Quantum geometric Langlands correspondence in positive characteristic: the GL_N case

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Abstract

Let C be a smooth connected projective curve of genus > 1 over an algebraically closed field k of characteristic $p > 0$, and $c \in \mathsf{k} \setminus \mathbb{F}_p$. Let Bun_N be the stack of rank N vector bundles on C and $\mathcal{L}_{\mathrm{det}}$ the line bundle on Bun_N given by determinant of derived global sections. In this thesis, we construct an equivalence of derived categories of modules for certain localizations of the twisted crystalline differential operator algebras $\mathcal{D}_{\mathrm{Bun}_N, \mathcal{L}_{\mathrm{det}}^c}$ and $\mathcal{D}_{\mathrm{Bun}_N, \mathcal{L}_{\mathrm{det}}^{-1/c}}$.

The first step of the argument is the same as that of [BB] for the non-quantum case: based on the Azumaya property of crystalline differential operators, the equivalence is constructed as a twisted version of Fourier–Mukai transform on the Hitchin fibration. However, there are some new ingredients. Along the way we introduce a generalization of p-curvature for line bundles with non-flat connections, and construct a Liouville vