\binom{-2p-2+\chi(u,v)}{j}\binom{p+1}{i}u_{[-2p-2+\chi(u,v)-j]}\left(v_{[i+j+k]}x\right) \pmod{J_{p,g}}.
\end{aligned}
$$

This implies that $\mathbb{C}\{u_{[-2p-2-\chi(u,v)]}v|[\epsilon_u]+[\epsilon_v]=[0]\}$ is a right ideal with respect to the products $u_{[k]}v$ for $k\leq -p-1$, and thus a right ideal with respect to $*_p$. Combining this with the calculation at the end of section 5.1.1 shows that $J_{p,g}$ is a right ideal of $(V_g, *_p)$.

5.1.3 The skew-symmetry formula for $Z(u,w)$

Recall the skew-symmetry formula [17],

$$Y(v,z)a = p(u,v)e^{zT}Y(u,-z)v$$

for all $u, v\in V$. In this section we recast this in terms of the modified field $Z(u,w)$.

If $x\in V_g$ then $Tx\equiv(-\Delta_x)x \pmod{J_{p,g}}$ by the definition of $J_{p,g}$, hence

$$
\begin{aligned}
T^{(k)}x &\equiv \binom{-\Delta_x}{k}x \\
\text{and}\qquad e^{zT}x &\equiv \sum_{j\in\mathbb{Z}_+}\binom{-\Delta_x}{j}xz^j=(1+z)^{-\Delta_x}x. \qquad (5.1.10)
\end{aligned}
$$

Here, and further, we often write $\equiv$ for $\equiv \operatorname{mod} J_{p,g}$.

Assume now that $[\epsilon_u]+[\epsilon_v]=[0]$; this implies that all $u_{(n)}v$ and $v_{(n)}u$ lie in V_g. From (5.1.10) we have

$$
\begin{aligned}
e^{zT}Y(u,-z)v &= \sum_{n\in\mathbb{Z}}(-z)^{-n-1}e^{zT}(u_{(n)}v) \\
&\equiv (1+z)^{-\Delta_u-\Delta_v}\sum_{n\in\mathbb{Z}}(-z)^{-n-1}(1+z)^{n+1}u_{(n)}v \\
&= (1+z)^{-\Delta_u-\Delta_v}Y\left(u,\frac{-z}{1+z}\right)v=(1+z)^{\epsilon_u-\Delta_v+p}Z\left(u,\frac{-z}{1+z}\right)v.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Z(v,z)u &= p(u,v)(1+z)^{\Delta_v+\epsilon_v+p}e^{zT}Y(u,-z)v \\
&\equiv p(u,v)(1+z)^{p-\gamma_u+\epsilon_u+\epsilon_v}Y\left(u,\frac{-z}{1+z}\right)v \qquad (5.1.11) \\
&\equiv p(u,v)(1+z)^{2p+\epsilon_u+\epsilon_v}Z\left(u,\frac{-z}{1+z}\right)v. \qquad (5.1.12)
\end{aligned}
$$

We expand (5.1.11) and (5.1.12) and equate coefficients to obtain the equations:

$$v_{[n]}u \equiv p(u,v)\sum_{j\in\mathbb{Z}_+}(-1)^{n+j+1}\binom{(p-\gamma_u+\epsilon_u+\epsilon_v)+1+n+j}{j}u_{(n+j)}v, \tag{5.1.13}$$

and $$v_{[n]}u \equiv p(u,v)\sum_{j\in\mathbb{Z}_+}(-1)^{n+j+1}\binom{2p+\epsilon_u+\epsilon_v+1+n+j}{j}u_{[n+j]}v, \tag{5.1.14}$$

respectively. We shall use (5.1.13) at the end of section 5.2.2, we need it there with $\epsilon_u \neq 0$. For now we use (5.1.14) and we only need it for $u, v \in V_g$. Let us write (5.1.14) in the slightly different form:

$$v_{[-p-1-m]}u \equiv p(u,v)(-1)^{p-m}\sum_{j\in\mathbb{Z}_+}\binom{p-m+j}{j}(-1)^j u_{[-p-1-m+j]}v, \tag{5.1.15}$$

for $u, v \in V_g$.

Substituting (5.1.15) into the definition of $v *_p u$ yields

$$v *_p u \equiv p(u,v)\sum_{m=0}^{p}\sum_{j\in\mathbb{Z}_+}\binom{-p-1}{m}\binom{p-m+j}{j}(-1)^{-p-m+j}u_{[-p-1-m+j]}v. \tag{5.1.16}$$

For $\alpha \in \mathbb{Z}$, let us consider the coefficient of $u_{[-p-1-\alpha]}v$ in (5.1.16). If $\alpha > p$ the coefficient is 0. If $\alpha \leq p$ the coefficient is

$$p(u,v)(-1)^{-p-\alpha}\sum_{j\in\mathbb{Z}_+}\binom{-p-1}{\alpha+j}\binom{p-\alpha}{j} = p(u,v)(-1)^{-p-\alpha}\binom{-\alpha-1}{p},$$

where we have used

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+}\binom{-p-1}{\alpha+j}\binom{p-\alpha}{j} &= [\xi^\alpha] : (1+\xi)^{-p-1}(1+\xi^{-1})^{p-\alpha} \\ &= [\xi^p] : (1+\xi)^{-p-1}(\xi+1)^{p-\alpha} = [\xi^p] : (1+\xi)^{-\alpha-1} \\ &= \binom{-\alpha-1}{p}.\end{aligned}$$

The terms in equation (5.1.16) with $0 \leq \alpha \leq p$ may be gathered together and, using $(-1)^n\binom{-m-1}{n} = (-1)^m\binom{-n-1}{m}$ for $m, n \in \mathbb{Z}_+$, reduced to $p(u,v)u *_p v$. The sum of the remaining terms (i.e., those with $\alpha < 0$) is

$$\begin{aligned}
p(u,v)\sum_{\alpha<0}(-1)^{-p-\alpha}\binom{-\alpha-1}{p}u_{[-p-1-\alpha]}v &= -p(u,v)\sum_{k\in\mathbb{Z}_+}(-1)^k\binom{p+k}{p}u_{[k]}v \\
&= -p(u,v)\sum_{k\in\mathbb{Z}_+}\binom{-p-1}{k}u_{[k]}v \\
&= -p(u,v)[u,v],
\end{aligned}$$

where $k = -p-1-\alpha$ and

$$\begin{aligned}
[u,v] = \mathrm{Res}_z(1+z)^{-p-1}Z(u,z)v &= \sum_{j\in\mathbb{Z}_+}\binom{-p-1}{j}u_{[j]}v \\
= \mathrm{Res}_z(1+z)^{\gamma_u-1}Y(u,z)v &= \sum_{j\in\mathbb{Z}_+}\binom{\gamma_u-1}{j}u_{(j)}v.
\end{aligned} \tag{5.1.17}$$

We have proved that if $u, v \in V_g$, then

$$u *_p v - p(u,v)v *_p u \equiv [u,v] \pmod{J_{p,g}}. \tag{5.1.18}$$

We call equation (5.1.18) the skew-symmetry formula.

5.1.4 The bracket $[\cdot,\cdot]$

In this section we prove that $J_{p,g}$ is a left ideal of $(V_g, *_p)$. This requires us to first prove some identities for $[\cdot,\cdot]$.

Lemma 5.1.5. *If $u, v_{[n]}x \in V_g$, (so that $\epsilon_u = 0$ and $[\epsilon_v] + [\epsilon_x] = [0]$), then*

$$[u, v_{[n]}x] = ([u,v])_{[n]}x + p(u,v)v_{[n]}([u,x]). \tag{5.1.19}$$

Proof. We begin with

$$\begin{aligned}
[u,(Z(v,w)x)] &= \mathrm{Res}_z(1+z)^{-p-1}Z(u,z)Z(v,w)x \\
&= p(u,v)\,\mathrm{Res}_z(1+z)^{-p-1}Z(v,w)Z(u,z)x \\
&\quad + \mathrm{Res}_z(1+z)^{-p-1}[Z(u,z),Z(v,w)]x \\
&= p(u,v)Z(v,w)[u,x] + \mathrm{Res}_z(1+z)^{-p-1}[Z(u,z),Z(v,w)]x.
\end{aligned}$$

Now we use (5.1.8) with $n = 0$ to expand the second term in the last line; it becomes

$$\begin{aligned}
&\mathrm{Res}_z(1+z)^{-p-1}\sum_{j\in\mathbb{Z}_+}(1+w)^{p+1+j-\chi(u,v)}Z(u_{[j]}v,w)x\partial_w^{(j)}(z,w)\\
&=\sum_{j\in\mathbb{Z}_+}(1+w)^{p+1+j-\chi(u,v)}Z(u_{[j]}v,w)x\,\mathrm{Res}_z(1+z)^{-p-1}\partial_w^{(j)}(z,w)\\
&=\sum_{j\in\mathbb{Z}_+}(1+w)^{p+1+j-\chi(u,v)}Z(u_{[j]}v,w)x\binom{-p-1}{j}(1+w)^{-p-1-j}\\
&=(1+w)^{-\chi(u,v)}Z([u,v],w)x.
\end{aligned}$$

So we have proved that

$$[u,(Z(v,w)x)]=p(u,v)Z(v,w)[u,x]+(1+w)^{-\chi(u,v)}Z([u,v],w)x.$$

If $\epsilon_u=0$, then $\chi(u,u')=0$ for all $u'\in V$. Extracting the w^{-n-1} coefficient yields equation (5.1.19). □

Let $u,v_{[n]}x\in V_g$, observe that $\chi(u_{(k)}v,x)=\chi(v,u_{(k)}x)=\chi(v,x)$. Using (5.1.18), (5.1.19) and the fact that $J_{p,g}$ is a right ideal of V_g, we have

$$\begin{aligned}
u*_p(v_{[-2p-2+\chi(v,x)]}x)&\equiv p(u,v)p(u,x)(v_{[-2p-2+\chi(v,x)]}x)*_p u+[u,(v_{[-2p-2+\chi(v,x)]}x)]\\
&\equiv p(u,v)p(u,x)(v_{[-2p-2+\chi(v,x)]}x)*_p u\\
&\quad+([u,v])_{[-2p-2+\chi(v,x)]}x+p(u,v)v_{[-2p-2+\chi(v,x)]}([u,x])\\
&\equiv 0.
\end{aligned}$$

Next observe that if $u,v\in V_g$ then

$$\begin{aligned}
[(T+H)u,v]&=\mathrm{Res}_z(1+z)^{\gamma_u-1}[Y(Tu,z)+\gamma_uY(u,z)]v\\
&=\mathrm{Res}_z\,\partial_z\,[(1+z)^{\gamma_u}Y(u,z)v]=0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
v*_p[(T+H)u]&\equiv p(u,v)[(T+H)u]*_p v-p(u,v)[(T+H)u,v]\\
&\equiv 0.
\end{aligned}$$

From the definition of $J_{p,g}$ in equation (5.0.8), and the remarks above we see that $(V_g)*_p(J_{p,g})\subseteq J_{p,g}$, i.e., $J_{p,g}$ is a left ideal in $(V_g,*_p)$. Since $J_{p,g}$ is also a right ideal, it is a 2-sided ideal.

5.1.5 $\mathrm{Zhu}_{p,g}(V)$ is associative

Let $u, v, x \in V_g$, we use (5.1.9) to expand $(u *_p v) *_p x$ directly and prove it equals $u *_p (b *_p x)$ modulo $J_{p,g}$. For brevity we put

$$D_r(v,x) = \sum_{m=0}^{p} \binom{-p-1}{m} v_{[-p-1-m+r]}x,$$

in particular, $D_0(v,x) = v *_p x$.

First we use (5.0.7) to expand $(u *_p v) *_p x$, then we apply equation (5.1.9) to each term of the form $(u_{[-p-1-m]}v)_{[-p-1-n]}x$. We obtain

$$\begin{aligned}(u *_p v) *_p x &= \sum_{m=0}^{p}\sum_{n=0}^{p} \binom{-p-1}{m}\binom{-p-1}{n}(u_{[-p-1-m]}v)_{[-p-1-n]}x \\ &\equiv \sum_{m=0}^{p}\sum_{i,j\in\mathbb{Z}_+} (-1)^j \binom{-p-1}{m}\binom{-p-1-m}{j}\binom{m}{i} u_{[-p-1-m-j]}D_{i+j}.\end{aligned} \tag{5.1.20}$$

We have omitted terms of the form $v_{[\cdot]}(u_{[\cdot]}x)$ here, because they lie in $J_{p,g}$ (see the remarks following equation (5.1.9)).

We change indices to $\beta = m + j$ and $\gamma = m - i$. Modulo $J_{p,g}$, the sum (5.1.20) becomes

$$\sum_{0\leq\gamma\leq m\leq\beta\leq p} (-1)^{\beta-m}\binom{-p-1}{m}\binom{-p-1-m}{\beta-m}\binom{m}{\gamma} u_{[-p-1-\beta]}D_{\beta-\gamma}. \tag{5.1.21}$$

The range of the indices in the summation (5.1.21) deserves explanation. Since $i \geq 0$, we have $\gamma = m - i \leq m$. Since $j \geq 0$, we have $\beta = m + j \geq m$. Since $\binom{m}{\gamma}$ appears in the summand, $\gamma \geq 0$. Terms with $\beta > p$ lie in $J_{p,g}$, so $\beta \leq p$. Thus the summation is over the range $0 \leq \gamma \leq m \leq \beta \leq p$.

Fix $\gamma, \beta \in \mathbb{Z}$ such that $0 \leq \gamma \leq \beta \leq p$. Then the coefficient of $u_{[-p-1-\beta]}D_{\beta-\gamma}$ in (5.1.21) is

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} \binom{-p-1}{m}\binom{p+\beta}{\beta-m}\binom{m}{\gamma} \\ =&\sum_{m\in\mathbb{Z}} \frac{(-p-1)\cdots(-p-m)}{m!}\cdot\frac{(p+\beta)!}{(\beta-m)!(p+m)!}\cdot\frac{m!}{\gamma!(m-\gamma)!} \\ =&\sum_{m\in\mathbb{Z}}(-1)^m\frac{(p+\beta)!}{(\beta-m)!p!}\cdot\frac{1}{\gamma!(m-\gamma)!} \\ =&\frac{(p+\beta)!}{p!\gamma!(\beta-\gamma)!}\sum_{m\in\mathbb{Z}}(-1)^m\binom{\beta-\gamma}{\beta-m}.\end{aligned}$$

The final sum here is just the sum of all the coefficients in the expansion of $(-1)^\beta(1-\xi)^{\beta-\gamma}$, which is 0 if $\gamma < \beta$ and 1 if $\gamma = \beta$. In the latter case we obtain

$$(-1)^\beta \frac{(p+\beta)!}{p!\beta!} = \binom{-p-1}{\beta}$$

as the coefficient. Therefore

$$\begin{aligned}(u *_p v) *_p x &\equiv \sum_{\beta=0}^{p} \binom{-p-1}{\beta} u_{[-p-1-\beta]}(v *_p x) \pmod{J_{p,g}} \\ &= u *_p (v *_p x).\end{aligned}$$

5.1.6 $\mathrm{Zhu}_{p,g}(V)$ is unital

By the vacuum axiom, we have $|0\rangle_{(n)}u = \delta_{n,-1}u$. Therefore

$$|0\rangle_{[n]}u = \sum_{j\in\mathbb{Z}_+} \binom{p}{j} |0\rangle_{(n+j)}u = \binom{p}{-n-1}u,$$

and hence

$$\begin{aligned}|0\rangle *_p u &= \sum_{m=0}^{p} \binom{-p-1}{m} |0\rangle_{[-p-1-m]}u \\ &= \sum_{m=0}^{p} \binom{-p-1}{m}\binom{p}{p+m} u = u.\end{aligned}$$

Note that $[|0\rangle, u] = 0$, so by skew-symmetry

$$u *_p |0\rangle \equiv |0\rangle *_p u \equiv u \pmod{J_{p,g}}$$

as well.

It is possible that $J_{p,g} = V_g$; in this case $\mathrm{Zhu}_{p,g}(V) = 0$. Suppose V has a nonzero g-twisted positive energy module M, we may assume that $M_0 \neq 0$ without loss of generality. The identity element $[|0\rangle] \in \mathrm{Zhu}_{0,g}(V)$ has nonzero action on M_0; hence $\mathrm{Zhu}_{0,g}(V) \neq 0$. The higher level Zhu algebras are all quotients of V_g by smaller ideals (see the next section), and so are also nonzero.

5.1.7 Homomorphisms between different Zhu algebras of V

To avoid confusion we write the level p n^{th} product as $u_{[n,p]}v$ in this section. Because

$$u_{[n,p]}v = \mathrm{Res}_z\, z^n(1+z)^{\gamma_u+p}Y(u,z)v,$$

we have

$$u_{[n,p]}v = u_{[n,p-1]}v + u_{[n+1,p-1]}v.$$

This implies $J_{p,g} \subseteq J_{p-1,g}$. Furthermore

$$\begin{aligned}
u *_p v &= \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-1-m,p]}v \\
&= \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-1-m,p-1]}v + \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-m,p-1]}v \\
&= \sum_{n=1}^{p+1} \binom{-p-1}{n-1} u_{[-p-n,p-1]}v + \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-m,p-1]}v \\
&\equiv \sum_{m=0}^{p} \binom{-p}{m} u_{[-p-m,p-1]}v \pmod{J_{p-1,g}} \\
&= u *_{p-1} v.
\end{aligned}$$

Hence, for all $p \geq 1$, the identity map on V induces a surjective homomorphism of associative algebras $\phi_p : \mathrm{Zhu}_{p,g}(V) \twoheadrightarrow \mathrm{Zhu}_{p-1,g}(V)$, i.e., the diagram

$$\begin{array}{ccc}
V_g & & \\
{\scriptstyle \pi_p}\downarrow & \searrow^{\pi_{p-1}} & \\
\mathrm{Zhu}_{p,g}(V) & \xrightarrow[\phi_p]{} & \mathrm{Zhu}_{p-1,g}(V)
\end{array} \tag{5.1.22}$$

commutes.

5.2 Representation theory

Let us write $\mathsf{PEMod}(g, V)$ for the category of g-twisted positive energy V-modules. Morphisms in $\mathsf{PEMod}(g, V)$ are linear maps $f : M_1 \to M_2$ such that

- $f(u^{M_1}_{(n)}x) = p(f, u)u^{M_2}_{(n)}f(x)$ for all $u \in V$, $x \in M_1$, $n \in [\gamma_u]$
- $\deg f(x) = \deg x$ for all $x \in M$.

If the degrees of all elements in a module are shifted by a fixed amount then the resulting module is essentially the same, but it is convenient for us not to identify such modules.

5.2.1 The Restriction Functor Ω_p

Definition 5.2.1 (Restriction functor Ω_p). For $M \in \mathsf{PEMod}(g, V)$ let $\Omega_p(M) = M_p$, endowed with an action of $\mathrm{Zhu}_{p,g}(V)$ via $[u]x = u^M_0 x$ (cf. section 5.0.5). If $f : M \to$

M' is a morphism in $\mathsf{PEMod}(g, V)$ then let $\Omega_p(f) = f|_{M_p}$. Ω_p is a functor from the category $\mathsf{PEMod}(g, V)$ to the category $\mathrm{Zhu}_{p,g}(V)$-mod of $\mathrm{Zhu}_{p,g}(V)$-modules.

The following lemma is adapted from [22], Lemma 6.1.1.

Lemma 5.2.1. *Let V be a VOSA with automorphism g, and let $M \in \mathsf{PEMod}(g, V)$. Fix $u, v \in V$ of homogeneous conformal weight, $x \in M$, $m \in [\epsilon_u]$, and $k \in [\epsilon_v]$. There exists $c \in V$ such that*

$$u^M_m(v^M_k x) = c^M_{m+k} x$$

(note c need not be of homogeneous conformal weight).

Proof. Consider equation (1.1.5) with u, v, x, m, k as above, and $n \in \mathbb{Z}$. Because $Y^M(u, z)$ and $Y^M(v, z)$ are quantum fields, there exist $\overline{m} \in [\epsilon_u]$ and $\overline{k} \in [\epsilon_v]$ such that $u^M_m x = v^M_k x = 0$ for $m \geq \overline{m}$, $k \geq \overline{k}$. The lemma obviously holds when $k \geq \overline{k}$.

Substitute $m = \overline{m}$ into (1.1.5) to get

$$\sum_{j\in\mathbb{Z}_+} \binom{\overline{m} + \Delta_u - 1}{j} (u_{(n+j)}v)^M_{\overline{m}+k} x = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} u^M_{\overline{m}+n-j} v^M_{k+j-n} x,$$

then put $k = \overline{k} + n - 1$ to obtain

$$\sum_{j\in\mathbb{Z}_+} \binom{\overline{m} + \Delta_u - 1}{j} (u_{(n+j)}v)^M_{\overline{m}+\overline{k}+n-1} x = u^M_{\overline{m}+n} v^M_{\overline{k}-1} x.$$

Hence the lemma is true for $k = \overline{k} - 1$ too, we simply let

$$c = \sum_{j\in\mathbb{Z}_+} \binom{\overline{m} + \Delta_u - 1}{j} u_{(m-\overline{m}+j)} v.$$

Now put $m = \overline{m}$ and $k = \overline{k} + n - 2$ to obtain

$$\sum_{j\in\mathbb{Z}_+} \binom{\overline{m} + \Delta_u - 1}{j} (u_{(n+j)}v)^M_{\overline{m}+\overline{k}+n-2} x = u^M_{\overline{m}+n} v^M_{\overline{k}-2} x - n u^M_{\overline{m}+n-1} v^M_{\overline{k}-1} x.$$

Since we can write $u^M_{\overline{m}+n-1} v^M_{\overline{k}-1} x$ as $c^M_{\overline{m}+\overline{k}+n-2} x$, we can now do the same for $u^M_{\overline{m}+n} v^M_{\overline{k}-2} x$.

The general case follows inductively, we write any term of the form $u^M_m v^M_k x$ as a linear combination of terms of the form $(c^i)^M_{r_i} x$. The graded structure of M implies that for all such terms $r_i = m + k$, so the linear combination can be taken to be the single term $c = \sum_i c^i$. □

Proposition 5.2.2. *If $M \in \mathsf{PEMod}(g, V)$ is irreducible then $\Omega_p(M)$ is either 0 or it is irreducible.*

Proof. Suppose to the contrary that $N = \Omega_p(M)$ has a proper $\mathrm{Zhu}_{p,g}(V)$-submodule N'. From Lemma 5.2.1 we see that the V-submodule M' of M generated by N' is the span of elements $u_n^M x$ where $u \in V$, $n \in [\epsilon_u]$, and $x \in N$. But then

$$M'_p = (V_g)_0 N' = \mathrm{Zhu}_{p,g}(V) N' = N' \subsetneq N = M_p,$$

which contradicts the irreducibility of M. □

Remark 5.2.3. Via the surjective homomorphisms ϕ_{j+1} (more generally $\phi_{P,Q}$ to be introduced in Section 5.4 for all $Q \leq P$), M_q acquires the structure of a $\mathrm{Zhu}_{p,g}(V)$-module for each $q \leq p$. Let $M \in \mathsf{PEMod}(g, V)$ be irreducible and $0 \leq q \leq p$. The $\mathrm{Zhu}_{p,g}(V)$-module M_q is either 0 or irreducible for the same reason as in Proposition 5.2.2. Assume V, and therefore M_0, has countable dimension. Since $L_0 : M_0 \to M_0$ commutes with u_0 for all $u \in V_g$, we have, by the Dixmier-Schur lemma, $L_0^M|_{M_0} = hI_{M_0}$ for some constant $h \in \mathbb{C}$. Now let $M_j \neq 0$. Because of Lemma 5.2.1, and the fact that M is irreducible, there exists $u \in V$ such that $u_{-j}^M|_{M_0} : M_0 \to M_j$ is nonzero. From (1.1.4) we see $L_0^M|_{M_j} = (h + j)I_{M_j}$. The conclusion is that those $\mathrm{Zhu}_{p,g}(V)$-modules M_q $(0 \leq q \leq p)$ which are nonzero are pairwise non-isomorphic. Because L_0 acts diagonally on each with distinct eigenvalues.

5.2.2 The Induction Functors M^p and L^p

Let us fix a $\mathrm{Zhu}_{p,g}(V)$-module N. In this section we construct (in a functorial manner) a module $M \in \mathsf{PEMod}(g, V)$ such that $M_p = N$.

Lemma 5.2.4. *The linear map* $\varphi_p : (\mathrm{Lie}^g V)_0 \to \mathrm{Zhu}_{p,g}(V)$ *defined by* $u_0 \mapsto [u]$ *is well defined and is a surjective homomorphism of Lie superalgebras, where* $(\mathrm{Lie}^g V)_0$ *has the Lie bracket (5.0.3), and* $\mathrm{Zhu}_{p,g}(V)$ *has the commutator bracket* $[u, v] = u *_p v - p(u, v) v *_p u$.

Furthermore, the following diagram (of Lie superalgebra homomorphisms) commutes:

$$\begin{array}{ccc} (\mathrm{Lie}^g V)_0 & & \\ \downarrow{\scriptstyle \varphi_p} & \searrow^{\varphi_{p-1}} & \\ \mathrm{Zhu}_{p,g}(V) & \xrightarrow[\phi_p]{} & \mathrm{Zhu}_{p-1,g}(V). \end{array} \tag{5.2.1}$$

Proof. Recall definition (5.0.2) of the Lie superalgebra $\mathrm{Lie}^g V$ as the quotient $Q/(T + H + \partial_t)Q$ where Q is spanned by elements of the form ut^m, $m \in [\epsilon_u]$. Let $f : V_g \to (\mathrm{Lie}^g V)_0$ be defined by $u \mapsto u_0$. The degree 0 piece of $(T + H + \partial_t)Q$ is spanned by elements of the form $(Tu)_0 + \Delta_u u_0$ where $u \in V_g$, hence it is contained in $f(J_{p,g})$. Therefore the canonical surjection $V_g \twoheadrightarrow \mathrm{Zhu}_{p,g}(V)$ factors to give a linear map $\varphi_p : (\mathrm{Lie}^g V)_0 \twoheadrightarrow \mathrm{Zhu}_{p,g}(V)$.

Let $u, v \in V_g$. Equation (5.0.3) with $m = n = 0$ is

$$[u_0, v_0] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_u - 1}{j} (u_{(j)} v)_0.$$

Equations (5.1.17) and (5.1.18), which are

$$[u,v] = \sum_{j\in\mathbb{Z}_+} \binom{\Delta_u - 1}{j} u_{(j)}v$$

and

$$u *_p v - p(u,v)v *_p u \equiv [u,v] \pmod{J_{p,g}},$$

respectively, combine to imply that $\varphi_p([u_0, v_0]) = u *_p v - p(u,v)v *_p u$, i.e., that φ_p is a Lie superalgebra homomorphism. Diagram (5.2.1) is commutative because the maps in question are all induced from the identity map on V_g. □

From now on we write $\mathfrak{g} = \mathrm{Lie}^g V$. Because of Lemma 5.2.4 our $\mathrm{Zhu}_{p,g}(V)$-module N is naturally a $\mathfrak{g}_0$-module. Consider the graded Lie subalgebra $\mathfrak{g}_+ = \mathfrak{g}_0 + \mathfrak{g}_{>p}$ where $\mathfrak{g}_{>p} = \oplus_{j>p}\mathfrak{g}_j$. We extend the representation of $\mathfrak{g}_0$ on N to a representation of $\mathfrak{g}_+$ by letting $\mathfrak{g}_{>p}$ act by 0. Then we induce from $\mathfrak{g}_+$ to $\mathfrak{g}$ to get the $\mathfrak{g}$-module

$$\tilde{M} = \mathrm{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} N = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+)} N.$$

We make $\tilde{M}$ into a graded $\mathfrak{g}$-module by declaring that $\deg N = p$ and that u_n lowers degree by n.

We write u_n^M for the image of u_n in $\mathrm{End}\,\tilde{M}$, and we put

$$Y^M(a,z) = \sum_{n\in[\epsilon_u]} u_n^M z^{-n-\Delta_a} \in (\mathrm{End}\,M)[[z,z^{-1}]]z^{-\gamma_u}.$$

We claim $Y^M(u,z)$ is a quantum field, i.e., that $u_n^M \mathbf{m} = 0$ for $n \gg 0$ for all $\mathbf{m} \in M$. The proof is by induction on the length of the monomial $\mathbf{m}$. By construction $u_n^M x = 0$ whenever $n > p$ and $x \in N$, this is the base case. Fix $s \in \mathbb{Z}_+$ and suppose that the claim holds for all length s monomials

$$\mathbf{m} = (v^1)_{n_1}^M (v^2)_{n_2}^M \cdots (v^s)_{n_s}^M x,$$

i.e., for all $v \in V$ we have $v_n^M \mathbf{m} = 0$ for $n \gg 0$. Let $u, v \in V$ and $k \in [\epsilon_v]$ be fixed now. By equation (5.0.3) we have

$$u_m^M v_k^M \mathbf{m} = v_k^M u_m^M \mathbf{m} + \sum_{j\in\mathbb{Z}_+} \binom{m+\Delta_u - 1}{j} (u_{(j)}v)_{m+k}^M \mathbf{m},$$

where the sum is finite because $Y(u,z)$ is a quantum field. By the inductive assumption, each of these finitely many terms vanishes for $m \gg 0$. Hence the claim holds for length $s+1$ monomials.

Let $F_r = \oplus_{j\geq r}\mathfrak{g}_j$ for $r \in \Gamma \cap \mathbb{R}_+$. We define a topology on $U(\mathfrak{g})$ by declaring the subsets $\{U(\mathfrak{g})F_r\}_{r\geq 0}$ to be a fundamental system of neighborhoods of 0. The multiplication in $U(\mathfrak{g})$ is continuous with respect to this topology, so $U(\mathfrak{g})$ becomes a topological algebra. Let

$$\hat{U} = \varprojlim (U(\mathfrak{g})/U(\mathfrak{g})F_r)$$

be the completion of $U(\mathfrak{g})$ with respect to this topology. Since each $U(\mathfrak{g})F_N$ is a left ideal, $\hat{U}$ is naturally a left $U(\mathfrak{g})$-module. We claim that $\hat{U}$ is also an algebra. We may identify $\hat{U}$ with the space of infinite sums of monomial elements of $U(\mathfrak{g})$ in which only finitely many terms lie outside $U(\mathfrak{g})F_N$ for each $N \geq 0$. Let $x = \sum_{i \in \mathbb{Z}_+} x^{(i)}$ and $y = \sum_{j \in \mathbb{Z}_+} y^{(j)}$ be two such sums. If $y^{(j)} \in U(\mathfrak{g})F_N$ then clearly $x^{(i)}y^{(j)} \in U(\mathfrak{g})F_N$. Using the commutation relations one may check the following: for each of the other finitely many $y^{(j)}$, there exists $N^{(j)}$ such that if $x^{(i)} \in U(\mathfrak{g})F_{N^{(j)}}$ then $x^{(i)}y^{(j)} \in U(\mathfrak{g})F_N$. So only finitely many $x^{(i)}y^{(j)}$ lie outside $U(\mathfrak{g})F_N$. The product of x and y is defined term-by term, and is a well-defined element of $\hat{U}$.

For $u, v \in V$, $m \in [\epsilon_u]$, $k \in [\epsilon_v]$, and $n \in \mathbb{Z}$, let

$$BI(u,v;m,k;n) = \sum_{j \in \mathbb{Z}_+} \binom{m + \Delta_u - 1}{j} (u_{(n+j)}v)_{m+k} - \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} \left[u_{m+n-j}v_{k+j-n} - (-1)^n p(u,v) v_{k-j}u_{m+j} \right], \tag{5.2.2}$$

which we may think of as an element of $\hat{U}$ because, for each r, all but finitely many terms lie in $U(\mathfrak{g})F_r$. Let $\mathcal{B} \subseteq \hat{U}$ be the span of the terms $|0\rangle_n - \delta_{n,0}1$ for $n \in \mathbb{Z}$ and $BI(u,v;m,k;n)$ as u, v, m, and k range over all their possible values, and let $S = \mathcal{B}\tilde{M} \subseteq \tilde{M}$. Because $Y(u,z)$, $Y^M(u,z)$, and $Y^M(v,z)$ are quantum fields, elements of $\mathcal{B}\tilde{M}$ are finite sums, hence S is well-defined[1].

In Lemma 2.26 of [4] it is proved by direct calculation that $[u_s, \mathcal{B}] \subset \mathcal{B}$ for all $u \in V$, $s \in [\epsilon_u]$. This implies that S is an $\mathfrak{g}$-submodule of $\tilde{M}$, so let $M = \tilde{M}/S$. On the quotient M, $Y^M(u,z)$ is a quantum field, and the vacuum and Borcherds identities are satisfied. Therefore M is a V-module.

Lemma 5.2.1, and the fact that $\mathfrak{g}_{>p}$ annihilates N, imply that M has no pieces of negative degree. Therefore M is actually a g-twisted positive energy V-module. We claim now that $M_p = N$.

Let $x \in N$ and $y = (u^1)^M_{n_1} \cdots (u^s)^M_{n_s} x \in M_p$ (so $\sum n_i = 0$). By Lemma 5.2.1, $y = s + x'$ where $s \in S_p$ and $x' = v^M_0 x \in N$ for some $v \in V_g$. Therefore $\tilde{M}_p = S_p + N$. Since we want to show that this sum is direct, it suffices to show that $S_p \cap N = 0$. Our strategy for proving this is a hybrid of the strategies of [7] and [4].

Following [7] we introduce a bilinear form $\langle \cdot, \cdot \rangle : N^* \times \tilde{M} \to \mathbb{C}$.

Definition 5.2.2. Let $\psi \in N^*$. If $x \in N$, set $\langle \psi, x \rangle = \psi(x) \in \mathbb{C}$, i.e., $\langle \cdot, \cdot \rangle$ restricts to the canonical form on $N^* \times N$. For each n such that $n \neq 0$ and $n \leq p$, fix an ordered basis B_n of $\mathfrak{g}_n$; their union B is a basis of a subspace of $\mathfrak{g}$ complementary

[1] We could, therefore, have avoided the introduction of the completion $\hat{U}$ here. However, $\hat{U}$ will appear in a later section.

to $\mathfrak{g}_+$. Let $s \geq 2$ be an integer, and let $a^1_{n_1}, \ldots a^s_{n_s} \in B$ be ordered lexicographically, i.e., $n_1 \leq n_2 \leq \ldots \leq n_s$ and if consecutive n_i are equal then the corresponding a^i are in increasing order in B_{n_i}. Suppose also that $\sum_i n_i = 0$. Choose $c \in V$ such that $c_{n_1+n_2} = a^1_{n_1} a^2_{n_2}$ (Lemma 5.2.1 states that such an element exists, it may not be unique though) and define

$$\left\langle \psi, (a^1)^M_{n_1} \cdots (a^s)^M_{n_s} x \right\rangle = \left\langle \psi, c^M_{n_1+n_2} (a^3)^M_{n_3} \cdots (a^s)^M_{n_s} x \right\rangle. \tag{5.2.3}$$

If $s = 2$ then $n_1 + n_2 = 0$ and $c^M_0 x \in N$. If $s \geq 3$ then equation (5.2.3) defines $\langle \cdot, \cdot \rangle$ inductively. If $q \neq p$ and $y \in \tilde{M}_q$ then set $\langle \psi, y \rangle = 0$. Finally we extend the definition to all elements of M linearly.

The bilinear form $\langle \cdot, \cdot \rangle$ is well-defined because of the PBW theorem, i.e., we defined it on a basis of $\tilde{M}$. In [7], choices of elements c were given by an explicit formula. We omit such a formula, since we do not require it.

For any $u, v \in V$ with $[\epsilon_u] + [\epsilon_v] = [0]$, and for all $\psi \in N^*$, $x \in N$ we show below that

$$\langle \psi, BI(u, v; p + 1 + \epsilon_u, -(p + 1 + \epsilon_u); -1) x \rangle = 0. \tag{5.2.4}$$

The proof uses the fact that N is a $\mathrm{Zhu}_{p,g}(V)$-module and it uses our particular choice of $\langle \cdot, \cdot \rangle$. Since it is fairly involved we give it at the end of this section and for now we continue with the main argument.

By Lemma 2.27 of [4], $\mathcal{B}$ is spanned by elements of the form $BI(u, v; m_u, k; -1)$, where u and v range over V, k ranges over $[\epsilon_v]$, and m_u is a fixed number in $[\epsilon_u]$ (which can be chosen as we please for each u). From equation (5.2.4) and this lemma it follows that $\langle \psi, \mathcal{B} N \rangle = 0$.

Recall the notion of a local pair. Let U be a vector superspace and let $u(w)$, $v(w)$ be $\mathrm{End}\, U$-valued quantum fields. We say the pair $(u(w), v(w))$ is *local* if there exists $n \in \mathbb{Z}_+$ such that

$$(z - w)^n [a(z), v(w)] = 0.$$

If $u(w) = \sum_{n \in [\gamma_u]} u_{(n)} w^{-n-1}$ and $v(w) = \sum_{n \in [\gamma_v]} v_{(n)} w^{-n-1}$ then locality of the pair $(u(w), v(w))$ is equivalent to the existence of a finite collection of quantum fields $c^j(w)$, $j = 0, 1, \ldots N$, such that

$$[u_{(m)}, v_{(k)}] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} c^j_{(m+k-j)}$$

(see [17] and [4]). In particular, the quantum fields $Y^M(u, w)$ are pairwise local because of equation (5.0.3). There is now the important lemma of Dong (see [17] for a proof).

Lemma 5.2.5 (Dong's Lemma). *Let $u(w)$, $v(w)$ and $x(w)$ be pairwise local quantum* *fields. Then $u(w)$ and the g-twisted n^{th} product*[2]

$$v(w)_{(n,g)}x(w) = \mathrm{Res}_z(zw^{-1})^{m+\Delta_v-1}\left[v(z)x(w)i_{z,w} - p(u,v)x(w)v(z)i_{w,z}\right](z-w)^n$$

also form a local pair.

Let[3]

$$\begin{aligned} BI^M(u,v;m;n;w) &= \sum_{k\in\epsilon_v} w^{-m-\Delta_u-k-\Delta_v+n-1}BI(u,v;m,k;n) \\ &= \sum_{j\in\mathbb{Z}_+}\binom{m+\Delta_u-1}{j}Y^M(u_{(n+j)}v,w)w^{-j} \\ &\quad - Y^M(u,z)_{(n,g)}Y^M(v,w). \end{aligned}$$

Dong's lemma, together with the finiteness of the sum in the second line, implies that $BI^M(u,v;m;n;w)$ is a quantum field which forms a local pair with any quantum field $Y^M(x,w)$.

The following is Lemma 2.23 of [4].

Lemma 5.2.6 (Uniqueness Lemma). *Let U be a vector superspace, $\mathcal{F} = \{u^i(w)|i \in I\}$ a collection of* End U*-valued quantum fields that are pairwise local, and $v(w)$ an* End U*-valued quantum field which forms a local pair with each element of $\mathcal{F}$. Suppose W is a* generating subspace *of U, meaning that U is spanned by vectors of the form $u^{i_1}_{(n_1)}\cdots u^{i_s}_{(n_s)}x$ where $x \in W$. If $v(w)W = 0$ then $v(w)U = 0$.*

Let us take $U = \tilde{M}$, $W = N$, $\mathcal{F} = \{Y^M(u,w)|u \in V\}$. Let $\psi \in N^*$ and write $v(w) = BI(u,v;m;n;w)$. We saw above that $\langle\psi, \mathcal{B}N\rangle = 0$, therefore Lemma 5.2.6 implies

$$\left\langle \psi, \mathcal{B}\tilde{M} \right\rangle = 0. \tag{5.2.5}$$

Strictly speaking this follows from a modification of Lemma 5.2.6 in which every occurrence of a vector $v \in U$ is replaced with $\langle\psi, v\rangle$. But the proof is formally identical to the actual proof given in [4].

Since $\langle\cdot,\cdot\rangle$ restricts to the canonical pairing of N^* with N, and (5.2.5) holds for all $\psi \in N^*$, we obtain $N \cap S_p = 0$.

Definition 5.2.3 (Induction functor M^p). If $N \in \mathrm{Zhu}_{p,g}(V)$-mod, we define $M^p(N) = M \in \mathsf{PEMod}(g,V)$ as constructed above. If $g : N \to N'$ is a homomorphism of $\mathrm{Zhu}_{p,g}(V)$-modules then we define $M^p(g) : M^p(N) \to M^p(N')$ as follows: g is also a homomorphism of $\mathfrak{g}_+$-modules. Let $\tilde{M} = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_+} N$ and $\tilde{M}' = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_+} N'$, $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_+}$ is a Lie algebra induction functor, so g lifts to a unique homomorphism $\tilde{g} : \tilde{M} \to \tilde{M}'$ of

[2]See Remark 2.20 of [4] for an explanation of g-twisted n^{th} products. In particular, of the difference between this definition and equation (5.1.5).

[3]We are abusing notation slightly by using the symbol BI, but with u_n replaced by u^M_n.

$\mathfrak{g}$-modules such that $\tilde{g}|_N : N \to N'$ coincides with g. Now

$$\tilde{g}(S) = \tilde{g}(\mathcal{B}\tilde{M}) = \mathcal{B}\tilde{g}(\tilde{M}) \subseteq \mathcal{B}\tilde{M}' = S'$$

so there is an induced map $g : M \to M'$ and we put $M^p(g) = g$. M^p is a functor from $\mathrm{Zhu}_{p,g}(V)$-mod to $\mathsf{PEMod}(g, V)$.

Let $I \subset \tilde{M}$ be the maximal graded $\mathfrak{g}$-submodule of M whose intersection with N is 0 and let $L = \tilde{M}/I$. Because S is a graded submodule of $\tilde{M}$ meeting N trivially we know I exists and contains S, which in turn implies that L is a V-module. The quotient map $\tilde{M}/S \twoheadrightarrow \tilde{M}/I$ is a homomorphism of V-modules. Consequently L is a g-twisted positive energy V-module with $L_p = N$, and such that every graded V-submodule of L intersects L_p non-trivially.

Definition 5.2.4. For any $N \in \mathrm{Zhu}_{p,g}(V)$-$\mathsf{mod}$, let $L^p(N) = L \in \mathsf{PEMod}(g, V)$. A homomorphism $g : N \to N'$ extends to a homomorphism $\tilde{g} : \tilde{M} \to \tilde{M}'$. Clearly $\tilde{g}(I) \subseteq I'$, we let $L^p(g)$ be the induced map $g : L^p(N) \to L^p(N')$. L^p is a functor from $\mathrm{Zhu}_{p,g}(V)$-mod to $\mathsf{PEMod}(g, V)$.

To complete the construction of the induction functor it remains to prove equation (5.2.4). This follows from the following lemma, which the rest of this section is devoted to proving.

Lemma 5.2.7. *Let N be a $\mathrm{Zhu}_{p,g}(V)$-module and let $\psi \in N^*$. For all $u, v \in V$ such that $[\epsilon_u] + [\epsilon_v] = 0$ and $x \in N$ we have*

$$\langle \psi, (u_{[-1]}v)_0^M x - \sum_{j \in \mathbb{Z}_+} u^M_{p+\epsilon_u - j} v^M_{-p-\epsilon_u + j} x \rangle = 0. \tag{5.2.6}$$

Equivalently

$$\langle \psi, (u_{[-1]}v)_0^M x \rangle = \sum_{k=0}^{p-\chi(u,v)} \langle \psi, u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x \rangle + \sum_{k=1}^{p} \langle \psi, u^M_{k+\epsilon_u} v^M_{-k-\epsilon_u} x \rangle, \tag{5.2.7}$$

because $v^M_n x = 0$ when $n > p$.

Evaluating $\langle \psi, u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x \rangle$ for $k - \epsilon_u > 0$ involves a choice of $c \in V$ such that $u_{-k+\epsilon_u} v_{k-\epsilon_u} = c_0$. We cannot evaluate $\langle \psi, u^M_{k+\epsilon_u} v^M_{-k-\epsilon_u} x \rangle$, where $k + \epsilon_u > 0$, in the same way, because the monomial is not ordered correctly. We must first use the commutation relations of $\mathfrak{g}$ to rewrite $u^M_{k+\epsilon_u} v^M_{-k-\epsilon_u}$ in terms of our PBW basis.

The following lemma gives us a choice of c to use henceforth.

Lemma 5.2.8. *Let V be a VOSA with automorphism g, and let $M \in \mathsf{PEMod}(g, V)$. Suppose $x \in M_p$, $u, v \in V$ satisfy $[\epsilon_u] + [\epsilon_v] = [0]$, and k is an integer such that $0 \leq k \leq p + \epsilon_u$. We have*

$$u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x = \sum_{m=0}^{p-k} \binom{-p-1-k}{m} (u_{[-p-1-k-m]}v)_0^M x. \tag{5.2.8}$$

Proof. We substitute equation (5.0.5) into the right hand side of equation (5.2.8), and remove terms of the form $u_n^M x$ for $n > p$, to obtain

$$\sum_{\substack{m,j\in\mathbb{Z}_+,\\ 0\leq j+m\leq p-k}} (-1)^j \binom{-p-1-k}{m}\binom{-p-1-k-m}{j} u^M_{-k-m-j+\epsilon_u} v^M_{k+m+j-\epsilon_u} x.$$

For an integer α such that $0 \leq \alpha \leq p-k$, the coefficient of $u^M_{-k-\alpha+\epsilon_u} v^M_{k+\alpha-\epsilon_u} x$ above is

$$\begin{aligned}\sum_{j=0}^{\alpha}(-1)^j\binom{-p-1-k}{\alpha-j}\binom{-p-1-k-\alpha+j}{j} &= \sum_{j=0}^{\alpha}\binom{-p-1-k}{\alpha-j}\binom{p+k+\alpha}{j}\\ &= [\xi^\alpha] : (1+\xi)^{-p-1-k}(1+\xi)^{p+k+\alpha}\\ &= [\xi^\alpha] : (1+\xi)^{\alpha-1}.\end{aligned}$$

This is 1 if $\alpha = 0$ and 0 if $\alpha > 0$. This proves the lemma. □

We now have

$$\langle \psi, u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x\rangle = \sum_{m=0}^{p-k}\binom{-p-1-k}{m}\langle \psi, (u_{[-p-1-k-m]}v)^M_0 x\rangle \tag{5.2.9}$$

for *all* $k \in \mathbb{Z}$. If $\epsilon_u = k = 0$ this equation follows from the fact that N is a $\mathrm{Zhu}_{p,g}(V)$-module; for other values of ϵ_u and k, it follows from Lemma 5.2.8 and Definition 5.2.2. For brevity let us omit $\langle \psi, \cdot\rangle$ for the rest of this section.

Armed with equation (5.2.9), we rewrite the first summation on the right hand side of equation (5.2.7) as

$$\sum_{k=0}^{p-\chi(u,v)}\sum_{m=0}^{p-k}\binom{-p-1-k}{m}(u_{[-p-1-k-m]}v)^M_0 x = \sum_{\alpha=0}^{p-\chi(u,v)}\binom{-p}{\alpha}(u_{[-p-1-\alpha]}v)^M_0 x. \tag{5.2.10}$$

To see this equality we first suppose $\chi(u,v) = 0$. For an integer α such that $0 \leq \alpha \leq p$, the coefficient of $(u_{[-p-1-\alpha]}v)^M_0$ is

$$\begin{aligned}\sum_{k=0}^{\alpha}\binom{-p-1-k}{\alpha-k} &= [\xi^\alpha] : (1+\xi)^{-p-1}\sum_{k=0}^{\infty}\xi^k(1+\xi)^{-k}\\ &= [\xi^\alpha] : (1+\xi)^{-p} = \binom{-p}{\alpha}\end{aligned} \tag{5.2.11}$$

(using the geometric series formula), so equation (5.2.10) is true. When $\chi(u,v) = 1$, equation (5.2.11) is valid for $0 \leq \alpha \leq p-1$, but not $\alpha = p$. However, N is a $\mathrm{Zhu}_{p,g}(V)$-module, so $(u_{[-2p-2+\chi(u,v)]}v)^M_0 x = (u_{[-2p-1]}v)^M_0 x = 0$ for $x \in N$. Therefore (5.2.10) holds if $\chi(u,v) = 1$ too.

Substituting equation (5.1.1) into the right hand side of equation (5.2.10) reduces it to

$$\sum_{\alpha=0}^{p-\chi(u,v)} \binom{-p}{\alpha} \left[\sum_{i\in\mathbb{Z}_+} \binom{\gamma_u + p}{i} (u_{(-p-1-\alpha+i)}v)_0^M \right] x$$

$$= \sum_{j\in\mathbb{Z}} \left[\sum_{\alpha=0}^{p-\chi(u,v)} \binom{-p}{\alpha} \binom{\gamma_u + p}{p+1+j+\alpha} \right] (u_{(j)}v)_0^M x. \tag{5.2.12}$$

Now we turn our attention to the second sum on the right hand side of equation (5.2.7). First apply equation (5.0.3) to get

$$\sum_{k=1}^{p} u_{k+\epsilon_u}^M v_{-k-\epsilon_u}^M x = p(u,v) \sum_{k=1}^{p} v_{-k-\epsilon_u}^M u_{k+\epsilon_u}^M x + \sum_{k=1}^{p} \sum_{j\in\mathbb{Z}_+} \binom{k+\gamma_u-1}{j} (u_{(j)}v)_0^M x. \tag{5.2.13}$$

We put $l = k - \chi(u,v)$ and use $\epsilon_u = -\epsilon_v - \chi(u,v)$ to write the first summation on the right hand side of (5.2.13) as

$$p(u,v) \sum_{l=1-\chi(u,v)}^{p-\chi(u,v)} v_{-l+\epsilon_v}^M u_{l-\epsilon_v}^M x.$$

We now repeat calculation (5.2.10)-(5.2.11) to reduce this summation to

$$p(u,v) \sum_{\alpha=0}^{p-\chi(u,v)} \left[\binom{-p}{\alpha} - \delta_{0,\chi(u,v)} \binom{-p-1}{\alpha} \right] (v_{[-p-1-\alpha]}u)_0^M x$$

$$= p(u,v) \sum_{\alpha=0}^{p-\chi(u,v)} \binom{-p-1+\chi(u,v)}{\alpha-1+\chi(u,v)} (v_{[-p-1-\alpha]}u)_0^M x. \tag{5.2.14}$$

Equation (5.1.13), together with the fact that N is a $\mathrm{Zhu}_{p,g}(V)$-module, implies

$$(v_{[-p-1-\alpha]}u)_0 x = p(u,v)(-1)^{p-\alpha} \sum_{j\in\mathbb{Z}} \binom{\alpha+\gamma_u-1+\chi(u,v)}{j+p+1+\alpha} (u_{(j)}v)_0^M x.$$

Plugging this into equation (5.2.14) yields

$$\sum_{j\in\mathbb{Z}} \left[\sum_{\alpha=0}^{p-\chi(u,v)} (-1)^{p+\alpha} \binom{-p-1+\chi(u,v)}{\alpha-1+\chi(u,v)} \binom{\alpha+\gamma_u-1+\chi(u,v)}{p+1+j+\alpha} \right] (u_{(j)}v)_0^M x. \tag{5.2.15}$$

Combining (5.2.12) with (5.2.15) renders the right hand side of equation (5.2.7) equal to

$$\sum_{k=1}^{p}\sum_{j\in\mathbb{Z}}\binom{k+\gamma_u-1}{j}(u_{(j)}v)_0^M x+\sum_{j\in\mathbb{Z}}\sum_{\alpha=0}^{p-\chi(u,v)}\left[\binom{-p}{\alpha}\binom{\gamma_u+p}{p+1+j+\alpha}\right.$$
$$\left.+(-1)^{p+\alpha}\binom{-p-1+\chi(u,v)}{\alpha-1+\chi(u,v)}\binom{\alpha+\gamma_u-1+\chi(u,v)}{p+1+j+\alpha}\right](u_{(j)}v)_0^M x.$$

The left hand side of (5.2.7) is

$$\sum_{j\in\mathbb{Z}}\binom{\gamma_u+p}{j+1}(u_{(j)}v)_0^M x.$$

Before proving the equality of these two expressions in general, we dispense with the special case $p=0$, $\chi(u,v)=1$. In this case the left hand side vanishes because N is a $\mathrm{Zhu}_{0,g}(V)$-module and the summation of the right hand side is over an empty range. We exclude this case from now on.

In the other cases we prove equality for the coefficient of $(u_{(j)}v)_0^M$ and thus for the whole expression.

We claim that

$$\sum_{k=1}^{p}\binom{k+\gamma-1}{j}=\binom{\gamma+p}{j+1}-\binom{\gamma}{j+1}. \tag{5.2.16}$$

Indeed the left hand side of (5.2.16) is

$$[\xi^j]:(1+\xi)^\gamma\sum_{k=0}^{p-1}(1+\xi)^k=[\xi^j]:(1+\xi)^\gamma\frac{1-(1+\xi)^p}{1-(1+\xi)}$$
$$=[\xi^j]:\frac{(1+\xi)^\gamma-(1+\xi)^{p+\gamma}}{-\xi},$$

which equals the right hand side.

It therefore remains to prove that

$$\binom{\gamma_u}{j+1}=\sum_{k=0}^{p-\chi(u,v)}\left[\binom{-p}{k}\binom{\gamma+p}{p+j+k+1}\right.$$
$$\left.+(-1)^{p+k}\binom{-p-1+\chi(u,v)}{k-1+\chi(u,v)}\binom{k+\gamma-1+\chi(u,v)}{p+1+j+k}\right] \tag{5.2.17}$$

for $p,j\in\mathbb{Z}_+$, γ arbitrary and $(p,\chi)\neq(0,1)$. Equation (5.2.17) is a special case of Lemma 5.2.9 below (with $(\gamma,n,X,Y)=(\gamma_u,j+1,p,p+1-\chi(u,v))$).

Lemma 5.2.9. *For $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}$, and $X, Y \in \mathbb{Z}$ with $X \geq 0$, $Y \geq 1$, let*

$$H_{\gamma,n}(X,Y) = \sum_{k=0}^{Y-1} \binom{-X}{k} \binom{\gamma + X}{n + X + k}$$
$$\text{and} \quad D_{\gamma,n}(X,Y) = \sum_{k=0}^{X-1} (-1)^{Y+k} \binom{-Y}{k} \binom{\gamma + k}{n + Y + k}.$$

Then

$$H_{\gamma,n}(X,Y) + D_{\gamma,n}(X,Y) = \binom{\gamma}{n}.$$

Proof. Fix γ and n, for a pair (X,Y) call the claim $P(X,Y)$. For $Y \geq 1$, $P(0,Y)$ is clear. Suppose $P(X,Y)$ holds for some (X,Y), we will deduce $P(X+1,Y)$, the claim follows by induction.

Using $\binom{\alpha+1}{s+1} = \binom{\alpha}{s+1} + \binom{\alpha}{s}$ we have

$$\begin{aligned}
H_{\gamma,n}(X+1,Y) &= \sum_{k=0}^{Y-1} \binom{-X-1}{k} \binom{\gamma + X}{n + X + 1 + k} + \sum_{k=0}^{Y-1} \binom{-X-1}{k} \binom{\gamma + X}{n + X + k} \\
&= \binom{-X-1}{Y-1} \binom{\gamma + X}{n + X + Y + 1} \\
&\quad + \sum_{k=0}^{Y-1} \left[\binom{-X-1}{k-1} + \binom{-X-1}{k} \right] \binom{\gamma + X}{n + X + k} \\
&= (-1)^{X+Y+1} \binom{-Y}{X} \binom{\gamma + X}{n + X + Y} + H_{\gamma,n}(X,Y).
\end{aligned}$$

Therefore

$$\begin{aligned}
H_{\gamma,n}(X,Y) + D_{\gamma,n}(X,Y) &= H_{\gamma,n}(X+1,Y) + D_{\gamma,n}(X,Y) \\
&\quad + (-1)^{X+Y} \binom{-Y}{X} \binom{\gamma + X}{n + X + Y} \\
&= H_{\gamma,n}(X+1,Y) + D_{\gamma,n}(X+1,Y).
\end{aligned}$$

□

5.2.3 Some properties of the functors Ω_p, M^p, and L^p

Definition 5.2.5. Let $M \in \mathsf{PEMod}(g, V)$. We say M is *normalized* if $M_0 \neq 0$. We say M is *p-irreducible*[4] if

- M is generated over V by M_p, i.e., $VM_p = M$ in light of Lemma 5.2.1.

[4]In [4] 0-irreducible V-modules are called *almost irreducible*.

- Every nonzero graded V-submodule of M has nonzero intersection with M_p.

If $N \in \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ then $L^p(N)$ is p-irreducible. If $M \in \mathsf{PEMod}(g, V)$ is irreducible then it is p-irreducible for each p such that $M_p \neq 0$.

Definition 5.2.6. Let $\Phi_p : \mathrm{Zhu}_{p-1,g}(V)\text{-}\mathsf{mod} \to \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ be the functor sending $N \in \mathrm{Zhu}_{p-1,g}(V)\text{-}\mathsf{mod}$ to the same vector space N with $\mathrm{Zhu}_{p,g}(V)$ acting by $ax = \phi_p(a)x$. We say $N \in \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ is *p-founded* if it is *not* in the image of Φ_p, i.e., the action of $\mathrm{Zhu}_{p,g}(V)$ on N does not factor to an action of $\mathrm{Zhu}_{p-1,g}(V)$.

Theorem 5.2.10.

- *For all $N \in \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$,*

$$[\Omega_p \circ M^p](N) \cong [\Omega_p \circ L^p](N) \cong N.$$

Indeed each of these compositions of functors is equivalent to the identity functor on $\mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$.

- *Let $M \in \mathsf{PEMod}(g, V)$ be p-irreducible. Then $[L^p \circ \Omega_p](M) = M$.*

- *Ω_p and L^p restrict to inverse equivalences between $\mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ and the full subcategory of $\mathsf{PEMod}(g, V)$ of p-irreducible V-modules.*

- *Ω_p and L^p further restrict to inverse equivalences between the full subcategory of $\mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ of p-founded $\mathrm{Zhu}_{p,g}(V)$-modules, and the full subcategory of $\mathsf{PEMod}(g, V)$ of normalized p-irreducible V-modules.*

Proof. The first claim follows from the results of the last section.

For the second claim, let $N = \Omega_p(M)$, $M' = M^p(N)$, and $L = L^p(N)$. Because M is generated over V by N, we have that M is a quotient of M', so let $M = M'/J$. The graded submodule J must have zero intersection with M'_p because $M_p \cong M'_p \cong N$. It must be maximal with this property because, by assumption, all nonzero graded submodules of M have nonzero intersection with M_p. Thus $M = M'/J = L$.

The third claim follows from the first two.

For the fourth claim, let $N \in \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod}$ and suppose $M = M^p(N)$ is not normalized, i.e., that $M_0 = 0$. We lower the degree of every vector in M by 1, and apply the functor Ω_{p-1}, to obtain N as a $\mathrm{Zhu}_{p-1,g}(V)$-module. This action of $\mathrm{Zhu}_{p-1,g}(V)$ on N is factored from the $\mathrm{Zhu}_{p,g}(V)$-action via ϕ_p. Therefore if N is p-founded then M is normalized.

On the other hand if N is not p-founded then $M_0 = 0$; this is implied by the following claim: Let $D_+ : \mathsf{PEMod}(g, V) \to \mathsf{PEMod}(g, V)$ be the functor that raises the degree of every vector in a module by 1. Then

$$\begin{array}{ccc}
\mathrm{Zhu}_{p-1,g}(V)\text{-}\mathsf{mod} & \xrightarrow{\Phi_p} & \mathrm{Zhu}_{p,g}(V)\text{-}\mathsf{mod} \\
{\scriptstyle L^{p-1}}\downarrow & & \downarrow{\scriptstyle L^p} \\
\mathsf{PEMod}(g, V) & \xrightarrow[D_+]{} & \mathsf{PEMod}(g, V)
\end{array}$$

commutes.

Let $N \in \mathrm{Zhu}_{p-1,g}(V)$-mod. Recall Lemma 5.2.4, and in particular the commutative diagram (5.2.1), it implies that N acquires the same $\mathfrak{g}_0$-module structure via $\mathrm{Zhu}_{p-1,g}(V)$ and via $\mathrm{Zhu}_{p,g}(V)$ (recall that $\mathfrak{g} = \mathrm{Lie}^g V$).

In the construction of $L^{p-1}(N)$ we induce from N as a module over $\mathfrak{g}_+ = \mathfrak{g}_0 + \mathfrak{g}_{>p-1}$ to get the $\mathfrak{g}$-module $\tilde{M}$. In the construction of $L^p(N)$ we induce from N as a module over $\mathfrak{g}'_+ = \mathfrak{g}_0 + \mathfrak{g}_{>p}$ to get the $\mathfrak{g}$-module $\tilde{M}'$. In each case there is a choice of grading, and these are different by our conventions; this is taken care of by D_+.

By the PBW theorem, both $\tilde{M}$ and $\tilde{M}'$ are spanned by monomials

$$(u^1)^M_{n_1} \cdots (u^s)^M_{n_s} x,$$

where $x \in N$, $u^i \in V$ and $n_1 \leq n_2 \leq \ldots \leq n_s$. For $\tilde{M}$, $n_s \leq p-1$ and for $\tilde{M}'$, $n_s \leq p$. There is a natural homomorphism of $\mathfrak{g}$-modules $\tilde{M}' \twoheadrightarrow \tilde{M}$ whose kernel is the span of monomials with $p - 1 < n_s \leq p$. In other words $\tilde{M} = \tilde{M}'/U(\mathfrak{g})K_pN$, where $K_p = \oplus_{p-1<j\leq p}\mathfrak{g}_j$.

We then obtain our V-modules $L^{p-1}(N)$ and $L^p(N)$ as quotients of $\tilde{M}$ and $\tilde{M}'$ by their respective maximal $\mathfrak{g}$-submodules having zero intersection with N. If we show that the V-submodule of $L^{p-1}(N)$ generated by K_pN has zero intersection with N, then we have $L^{p-1}(N) \cong L^p(N)$. By Lemma 5.2.1, this intersection is $\oplus_{p-1<j\leq p}\mathfrak{g}_{-j}\mathfrak{g}_jN \cap N$. Hence it suffices to show that $u_{-j}v_jx = 0$ for all $u, v \in V$ such that $[\epsilon_u] + [\epsilon_v] = [0]$, $x \in N$, and $p - 1 < j \leq p$ (here we write u_j in place of the more cumbersome $a_j^{L^{p-1}(N)}$).

Lemma 5.2.8 implies that $u_{-j}v_jx$ is a linear combination of terms of the form $(u_{[n]}v)_0x$ where

$$n \leq -p - 1 - j + \epsilon_u < -2p + \epsilon_u \leq -2(p-1) - 2 + \chi(u, v).$$

This inequality implies that $u_{-j}v_jx = 0$ because N is a $\mathrm{Zhu}_{p-1,g}(V)$-module. □

Definition 5.2.7. The VOSA V with automorphism g is *g-rational* if

- $\mathsf{PEMod}(g, V)$ has finitely many irreducible objects up to isomorphism and degree shifts.
- The graded subspaces M_n of an irreducible $M \in \mathsf{PEMod}(g, V)$ are each finite dimensional.
- Any object of $\mathsf{PEMod}(g, V)$ is a direct sum of irreducible objects.

Proposition 5.2.11. *If V is g-rational then each $\mathrm{Zhu}_{p,g}(V)$ is a finite dimensional semisimple algebra.*

Proof. Let $N \in \mathrm{Zhu}_{p,g}(V)$-mod and put $M = L^p(N)$. Then M decomposes into a direct sum of irreducible g-twisted positive energy V-modules and so $N = \Omega_p(M)$ is a direct sum of irreducible $\mathrm{Zhu}_{p,g}(V)$-modules by Proposition 5.2.2. Thus $\mathrm{Zhu}_{p,g}(V)$ is semisimple.

In particular $\mathrm{Zhu}_{p,g}(V)$ itself decomposes into a direct sum of irreducible modules (left ideals). We can decompose the unit element as a finite sum $|0\rangle = e_1 + \cdots + e_n$ where each e_i lies in a distinct ideal. These are precisely the nonzero ideals in the sum. Because of rationality each ideal has finite dimension and thus $\mathrm{Zhu}_{p,g}(V)$ itself has finite dimension. □

We can prove a converse to this result if we assume V has nonzero central charge c.

Lemma 5.2.12. *Let V be a VOSA with automorphism g, and with nonzero central charge. Let $M \in \mathsf{PEMod}(g, V)$. Then*

- *M is not bounded (we say a graded V-module M is* bounded *if M_n is nonzero for only finitely many n).*
- *If M is irreducible then $L^M_{-1} : M_n \to M_{n+1}$ is injective for $n \gg 0$.*
- *If M is irreducible then in each coset $[n] \in \mathbb{R}/\mathbb{Z}$ we have $M_n = 0$ for all $n \in [n]$ or else $M_n \neq 0$ for $n \in [n]$, $n \gg 0$.*

Proof. Suppose M is bounded, then there exists $N > 0$ such that $L^M_n M = 0$ whenever $|n| \geq N$. For any $x \in M$ we have

$$\begin{aligned} (2N+1)L^M_1 x &= [L^M_{N+1}, L^M_{-N}]x = 0 \\ \text{and} \quad (2N+1)L^M_{-1} x &= [L^M_N, L^M_{-N-1}]x = 0, \\ \text{hence} \quad 2L^M_0 x &= [L^M_1, L^M_{-1}]x = 0. \end{aligned}$$

But also

$$0 = [L^M_N, L^M_{-N}]x = 2NL^M_0 x + \frac{N^3 - N}{12} cx.$$

So $c = 0$, which is a contradiction.

Now assume M is irreducible. Let $x \in M$ be of homogeneous degree, suppose $L^M_{-1} x = 0$, and let $u \in V$ be of homogeneous conformal weight. Either $Y^M(u, z)x = 0$, or there is some $u^M_{(k)} x \neq 0$. Suppose the latter case holds, and let $k \in [\gamma_u]$ be maximal with this property. Then

$$-(k+1)u^M_{(k)} x = [L^M_{-1}, u^M_{(k+1)}]x = 0,$$

so we must have $k = -1$. If $[\gamma_u] \neq [0]$ we have a contradiction, so $Y^M(u, z)x = 0$. If $[\gamma_u] = [0]$ then we have $u^M_{(n)} x = 0$ for $n \geq 0$. In either case we have $u^M_m x \neq 0$ only if $m < -\Delta_u + 1 \leq 1$.

Therefore the elements of the V-submodule $Vx \subseteq M$ have degrees at least $\deg x - 1$. Let d be the lowest degree that occurs among the elements of M. If $\deg x > d + 1$ then Vx is a proper submodule of M, which contradicts irreducibility. Thus there exists $N = d + 1 > 0$ such that if $\deg x \geq N$, then $L^M_{-1} x \neq 0$.

The final statement follows from the first two. M is unbounded, i.e., there is a nonzero M_n for some $n \geq N$. But then L_{-1}^M is injective on M_j for $j \geq n$. Hence $M_n, M_{n+1}, M_{n+2}, \ldots \neq 0$. □

Theorem 5.2.13. *Let V be a VOSA with automorphism g, and with nonzero central charge. If $\mathrm{Zhu}_{P,g}(V)$ is a finite dimensional semisimple algebra for each $P \in \mathbb{R}_+$ then V is g-rational.*

For this theorem and its proof we really do use that all the Zhu algebras $\mathrm{Zhu}_{P,g}(V)$ (even for $P \notin \mathbb{Z}_+$) are finite dimensional and semisimple. The general definition of the Zhu algebras is given in Section 5.4.

Proof. The normalized irreducible g-twisted positive energy V-modules are in bijection with the irreducible $\mathrm{Zhu}_{0,g}(V)$-modules. Therefore, up to isomorphism and degree-shifts, there are finitely many irreducible objects in $\mathsf{PEMod}(g, V)$. For each such normalized irreducible M, M_P is an irreducible $\mathrm{Zhu}_{P,g}(V)$-module, hence finite dimensional because $\mathrm{Zhu}_{P,g}(V)$ is semisimple.

Let $M \in \mathsf{PEMod}(g, V)$ be a normalized irreducible object. By the argument from Remark 5.2.3, $L_0|_{M_n} = \lambda + n$ for all $n \in \mathbb{R}_+$, for some $\lambda \in \mathbb{C}$. By Lemma 5.2.12, M has some graded piece M_n (with $n \in \mathbb{Z}$ in fact) such that it and all higher integer graded pieces of M are nonzero. Therefore there exists $K \in \mathbb{Z}_+$ with the following property: for all irreducible normalized $M \in \mathsf{PEMod}(g, V)$ and $n \in \mathbb{Z}_+$ such that $\mathrm{Re}(L_0|_{M_n}) > K$, we have $M_n \neq 0$.

Let $M \in \mathsf{PEMod}(g, V)$ be normalized, and let $N = \Omega_0(M)$. Since $\mathrm{Zhu}_{0,g}(V)$ is semisimple, we have the direct sum of submodules $N = N' \oplus N''$ where N' is irreducible. We have $L_0|_{N'} = \lambda$ a scalar.

Let $M' = VN' \subseteq M$, and let W be the irreducible quotient of M'. Let K be as above and let $p > K - \mathrm{Re}(\lambda)$ be an integer. We have

$$W \cong L^p(W_p) = M^p(W_p)/J,$$

where J is the unique maximal ideal such that $J_p = 0$. Either $J = 0$ or the irreducible quotient $\overline{J}$ of J (which satisfies $\overline{J}_p = 0$) is zero because of our choice of p. Either way, $J = 0$ and so $M^p(W_p) \cong W \cong L^p(W_p)$.

The canonical surjection $M' \twoheadrightarrow W$ induces $M'_p \twoheadrightarrow W_p$. Because $\mathrm{Zhu}_{p,g}(V)$ is semisimple, we obtain an inclusion $W_p \hookrightarrow M'_p$, so we regard $W_p \subseteq M_p$ now. There are homomorphisms of V-modules

$$M^p(W_p) \to VW_p \to L^p(W_p),$$

these must each be isomorphisms. Therefore $VW_p \subseteq M'$ is an irreducible V-module (isomorphic to W).

By Lemma 5.2.1, $VW_p \cap M_0 \subseteq N'$. The submodule $\ker(M' \twoheadrightarrow W) \subseteq M'$ cannot contain N', for then it would equal all of M'. Hence VW_p has nonzero intersection with N'. But N' is irreducible, so $VW_p \cap N' = N'$. Hence $M' = VW_p$, an irreducible submodule of M with $M'_0 = N'$.

We apply this argument to each irreducible summand of N to obtain a sum (which is obviously direct) of irreducible V-modules $M^1 \oplus \cdots \oplus M^n \subseteq M$. If this inclusion is an equality then we are done. If not, then we let $P \in \mathbb{R}_+$ be minimal such that $(M/(M^1 \oplus \cdots \oplus M^n))_P \neq 0$. Because $\mathrm{Zhu}_{P,g}(V)$ is semisimple, we have $\Omega_P(M) = (M^1 \oplus \cdots \oplus M^n)_P \oplus N^{(P)}$ for some $\mathrm{Zhu}_{P,g}(V)$-module $N^{(P)}$. We repeat the arguments above on $N^{(P)}$, obtaining further irreducible summands. Ultimately, we can write M as a (direct) sum of irreducible V-modules in this way. □

5.3 Computation of Zhu algebras

5.3.1 Alternative constructions of $\mathrm{Zhu}_{p,g}(V)$

Let V be a VOSA with automorphism g and recall $\mathfrak{g} = \mathrm{Lie}^g V$ the associated Lie superalgebra. In section 5.2.2 we introduced the completed universal enveloping algebra $\hat{U}$, and its subspace $\mathcal{B}$. Let $\hat{\mathcal{B}}$ denote the closure of $\mathcal{B}$ in $\hat{U}$.

Lemma 2.26 of [4] states that $[u_s, \mathcal{B}] \subseteq \mathcal{B}$ for all $u \in V$, $s \in [\epsilon_u]$. It follows that $\hat{U}\hat{\mathcal{B}}$ is a (graded) 2-sided ideal in $\hat{U}$. Therefore we may define

$$U_g(V) = \hat{U}/\hat{U}\hat{\mathcal{B}},$$

a graded unital associative algebra.

The degree 0 piece $(U_g(V))_0$ is a subalgebra; its subspace $(U_g(V)(U_g(V)_{>p}))_0$ is a 2-sided ideal. Therefore we have another unital associative algebra[5]

$$W_{p,g}(V) = \frac{(U_g(V))_0}{(U_g(V)(U_g(V)_{>p}))_0}.$$

We also define

$$Q_{p,g}(V) = \frac{U_g(V)}{U_g(V)(U_g(V)_{>p})}.$$

Since $U_g(V)(U_g(V)_{>p})$ is a left ideal in $U_g(V)$ but not a 2-sided ideal, $Q_{p,g}(V)$ is a left $U_g(V)$-module but not an algebra.

Let us mimic the construction of section 5.2.2 on the $\mathfrak{g}_0$-module $N = U(\mathfrak{g}_0)$. We obtain the $U(\mathfrak{g})$-module $\tilde{M} = U(\mathfrak{g})/U(\mathfrak{g})(U(\mathfrak{g})_{>p})$, and then the quotient $\tilde{M}/\mathcal{B}\tilde{M} \in \mathsf{PEMod}(g, V)$. One may check that $\tilde{M}/\mathcal{B}\tilde{M} \cong Q_{p,g}(V)$ (having degree-shifted the latter module to put the unit element in degree p), thus $(\tilde{M}/\mathcal{B}\tilde{M})_p \cong W_{p,g}(V)$. We claim that $W_{p,g}(V) \cong \mathrm{Zhu}_{p,g}(V)$.

Now let $M \in \mathsf{PEMod}(g, V)$. Since M is a $U(\mathfrak{g})$-module such that $u_n^M x = 0$ for $n \gg 0$, it makes sense to regard M as a $\hat{U}$-module. Because M is a V-module, it is a $U_g(V)$-module. Each graded piece of M is a $U_g(V)_0$-module. Furthermore $(U_g(V)U_g(V)_{>p})_0$ annihilates M_p, so M_p is a $W_{p,g}(V)$-module. In particular we may induce the adjoint representation of $\mathrm{Zhu}_{p,g}(V)$ to a g-twisted positive energy V-module with p^{th} graded

[5] In both cases the unit element 1 might coincide with 0 though.

piece equal to $\mathrm{Zhu}_{p,g}(V)$. Thus we see that $\mathrm{Zhu}_{p,g}(V)$ is a $W_{p,g}(V)$-module. This provides us with a homomorphism $W_{p,g}(V) \to \mathrm{Zhu}_{p,g}(V)$ taking $[u_0] \mapsto [a]$.

On the other hand $Q_{p,g}(V) \in \mathsf{PEMod}(g,V)$ and its degree p piece is $W_{p,g}(V)$, which is therefore a $\mathrm{Zhu}_{p,g}(V)$-module. We obtain a homomorphism $\mathrm{Zhu}_{p,g}(V) \to W_{p,g}(V)$ taking $[a] \mapsto [u_0]$. Thus $W_{p,g}(V) \cong \mathrm{Zhu}_{p,g}(V)$.

It is clear that $W_{p,g}(V)$ is associative and unital. It is straightforward to see that there is a restriction functor $M \mapsto M_p$ from $\mathsf{PEMod}(g,V)$ to $W_{p,g}(V)$-mod. However it is not clear from its definition that $W_{p,g}(V)$ has an associated induction functor with the desired properties. This is the advantage that the construction $\mathrm{Zhu}_{p,g}(V)$ has over $W_{p,g}(V)$.

In section 5.1.7 we noted that the set of all Zhu algebras of a fixed vertex algebra V, with the maps ϕ_p, forms a directed system. The set of algebras $W_{p,g}(V)$, with the natural projections, forms an isomorphic directed system. The inverse limit of this directed system is equal to the completion of $U_g(V)_0$ with respect to the system $(U_g(V)U_g(V)_{>p})_0$ of neighborhoods of 0. We conclude that

$$\varprojlim \mathrm{Zhu}_{p,g}(V) \cong \widehat{U_g(V)}_0.$$

5.3.2 A simplified construction for universal enveloping vertex superalgebras

Recall the notion of Lie conformal superalgebra and universal enveloping vertex superalgebra from Section 3.5.2. In this section for simplicity we deal only with the untwisted unsuper case. We compute higher level Zhu algebras for a universal enveloping vertex algebra.

Proposition 5.3.1. *The level p Zhu algebra of $V(R)$ is isomorphic to*

$$Z_p(V(R)) = \frac{U(\mathrm{Lie}\,R)_0}{(U(\mathrm{Lie}\,R)U(\mathrm{Lie}\,R)_{>p})_0}.$$

Proof. The idea is the same as for the construction of $W_{p,g}(V)$. We apply the functor M^p to the left adjoint representation of $\mathrm{Zhu}_p(V(R))$, to get a positive energy $V(R)$-module M such that $M_p = \mathrm{Zhu}_p(V(R))$. By Lemma 3.5.1, M is a restricted $\mathrm{Lie}\,R$-module, and thus a $U(\mathrm{Lie}\,R)$-module.

Now, $M_p = \mathrm{Zhu}_p(V(R))$ is a $U(\mathrm{Lie}\,R)_0$-module on which $(U(\mathrm{Lie}\,R)U(\mathrm{Lie}\,R)_{>p})_0$ acts by 0. Therefore $\mathrm{Zhu}_p(V(R))$ is a $Z_p(V(R))$-module, and there is an algebra homomorphism $Z_p(V(R)) \to \mathrm{Zhu}_p(V(R))$.

On the other hand, $M = U(\mathrm{Lie}\,R)/U(\mathrm{Lie}\,R)U(\mathrm{Lie}\,R)_{>p}$ is a restricted $\mathrm{Lie}\,R$-module and therefore a $V(R)$-module. If we shift the grading on M to put $U(\mathrm{Lie}\,R)_0$ in degree p, then M is a positive energy $V(R)$-module. Since $M_p = Z_p(V(R))$, we have that $Z_p(V(R))$ is a $\mathrm{Zhu}_p(V(R))$-module. There is an algebra homomorphism $\mathrm{Zhu}_p(V(R)) \to Z_p(V(R))$.

The homomorphisms we constructed are mutually inverse, hence they are isomorphisms. □

Example 5.3.1. Let $U^k(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}})/(K = k)$, then

$$\mathrm{Zhu}_p(V^k(\mathfrak{g})) \cong U^k(\hat{\mathfrak{g}})_0/(U^k(\hat{\mathfrak{g}})U^k(\hat{\mathfrak{g}})_{>p})_0.$$

When $p = 0$ we can simplify the right hand side to $U^k((\hat{\mathfrak{g}})_0) \cong U(\mathfrak{g})$. The isomorphism $\mathrm{Zhu}(V^k(\mathfrak{g})) \cong U(\mathfrak{g})$ was proved in [12] by a different method.

In the same way we obtain $\mathrm{Zhu}_0(\mathrm{Vir}^c) \cong \mathbb{C}[x]$. Again this was proved in [12].

From the PBW theorem, we see that $\mathrm{Zhu}_1(\mathrm{Vir}^c)$ is spanned by the monomials L_0^k and $L_{-1}L_0^kL_1$ for $k \in \mathbb{Z}_+$. Indeed, this algebra is generated by the elements $L = L_0$ and $A = L_{-1}L_1$. We have

$$\begin{aligned} AL = L_{-1}L_1L_0 &= L_{-1}L_0L_1 + L_{-1}L_1 \\ &= L_0L_{-1}L_1 - L_{-1}L_1 + L_{-1}L_1 = L_0L_{-1}L_1 = LA. \end{aligned}$$

Hence $\mathrm{Zhu}_1(\mathrm{Vir}^c)$ is commutative, hence a quotient of $\mathbb{C}[A, L]$. We also have

$$A^2 = L_{-1}L_1L_{-1}L_1 = L_{-1}L_{-1}L_1L_1 + 2L_{-1}L_0L_1 = 2L_0L_{-1}L_1 + 2L_{-1}L_1 = 2LA + 2A.$$

Therefore

$$\mathrm{Zhu}_1(\mathrm{Vir}^c) \cong \mathbb{C}[A, L]/\left(A^2 - 2LA - 2A\right)$$

The higher level Zhu algebras of Vir^c may be computed explicitly in a similar way, but are not commutative in general.

Declaring $\{(U(\mathrm{Lie}\, R)U(\mathrm{Lie}\, R)_{>r})_0\}_{r\geq 0}$ to be a fundamental system of neighborhoods of $\{0\}$ equips $(U(\mathrm{Lie}\, R))_0$ with a topology. We have

$$\varprojlim \mathrm{Zhu}_p(V(R)) \cong (\widehat{U(\mathrm{Lie}\, R)})_0,$$

where the right hand side is the topological completion.

5.3.3 Results for rational vertex algebras

In [12], Frenkel and Zhu noted the following: If V is a vertex algebra and $I \subseteq V$ is an ideal, then

$$\mathrm{Zhu}(V/I) \cong \mathrm{Zhu}(V)/\,\mathrm{Zhu}(I),$$

where $\mathrm{Zhu}(I)$ denotes the image of I in $\mathrm{Zhu}(V)$ under the canonical map $V \to \mathrm{Zhu}(V)$. The same is true for $\mathrm{Zhu}_p(V/I)$, and the proof is the same.

Consider the simple vertex algebra $V_k(\mathfrak{g}) = V^k(\mathfrak{g})/I$, where $k \in \mathbb{Z}_+$, and I is the unique maximal ideal. It is known that I is generated (under left multiplication) by the element $(e_\theta)_{(-1)}^{k+1}|0\rangle$, where e_θ is the highest root vector in $\mathfrak{g}$. Also, it is known that $V_k(\mathfrak{g})$ is rational. It was shown in [12] that $\mathrm{Zhu}_0(I) = (e_\theta^{k+1}) \subseteq U(\mathfrak{g}) = \mathrm{Zhu}_0(V^k(\mathfrak{g}))$, and so

$$\mathrm{Zhu}_0(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/(e_\theta^{k+1}).$$

For specific choices of $\mathfrak{g}$ and k this quotient can be computed explicitly. For example if $\mathfrak{g} = \mathfrak{sl}_2$ and $k = 1$, one may compute

$$\mathrm{Zhu}_0(V_1(\mathfrak{sl}_2)) \cong \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}.$$

This algebra has the irreducible representations $\mathbb{C}$ and $\mathbb{C}^2$ which correspond to the degree zero pieces of the two integrable $\hat{\mathfrak{sl}}_2$-modules at level 1, as one expects (see [16]).

In the same way $\mathrm{Zhu}_p(V_k(\mathfrak{g})) = \mathrm{Zhu}_p(V^k(\mathfrak{g}))/\mathrm{Zhu}_p(I)$. Unfortunately, it seems to be much more difficult to compute $\mathrm{Zhu}_p(I)$ explicitly when $p > 0$.

Let $k \in \mathbb{Z}_+$. The normalized positive energy $V_k(\mathfrak{g})$-modules $\{M^1, M^2, \dots M^r\}$ are exactly the integrable $\hat{\mathfrak{g}}$-modules. Since $V_k(\mathfrak{g})$ is rational, $\mathrm{Zhu}_p(V_k(\mathfrak{g}))$ is finite-dimensional and semisimple for all $p \in \mathbb{Z}_+$. The graded pieces M_n^i, where $1 \le i \le r$ and $0 \le n \le p$, form a complete list of irreducible $\mathrm{Zhu}_p(V_k(\mathfrak{g}))$-modules. None of the modules M_n^i are duplicates, because $M_{n_1}^i$ and $M_{n_2}^i$ have different eigenvalues of L_0 for $n_1 \ne n_2$, and

$$M_{n_1}^{i_1} \cong M_{n_2}^{i_2} \Rightarrow M^{i_1} \cong M^{i_2} \quad \text{up to degree shifts.}$$

Therefore, if $d_n^i = \dim M_n^i$,

$$\mathrm{Zhu}_p(V_k(\mathfrak{g})) \cong \bigoplus_{\substack{1\le i\le r\\ 0\le n\le p}} \mathrm{Mat}_{d_n^i}\mathbb{C} \quad \text{and} \quad \phi_p : \bigoplus_{\substack{1\le i\le r\\ 0\le n\le p}} \mathrm{Mat}_{d_n^i}\mathbb{C} \to \bigoplus_{\substack{1\le i\le r\\ 0\le n\le p-1}} \mathrm{Mat}_{d_n^i}\mathbb{C}$$

is the natural projection. The inverse limit is now just the direct product

$$\varprojlim \mathrm{Zhu}_p(V_k(\mathfrak{g})) \cong \prod_{\substack{1\le i\le r\\ n\in\mathbb{Z}_+}} \mathrm{Mat}_{d_n^i}\mathbb{C}.$$

These comments apply to any rational vertex algebra with a Virasoro element, and for which we know the graded dimensions of its irreducible modules.

5.4 Extension to non-integer P

In several places we promised to describe the construction of $\mathrm{Zhu}_{P,g}(V)$ for $P \notin \mathbb{Z}_+$. In particular we require them for Theorem 5.2.13. We have seen that in an indecomposable g-twisted positive energy V-module, the degrees of the graded pieces lie in a single coset of Γ. Thus it makes sense to define Zhu algebras for all $P \in \Gamma \cap \mathbb{R}_+$. However, it is more convenient for us to adopt a system in which $\mathrm{Zhu}_{P,g}(V)$ is defined for arbitrary $P \in \mathbb{R}_+$, and such that if $Q < P$ and $[Q, P] \cap \Gamma$ is empty then the level P and level Q Zhu algebras are isomorphic. Such a system allows us to state that 'for every twisted positive energy V-module M, M_P is a $\mathrm{Zhu}_{P,g}(V)$-module' without qualification. In this section we reserve the lower case p to stand for the integer part $p = \lfloor P \rfloor$.

5.4.1 New notations

Let $u, v \in V$ such that $[\epsilon_u] + [\epsilon_v] = [0]$, and let $x \in M_P$. We repeat the calculation of section 5.0.5, with $p + 1 + \epsilon_u$ replaced by P_u, which we define to be the smallest element of $[\epsilon_u]$ that is strictly greater than P.

Equation (5.0.5) becomes

$$\sum_{j \in \mathbb{Z}_+} \binom{P_u + \Delta_u - 1}{j} (u_{(n+j)}v)_0^M x = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} u^M_{P_u+n-j} v^M_{-P_u-n+j} x. \tag{5.4.1}$$

The right hand side equals zero when $n < -P - P_u$, so we define $N_u = N([\epsilon_u], P)$ to be the largest integer strictly less than $-P - P_u$.

The modified state-field correspondence should now be

$$Z(u, z) = (1 + z)^{\xi_u} Y(u, z) = \sum_{n \in \mathbb{Z}} u_{[n]} z^{-n-1},$$

where $\xi_u = P_u + \Delta_u - 1$ is the largest element of $[\gamma_u]$ that does not exceed $P + \Delta_u$. We have

$$u_{[n]}v = \sum_{j \in \mathbb{Z}_+} \binom{\xi_u}{j} u_{(n+j)}v.$$

In the case that $\epsilon_u = \epsilon_v = 0$, we have simpler expressions for these quantities:

$$P_u = \lfloor P \rfloor + 1, \quad \xi_u = \lfloor P \rfloor + \Delta_u \quad \text{and} \quad N_u = -2 \lfloor P \rfloor - 2. \tag{5.4.2}$$

Writing $p = \lfloor P \rfloor$ makes many formulas from the rest of the paper remain true verbatim. For example, for $u, v \in V_g$ we define $u *_P v$ to be

$$u *_P v = \sum_{m=0}^{p} \binom{-p-1}{m} u_{[-p-1-m]}v. \tag{5.4.3}$$

We find $(u *_P v)_0^M = u_0^M v_0^M$ on M_P.

The subspace $J_{P,g}$ is now defined to be

$$J_{P,g} = \mathbb{C}\{(T + H)u | [\epsilon_u] = 0\} + \mathbb{C}\{u_{[N_u]}v | [\epsilon_u] + [\epsilon_v] = [0]\}.$$

Remark 5.4.1. Although the product $*_P$ depends only on $p = \lfloor P \rfloor$, the algebra $\mathrm{Zhu}_{P,g}(V)$ is not necessarily isomorphic to $\mathrm{Zhu}_{p,g}(V)$ because $J_{P,g}$ and $J_{p,g}$ may differ. For example let V be a VOSA and g an order two automorphism (such as the superspap).

- $P = 0$: If $\epsilon_u = 0$ then $P_u = 1$, and so $N_u = -2$. If $\epsilon_u = -\frac{1}{2}$ then $P_u = \frac{1}{2}$, and so $N_a = -1$.
- $P = \frac{1}{2}$: If $\epsilon_u = 0$ then $P_u = 1$ and so $N_u = -2$. If $\epsilon_u = -\frac{1}{2}$ then $P_u = \frac{3}{2}$, and so $N_u = -3$.

Thus $J_{1/2,g}$ is strictly smaller than $J_{0,g}$.

The inclusion

$$((T + H)V_g) *_P V_g \subseteq J_{P,g} \tag{5.4.4}$$

is proved just as before.

5.4.2 The Borcherds identity

The derivation of the modified Borcherds identity follows the same course as the $P \in \mathbb{Z}_+$ case. It is made more complicated by the piecewise nature of the functions P_u, N_u, and ξ_u. Just as we introduced a function $\chi(u, v)$ at the end of section 5.0.4 that related $\epsilon_{u_{(n)}v}$ to ϵ_u and ϵ_v, so now we introduce $\sigma(u, v)$ relating $\xi_{u_{(n)}v}$ to ξ_u and ξ_v.

Definition 5.4.1.

$$\sigma(u, v) = \xi_{u_{(-1)}v} + p - \xi_u - \xi_v. \tag{5.4.5}$$

We have

$$\xi_{u_{(n)}v} = \xi_u + \xi_v - p - n - 1 + \sigma(u, v). \tag{5.4.6}$$

In Lemma 5.1.5 we used the fact that $\chi(u, v) = 0$ whenever $\epsilon_u = 0$. For the general case we need the same fact for σ.

Lemma 5.4.2. *If $\epsilon_u = 0$, then $\sigma(u, v) = 0$ for all $v \in V$.*

Proof. $\xi_{u_{(-1)}v}$ is the largest element of $[\gamma_{u_{(-1)}v}] = [\gamma_u] + [\gamma_v] = [\Delta_u] + [\gamma_v]$ that does not exceed $P + \Delta_{u_{(-1)}v} = P + \Delta_u + \Delta_v$. But this is just Δ_u plus the largest element of $[\gamma_v]$ that does not exceed $P + \Delta_v$, viz. $\Delta_u + \xi_v$. Now, $\epsilon_u = 0$ implies $\xi_u = p + \Delta_u$, so we have $\xi_{u_{(-1)}v} = \xi_u + \xi_v - p$. Therefore $\sigma(u, v) = 0$. □

Lemma 5.4.3. *If $[\epsilon_u] + [\epsilon_v] = [0]$, then*

$$N_u = -2p - 2 + \sigma(u, v). \tag{5.4.7}$$

Proof. Recall that P_u is defined to be the smallest element of $[\epsilon_u]$ strictly greater than P. As a function of P, we have $P_u(P + 1) = P_u(P) + 1$. On the interval $P \in [0, 1)$ we have

$$P_u = \begin{cases} \epsilon_u + 1 & \text{if } 0 \leq P < 1 + \epsilon_u, \\ \epsilon_u + 2 & \text{if } 1 + \epsilon_u \leq P < 1. \end{cases}$$

Consequently $N(P) = N_u = -\lfloor P + P_u \rfloor - 1$ satisfies $N(P+1) = N(P) - 2$, and on the interval $[0,1)$ we have

$$N = \begin{cases} -1 & \text{if } 0 \leq P < E, \\ -2 & \text{if } E \leq P < E', \\ -3 & \text{if } E \leq P < 1, \end{cases}$$

where $E = \min\{1 + \epsilon_u, -\epsilon_u\}$, and $E' = \max\{1 + \epsilon_u, -\epsilon_u\}$.

Since $[\epsilon_u] + [\epsilon_v] = [0]$, we have $\epsilon_{u_{(n)}v} = 0$, hence $\xi_{u_{(n)}v} = p + \gamma_{u_{(n)}v}$ (cf. equation (5.4.2)). Therefore

$$\begin{aligned} \sigma(u,v) &= \xi_{u_{(-1)}v} + p - \xi_u - \xi_v \\ &= p + \gamma_{u_{(-1)}v} + p - (p + \gamma_u) - (p + \gamma_v) + (p + \gamma_u - \xi_u) + (p + \gamma_v - \xi_v) \\ &= (p + \gamma_u - \xi_u) + (p + \gamma_v - \xi_v) + \chi(u,v). \end{aligned}$$

As a function of P, $p + \gamma_u - \xi_u$ is periodic with period 1. On the interval $P \in [0,1)$, $p(P) = 0$, and

$$p + \gamma_u - \xi_u = \begin{cases} 0 & \text{if } 0 \leq P < 1 + \epsilon_u, \\ -1 & \text{if } 1 + \epsilon_u \leq P < 1. \end{cases}$$

There are now two cases. First suppose $\epsilon_u = \epsilon_v = 0$, then $p + \gamma_u - \xi_u = p + \gamma_v - \xi_v = 0$ and $\chi(u,v) = 0$, hence $\sigma(u,v) = 0$. Meanwhile $N_u = -2p - 2$ so we are done.

Now suppose that $\epsilon_u \neq 0$, hence $\epsilon_u + \epsilon_v = -1$ and $\chi(u,v) = 1$. Suppose without loss of generality that $\epsilon_u \geq \epsilon_v$. Then

$$(p + \gamma_u - \xi_u) + (p + \gamma_v - \xi_v) = \begin{cases} 0 & \text{if } 0 \leq P < 1 + \epsilon_v, \\ -1 & \text{if } 1 + \epsilon_v \leq P < 1 + \epsilon_u, \\ -2 & \text{if } 1 + \epsilon_u \leq P < 1. \end{cases}$$

Thus $N_u = -2p - 2 + \sigma(u,v)$ for $P \in [0,1)$, and hence for all P. □

We can rewrite the modified n^{th}-product identity, the modified Borcherds identity, and equation (5.1.9) for the general case. In each formula we simply replace χ with σ. Because of Lemma 5.4.3, the statements made at the end of section 5.1.2, with χ replaced by σ, are true in the general case. We conclude that $J_{P,g}$ is a right ideal of $(V_g, *_P)$.

5.4.3 Skew-symmetry

The calculations of section 5.1.3 work with ξ_u in place of $p + \gamma_u$. Equation (5.1.13) now has $(\xi_v - \Delta_v - \Delta_u)$ replacing $(p - \gamma_u + \epsilon_u + \epsilon_v)$ in the top of the binomial coefficient. The other important formula of the section is (5.1.15); it remains unchanged. The skew-symmetry formula holds as before, with $[\cdot,\cdot]$ defined as before. The proof that $J_{P,g}$ is a left ideal goes as before. Thus $\mathrm{Zhu}_{P,g}(V)$ is well-defined.

5.4.4 Remaining properties of $\mathrm{Zhu}_{P,g}(V)$

The sections on associativity and unitality carry over verbatim.

Let $P \geq Q \geq 0$. We claim that the identity map on V_g induces a surjective algebra homomorphism

$$\phi_{P,Q} : \mathrm{Zhu}_{P,g}(V) \twoheadrightarrow \mathrm{Zhu}_{Q,g}(V).$$

To prove this, we first assume that $P - Q \leq 1$.

Suppose $Q < \gamma - \Delta_u \leq P$ for some $\gamma \in [\gamma_u]$. Then $\xi_{u,P} = \xi_{u,Q} + 1$ and

$$u_{[n,P]}v = u_{[n,Q]}v + u_{[n+1,Q]}v.$$

In Lemma 5.4.3 we wrote down a formula for $N_{u,P}$ explicitly as a function of P. It has a jump discontinuity of size 1 at each $\epsilon \in [\epsilon_u]$. At $P = \epsilon$, it takes the lower value. Hence $N_{u,P} \leq N_{u,Q} - 1$. This implies that $J_{p,g} \subseteq J_{Q,g}$.

Now suppose that there is no such $\gamma \in [\gamma_u]$. Then $\xi_{u,P} = \xi_{u,Q}$, hence $u_{[n,P]}v = u_{[n,Q]}v$. Because $N_{u,P} \leq N_{u,Q}$, we have $J_{P,g} \subseteq J_{Q,g}$.

The main calculation of section 5.1.7 shows that $u_{[n,P]}v \equiv u_{[n,Q]}v \pmod{J_{Q,g}}$. So $\phi_{P,Q}$ makes sense for $P - Q \leq 1$. It is defined for more widely separated P and Q by composing the maps defined above. This definition is sound because all the maps are induced by the identity on V_g.

5.4.5 Representation theory

Section 5.2.1 carries over, essentially unmodified, to the general case. The same goes for section 5.2.2. Equation (5.2.4), must be replaced by

$$\langle \psi, BI(u, v; P_u, -P_u; -1)x \rangle = 0, \tag{5.4.8}$$

which is proved in the next section.

The functorial statements of section 5.2.3 carry over to the general case as well. Now we define a P-founded $\mathrm{Zhu}_{P,g}(V)$-module to be one that does not factor to a $\mathrm{Zhu}_{Q,g}(V)$-modules for *each* $Q < P$. The proof of Theorem 5.2.10, which uses Lemma 5.2.8, is easily extended to the general case, using Lemma 5.4.4 instead. The remainder of section 5.2 is the same for $P \notin \mathbb{Z}_+$ as it was for $P \in \mathbb{Z}_+$.

5.4.6 Proof of equation (5.4.8)

Lemma 5.2.8 generalizes to the following.

Lemma 5.4.4. *Let V be VOSA with automorphism g, and let $M \in \mathsf{PEMod}(g, V)$. Suppose $x \in M_P$, $u, v \in V$ such that $[\epsilon_u] + [\epsilon_v] = [0]$. Also let P_u and N_u be as in section 5.4.1. Define*

$$R_u = R([\epsilon_u], P) = P_u - \epsilon_u = \lfloor P - \epsilon_u \rfloor + 1, \tag{5.4.9}$$

and let k be an integer such that $0 \leq k \leq P_u - 1$. We have

$$u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x = \sum_{m \in \mathbb{Z}_+} \binom{-R_u - k}{m} (u_{[-R_u-k-m]} v)^M_0 x. \tag{5.4.10}$$

Now equation (5.2.9) is replaced by

$$\langle \psi, u^M_{-k+\epsilon_u} v^M_{k-\epsilon_u} x \rangle = \sum_{m \in \mathbb{Z}_+} \binom{-R_u - k}{m} \langle \psi, (u_{[-R_u-k-m]} v)^M_0 x \rangle. \tag{5.4.11}$$

We may use this equation to reduce (5.4.8) to a combinatorial identity as before. This time we arrive at

$$\begin{aligned}\binom{\gamma_u}{j+1} = &\sum_{k=0}^{R_v - \chi(u,v) - 1} \binom{-R_u + 1}{k} \binom{\xi_u}{R_u + j + k} \\ &+ \sum_{k=0}^{R_u - \chi(u,v) - 1} (-1)^{R_v + k + 1} \binom{-R_v + \chi(u,v)}{k - 1 + \chi(u,v)} \binom{k + \gamma_u - 1 + \chi(u,v)}{R_v + j + k}.\end{aligned} \tag{5.4.12}$$

If $\chi(u,v) = 0$, then $P_u = P_v = p + 1 = R_u = R_v$ and $\xi_u = p + \gamma_u$. Substituting these values reduces (5.4.12) to (5.2.17), which we have already proved. If $\chi(u,v) = 1$, we have

$$\xi_u = P_u + \Delta_u - 1 = P_u + \gamma_u - \epsilon_u - 1 = R_u + \gamma_u - 1, \tag{5.4.13}$$

and the same equations for b. Substituting these values reduces (5.4.12) to Lemma 5.2.9 with $(\gamma, n, X, Y) = (\gamma_u, j + 1, R_u - 1, R_v - 1)$.

Chapter 6

Higher coefficients of conformal blocks

In this chapter we prove some formulas that are required in Chapter 4. Recall that a conformal block $S \in \mathcal{C}(g,h)$, attached to a C_2-cofinite VOSA with a finite group G of automorphisms, satisfies a certain ODE and therefore has a 'Frobenius expansion'

$$\begin{aligned} S(u,\tau) &= \sum_{i=0}^{R} (\log q)^i S_i(u,\tau), \\ \text{where} \quad S_i(u,\tau) &= \sum_{j=1}^{b(i)} q^{\lambda_{ij}} S_{ij}(u,\tau), \\ \text{where} \quad S_{ij}(u,\tau) &= \sum_{n=0}^{\infty} C_{i,j,n}(u) q^{n/|G|}, \end{aligned}$$

The lowest order coefficients descend to functions on $\mathrm{Zhu}_g(V)$ and in Chapter 2 we make use of facts about these functions proved in Propositions 2.4.3 and 2.7.1. The arguments of Chapter 4 require generalizations of these propositions, which are recovered as special cases of the more general Propositions 6.0.5 and 6.0.6 below.

Proposition 6.0.5. *Let $S \in \mathcal{C}(g,h)$ with Frobenius expansion (2.3.2)and let $f = C_{i,j,P}$. We have*

- *f annihilates $J_{P,g}(V)$ and hence descends to a map $\mathrm{Zhu}_{P,g}(V) \to \mathbb{C}$.*
- *$f(u *_P v) = \delta_{\lambda(u)\lambda(v),1} p(u,v) \lambda(u)^{-1} f(v *_P u)$ for all $u, v \in \mathrm{Zhu}_{P,g}(V)$.*

Proof. As in Section 5.4 we reserve the lower case p to denote $\lfloor P \rfloor$.

As in the $P = 0$ case we note that any $u \in V$ such that $\mu(u) \neq 1$ lies in $\mathcal{O}(g,h)$, and so $f(u) = 0$. If $\mu(u) = 1$ then $2\pi i(L_{-1} + L_0)u = \tilde{\omega}_{([0])} u \in \mathcal{O}(g,h)$ is annihilated by f.

C_{ijP} annihilates $u_{[N_u,P]}v$ for $\epsilon_u = 0$

Let $\mu(u) = \lambda(u) = \mu(v) = \lambda(v) = 1$. Recall that $S_{ij}(u,\tau)$ is a series in powers of $\overline{q} = q^{1/|G|}$. So (for possibly fractional values of P) we have

$$\begin{aligned}
[q^P] : S_{ij}(X_2(u,v),q) = {} & [q^P] : S_{ij}(\mathrm{Res}_z\, \partial_z P(z,q) Y[u,z]v dz, \tau) \\
& - [q^P] : G_2(q) S_{ij}(u_{([0])}v, \tau) \\
= {} & \sum_{n\in\mathbb{Z}_+} C_{i,j,P-n}([q^n] : \mathrm{Res}_z\, \partial_z P(z,q) Y[u,z]v dz) \\
= {} & -(2\pi i)^2 C_{i,j,P}(\mathrm{Res}_z \frac{e^{2\pi i z}}{(e^{2\pi i z}-1)^2} Y[u,z]v dz) \\
& + 2\pi i \sum_{n\le p} C_{i,j,P-n}(\mathrm{Res}_z \sum_{k|n} \partial_z (e^{-2\pi i k z} - e^{2\pi i k z}) Y[u,z]v dz),
\end{aligned}$$

which vanishes for all u, v by definition of conformal block.

Let $A = \prod_{1\le m\le p}(L_{[-1]}^2 - (2\pi i m)^2)u$, then we have

$$\begin{aligned}
& \mathrm{Res}_z \sum_{k|n} \partial_z (e^{-2\pi i k z} - e^{2\pi i k z}) Y[A,z]v dz \\
& = \mathrm{Res}_z \sum_{k|n} \partial_z \prod_{1\le m\le p} (\partial_z^2 - (2\pi i m)^2)(e^{-2\pi i k z} - e^{2\pi i k z}) Y[u,z]v dz.
\end{aligned}$$

This vanishes for each $n \in \mathbb{Z}$ such that $1 \le n \le p$ because

$$(\partial_z^2 - (2\pi i m)^2)(e^{-2\pi i k z} - e^{2\pi i k z}) = (2\pi i)^2 (k^2 - m^2)(e^{-2\pi i k z} - e^{2\pi i k z}),$$

which vanishes if $k = m$. From all this we have

$$C_{i,j,P}\left(\mathrm{Res}_z \prod_{1\le m\le p} (\partial_z^2 - (2\pi i m)^2) \frac{e^{2\pi i z}}{(e^{2\pi i z}-1)^2} Y[u,z]v dz\right) = 0. \tag{6.0.1}$$

Observe that

$$(\partial_z^2 - (2\pi i m)^2) \frac{e^{2\pi i m z}}{(e^{2\pi i z}-1)^{2m}} = 2m(2m+1)(2\pi i)^2 \frac{e^{2\pi i (m+1) z}}{(e^{2\pi i z}-1)^{2(m+1)}},$$

so by induction on p we have

$$\prod_{1\le m\le p} (\partial_z^2 - (2\pi i m)^2) \frac{e^{2\pi i z}}{(e^{2\pi i z}-1)^2} = (2p+1)!(2\pi i)^{2p} \frac{e^{2\pi i (p+1) z}}{(e^{2\pi i z}-1)^{2(p+1)}}. \tag{6.0.2}$$

Therefore

$$
\begin{aligned}
0 &= C_{ijP}(\operatorname{Res}_z \frac{e^{2\pi i(p+1)z}}{(e^{2\pi iz}-1)^{2(p+1)}} Y[u,z]vdz) \\
&= C_{ijP}(\operatorname{Res}_w w^{-2p-2}(1+w)^{\Delta_u+p} Y(u,z)vdw) \\
&= C_{ijP}(u \circ_{(-2p-2,P)} v).
\end{aligned}
$$

Recall from Section 5.4 that when $\epsilon_u = 0$, we have $N_u = -2p-2$. Hence $C_{i,j,P}$ does annihilate $u_{[N_u]}v$.

<u>C_{ijP} annihilates $u_{[N_u,P]}v$ for $\epsilon_u \neq 0$</u>

Let $(\mu, \lambda) := (\mu(u), \lambda(u)) \neq (1,1)$. We have

$$S_{ij}(\operatorname{Res}_z P^{\mu,\lambda}(z,q) Y[u,z]vdz) = 0,$$

hence

$$
\begin{aligned}
0 &= [q^P] : S_{ij}(\operatorname{Res}_z P^{\mu,\lambda}(z,q) Y[u,z]vdz, \tau) \\
&= \sum_{n\in\mathbb{Z}_+} C_{i,j,P-n}([q^n] : \operatorname{Res}_z P^{\mu,\lambda}(z,q) Y[u,z]vdz) \\
&= C_{i,j,P}(\operatorname{Res}_z \left[\frac{2\pi i\delta}{1-\lambda} - 2\pi i \frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1}\right] Y[u,z]v) \\
&\quad + 2\pi i \sum_{m\in\mathbb{Z}_{>0}} \left\{ \lambda^m \sum_{n\in[\epsilon]_{>0}} C_{i,j,P-mn}(\operatorname{Res}_z e^{2\pi inz} Y[u,z]v) \right. \\
&\qquad\qquad \left. -\lambda^{-m} \sum_{n\in[\epsilon]_{<0}} C_{i,j,P+mn}(\operatorname{Res}_z e^{2\pi inz} Y[u,z]v) \right\}.
\end{aligned}
\tag{6.0.3}
$$

Put $\mathcal{S} = \{m \in [\epsilon] | |m| \leq P\}$,

$$A = \prod_{m\in\mathcal{S}} (-L_{[-1]} - 2\pi im)u \quad \text{and} \quad \mathbb{D} = \prod_{m\in\mathcal{S}} (\partial_z - 2\pi im).$$

If we substitute A in place of u in the equation above, and use $C_{i,j,P}(u_{([0])}v) = 0$, we obtain

$$C_{i,j,P}\left(\operatorname{Res}_z \mathbb{D} \frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1} Y[u,z]vdz\right) = 0.$$

One may verify by direct calculation that

$$(\partial_z - 2\pi ic)\frac{e^{2\pi i\gamma z}}{(e^{2\pi iz}-1)^m} = 2\pi i \frac{e^{2\pi i\gamma z}[(\gamma - c)(e^{2\pi iz}-1) - me^{2\pi iz}]}{(e^{2\pi iz}-1)^{m+1}}, \tag{6.0.4}$$

for $m \in \mathbb{Z}_+$ and arbitrary c, γ. Hence for $m \in \mathbb{Z}_{>0}$,

$$(\partial_z - 2\pi i(-m+1+\epsilon))(\partial_z - 2\pi i(m+\epsilon))\frac{e^{2\pi i(m+\epsilon)z}}{(e^{2\pi iz}-1)^{2m-1}}$$
$$= (2\pi i)^2 2m(2m-1)\frac{e^{2\pi i(m+1+\epsilon)z}}{(e^{2\pi iz}-1)^{2(m+1)-1}}.$$

Therefore if we put $\mathcal{S}_0 = \{m \in [\epsilon] | -p < m \leq p\}$, where $p = \lfloor P \rfloor$ as usual, and $\mathbb{D}_0 = \prod_{m\in\mathcal{S}_0}(\partial_z - 2\pi i m)$, then

$$\mathbb{D}_0\frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1} = (2\pi i)^{2p}(2p)!\frac{e^{2\pi i(p+1+\epsilon)z}}{(e^{2\pi iz}-1)^{2p+1}}. \tag{6.0.5}$$

We now distinguish four cases, and define the set $\mathcal{S}'$ by:

A	if $0 \leq P - \lfloor P \rfloor < \min\{1+\epsilon_u, -\epsilon_u\}$,	$\mathcal{S}' = \emptyset$;
B	if $1+\epsilon_u \leq P - \lfloor P \rfloor < -\epsilon_u$,	$\mathcal{S}' = \{p+1+\epsilon_u\}$;
C	if $-\epsilon_u \leq P - \lfloor P \rfloor < 1+\epsilon_u$,	$\mathcal{S}' = \{-p+\epsilon_u\}$;
D	if $\max\{1+\epsilon_u, -\epsilon_u\} \leq P - \lfloor P \rfloor < 1$,	$\mathcal{S}' = \{p+1+\epsilon_u, -p+\epsilon_u\}$.

so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}'$.

Applying (6.0.4) to (6.0.5) yields that

$$\mathbb{D}\frac{e^{2\pi i(1+\epsilon)z}}{e^{2\pi iz}-1} \quad \text{is a nonzero multiple of} \quad \frac{e^{2\pi i\alpha z}}{(e^{2\pi iz}-1)^{\beta}}$$

where α, β take the values

	α	β
A	$p+1+\epsilon$	$2p+1$,
B	$p+2+\epsilon$	$2p+2$,
C	$p+1+\epsilon$	$2p+2$,
D	$p+2+\epsilon$	$2p+3$.

In the definition of $J_{P,g}$ we introduced P_u, ξ_u and N_u. By definition P_u is the smallest element of $[\epsilon_u]$ that is strictly greater than P, $\xi_u = P_u + \Delta_u - 1$, and N_u is the largest integer strictly less than $-P - P_u$. One may check by cases that $\alpha = P_u$ and $\beta = -N_u$. So we see that

$$\operatorname{Res}_z \mathbb{D}\frac{e^{2\pi i(1+\epsilon_u)z}}{(e^{2\pi iz}-1)}Y[u,z]v dz = \operatorname{Res}_z e^{2\pi i P_u z}(e^{2\pi iz}-1)^{N_u}Y[u,z]v dz$$
$$= \operatorname{Res}_w w^{N_u}(1+w)^{\xi_u}Y(u,w)v dw = u_{(N_u,P)}v.$$

Therefore $C_{i,j,P}$ annihilates $u_{(N_u,P)}v$, and thus all of $J_{P,g}$.

Part II

Since $u *_P v - p(u,v) v *_P a = u_{([0])}v$, the second part of the proposition follows by exactly the same argument as in proposition 2.4.3. ☐

Proposition 6.0.6. *Let $S \in \mathcal{C}(g,h)$ with Frobenius expansion (2.3.2), then*

$$C_{i,j,P}((\omega - \frac{c}{24} - \lambda_{ij} - P)^{R+1-i} *_P u) = 0 \quad \textit{for all } u \in V,$$

where R is the maximal power of $\log(q)$ that appears in the expansion of S.

Proof. Recall equation (2.2.2) which states

$$(2\pi i)^2 q \frac{d}{dq} S(u,\tau) = -S(\operatorname{Res}_z P(z,q)L[z]udz, \tau).$$

Let D denote the operator $(2\pi i)^2 q\frac{d}{dq}$. Define the operator $\sigma : \mathcal{V}[G_2(q)] \to \mathcal{V}[G_2(q)]$ by $\sigma(a) = -\operatorname{Res}_z P(z,q)L[z]udz$ for $u \in V$ and $\sigma(f(\tau)u) = f(\tau)\sigma(u) + Df(\tau)u$. Then we have

$$(\sigma - D)S(v,\tau) = 0 \tag{6.0.6}$$

for all $v \in \mathcal{V}$.

In terms of the expansion

$$S(u,\tau) = \sum_{i=0}^{R} \log^i(q) S_i(u,\tau),$$

equation (6.0.6) implies

$$0 = (D-\sigma)S(u,\tau) = \sum_{i=0}^{R} \log^i(q)(D-\sigma)S_i(u,\tau) + (2\pi i)^2 \sum_{i=0}^{R} i \log^{i-1}(q) S_i(u,\tau),$$

so $(D-\sigma)S_R(u,\tau) = 0$ and

$$(D-\sigma)S_{i-1}(u,\tau) + (2\pi i)^2 i S_i(u,\tau) = 0$$

for $i = 0, 1, \dots R-1$. We deduce that

$$(D-\sigma)^k_{\leftarrow} S_{R-k}(u,\tau) = 0, \tag{6.0.7}$$

where the $\leftarrow$ subscript indicates to expand the binomial term and move the D operators to the left, then evaluate.

The desired result follows from (6.0.7) combined with lemma 6.0.7 below, applied to $F(u,\tau) = S_{ij}(u,\tau)$. ☐

Lemma 6.0.7. *Let*

$$F(u,\tau) = \sum_{n\geq 0} q^{\lambda+n} F_n(u)$$

be some series whose coefficients are linear functions on V. If we put $F'(u,\tau) = (D-\sigma)F(u,\tau)$ then

$$F'_P(u) = F_P((\omega - c/24 - \lambda - P) *_P u).$$

Proof. We begin by extracting the q^P coefficient of $DF(u,\tau) - F'(u,\tau) = \sigma F(u,\tau)$:

$$\begin{aligned}(2\pi i)^2(\lambda+P)C_P(u) - C'_P(u) &= \sum_{k\in\mathbb{Z}_+} C_{P-k}([q^k] : \mathrm{Res}_z -P(z,q)L[z]udz) \\ &= C_P(2\pi i\,\mathrm{Res}_z \frac{e^{2\pi iz}}{e^{2\pi iz}-1}L[z]udz) \\ &\quad - \sum_{m,n\in\mathbb{Z}_{>0}} C_{P-mn}(2\pi i\,\mathrm{Res}_z(e^{2\pi inz} - e^{-2\pi inz})L[z]udz).\end{aligned} \tag{6.0.8}$$

Since

$$2\pi i\,\mathrm{Res}_z \frac{e^{2\pi iz}}{e^{2\pi iz}-1}L[z]udz = (\omega - \frac{c}{24}) *_0 a,$$

it suffices for us to show that

$$C_P(u *_P v - u *_0 v) = 2\pi i \sum_{m,n\in\mathbb{Z}_{>0}} C_{P-mn}(\mathrm{Res}_z(e^{-2\pi inz} - e^{2\pi inz})Y[u,z]vdz). \tag{6.0.9}$$

To establish (6.0.9) we begin with another claim: that

$$u *_P v - u *_{P-1} v = \frac{(-1)^{p-1}}{p}\binom{2p-1}{p-1}\mathrm{Res}_z\,\partial_z\left[\frac{e^{2\pi ipz}}{(e^{2\pi iz}-1)^{2p}}\right]Y[u,z]vdz. \tag{6.0.10}$$

Since

$$u *_P v = 2\pi i\,\mathrm{Res}_w \sum_{m=0}^{p}\binom{-p-1}{m}\frac{(1+w)^{p+\Delta_u}}{w^{p+m+1}}Y(u,w)vdw,$$

we have $u *_P v - u *_{P-1} v = \mathrm{Res}_w\,\mathcal{A}(w)Y(u,w)vdw$ where

$$\begin{aligned}\mathcal{A} &= \sum_{m=0}^{p}\binom{-p-1}{m}\frac{(1+w)^{p+\Delta_u}}{w^{p+m+1}} - \sum_{m=0}^{p-1}\binom{-p}{m}\frac{(1+w)^{p+\Delta_u-1}}{w^{p+m}} \\ &= \binom{-p-1}{p}\frac{(1+w)^{p+\Delta_u}}{w^{2p+1}} \\ &\quad + \frac{(1+w)^{p+\Delta_u-1}}{w^{2p+1}}\sum_{m=0}^{p-1}\left[\binom{-p-1}{m}(1+w)w^{p-m} - \binom{-p}{m}w^{p-m+1}\right],\end{aligned}$$

the sum telescopes, reducing $\mathcal{A}(w)$ to

$$\binom{-p-1}{p}\frac{(1+w)^{p+\Delta_u}}{w^{2p+1}}+\binom{-p-1}{p-1}\frac{(1+w)^{p+\Delta_u}}{w^{2p}}.$$

Consequently $\mathrm{Res}_w\,\mathcal{A}(w)Y(u,w)vdw$ reduces to

$$\mathrm{Res}_z\left\{\binom{-p-1}{p}\frac{e^{2\pi ipz}}{(e^{2\pi iz}-1)^{2p+1}}+\binom{-p-1}{p-1}\frac{e^{2\pi ipz}}{(e^{2\pi iz}-1)^{2p}}\right\}Y[u,z]vdz$$

which equals the right hand side of (6.0.10) as required.

Now we use (6.0.2) to express the right hand side of (6.0.10) as

$$\frac{(-1)^{p-1}}{(2\pi i)^{2p-1}(p!)^2}\,\mathrm{Res}_z\,\partial_z\left[\prod_{1\le m\le p-1}(\partial_z^2-(2\pi im)^2)\right]\frac{e^{2\pi iz}}{(e^{2\pi iz}-1)^2}Y[u,z]vdz.$$

Therefore

$$u *_P b - u *_0 b = \mathrm{Res}_z\,\mathbb{K}\frac{e^{2\pi iz}}{(e^{2\pi iz}-1)^2}Y[u,z]vdz \tag{6.0.11}$$

where

$$\mathbb{K}=\sum_{q=1}^{p}\frac{(-1)^{q-1}}{(2\pi i)^{2q-1}(q!)^2}\partial_z\prod_{1\le m\le q-1}(\partial_z^2-(2\pi im)^2).$$

Recall that

$$(2\pi i)^2C_P(\mathrm{Res}_z\,\frac{e^{2\pi iz}}{(e^{2\pi iz}-1)^2}Y[u,z]vdz)$$
$$=2\pi i\sum_{m,n\ge1}C_{P-mn}(\mathrm{Res}_z\,\partial_z(e^{-2\pi inz}-e^{2\pi inz})Y[u,z]vdz),$$

from this and (6.0.11) it follows that

$$C_P(u *_P b - u *_0 v)=2\pi i\sum_{m,n\in\mathbb{Z}_{>0}}C_{P-mn}(\mathrm{Res}_z\,\mathbb{K}\partial_z(e^{-2\pi inz}-e^{2\pi inz})Y[u,z]vdz).$$

Equation (6.0.9), and thence the desired result, will follow from the claim that $\mathbb{K}(e^{-2\pi inz}-e^{2\pi inz})=(e^{-2\pi inz}-e^{2\pi inz})$ for $1\le n\le p$. This in turn reduces to the following combinatorial fact:

$$1=\sum_{k=1}^{p}\frac{(-1)^kn}{(k!)^2}(n-k+1)\cdots(n+k-1).$$

This identity is proved by writing the summand as a product of binomial coefficients and using basic generating function methods. □

Bibliography

[1] D. Adamovic and A. Milas. Vertex Operator Algebras associated to Modular Invariant Representations for $A_1^{(1)}$. *Math. Res. Lett.*, 2(5):563–575, 1995.

[2] Y. Arike. Some Remarks on Symmetric Linear Functions and Pseudotrace Maps. *Proc. Japan Acad. Ser. A Math. Sci.*, 86(7):119–124, 2010.

[3] R. Borcherds. Vertex Algebras, Kac-Moody Algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986.

[4] A. De Sole and V. Kac. Finite vs Affine W-algebras. *Jpn. J. Math.*, 1:137–261, 2006.

[5] C. Dong, H. Li, and G. Mason. Vertex Operator Algebras associated to Admissible Representations of $\mathfrak{sl}_2$. *Comm. Math. Phys.*, 184(1):65–93, 1997.

[6] C. Dong, H. Li, and G. Mason. Twisted Representations of Vertex Operator Algebras. *Math. Ann.*, 310:571–600, 1998.

[7] C. Dong, H. Li, and G. Mason. Vertex Operator Algebras and Associative Algebras. *J. Algebra*, 206:67–96, 1998.

[8] C. Dong, H. Li, and G. Mason. Modular-Invariance of Trace Functions in Orbifold Theory and Generalized Moonshine. *Commun. Math. Phys.*, 214:1–56, 2000.

[9] C. Dong and Z. Zhao. Modularity in Orbifold Theory for Vertex Operator Superalgebras. *Comm. Math. Phys.*, 260:227–256, 2005.

[10] E. Frenkel and D. Ben Zvi. *Vertex Algebras and Algebraic Curves*, volume 88 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2$^{\text{nd}}$ edition, 2004.

[11] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster.* Academic Press, Boston, MA, 1988.

[12] I. Frenkel and Y. Zhu. Vertex Operator Algebras associated to Representations of Affine and Virasoro Algebras. *Duke Math. J.*, 66(1):123–168, 1992.

[13] J. Igusa. *Theta Functions.* Springer-Verlag, New York, NY, 1972.

[14] V. Kac. Lie Superalgebras. *Adv. in Math.*, 26(1):8–96, 1977.

[15] V. Kac. An elucidation of: "Infinite-dimensional Algebras, Dedekind's η-function, Classical Möbius Function and the Very Strange Formula". $E_8^{(1)}$ and the Cube Root of the Modular Invariant j. *Adv. in Math.*, 35(3):264–273, 1980.

[16] V. Kac. *Infinite Dimensional Lie Algebras.* Cambridge University Press, Cambridge, 3rd edition, 1990.

[17] V. Kac. *Vertex Algebras for Beginners.* AMS, University Lecture Notes, Vol. 10, Providence, RI, 2nd edition, 1998.

[18] V. Kac and D. Peterson. Infinite-Dimensional Lie Algebras, Theta Functions and Modular Forms. *Adv. in Math.*, 53:125–264, 1984.

[19] V. Kac and M. Wakimoto. Unitarizable Highest Weight Representations of the Virasoro, Neveu-Schwarz and Ramond Algebras. In *Conformal Groups and related Symmetries*, number 261 in Lecture Notes in Physics, pages 345–372, Berlin, 1986. Springer.

[20] V. Kac and M. Wakimoto. Modular Invariant Representations of Infinite-Dimensional Lie Algebras and Superalgebras. *Proc. Nat. Acad. Sci. U.S.A.*, 85(14):4956–4960, 1988.

[21] S. Lang. *Elliptic Functions.* Springer-Verlag, Graduate Texts in Mathematics, 112, New York, NY, 2nd edition, 1987.

[22] H. Li. *Representation Theory and Tensor Product Theory for Vertex Operator Algebras.* PhD thesis, Rutgers University, 1994.

[23] H. Li. Local Systems of Twisted Vertex Operators, Vertex Operator Superalgebras and Twisted Modules. In *Moonshine, the Monster, and Related Topics*, number 193 in Contemp. Math., pages 203–236, Providence, RI, 1996. Amer. Math. Soc.

[24] A. Milas. Characters, Supercharacters and Weber Modular Functions. *J. Reine Angew. Math.*, 608:35–64, 2007.

[25] M. Miyamoto. Modular Invariance of Vertex Operator Algebras satisfying C_2-Cofiniteness. *Duke Math. J.*, 122(1):51–91, 2004.

[26] O. Perše. Vertex Operator Algebras associated to certain Admissible Modules for Affine Lie Algebras of type A. *Glas. Mat. Ser. III*, 43(1):41–57, 2008.

[27] J. van Ekeren. Higher Level Twisted Zhu Algebras. *J. Math. Phys.*, 52(5):052302, 2011.

[28] E. Whittaker and G. Watson. *A Course of Modern Analysis.* Cambridge University Press, Cambridge, 4th edition, 1996.

[29] Y. Zhu. Modular Invariance of Characters of Vertex Operator Algebras. *J. Amer. Math. Soc.*, 9:237–301, 1996.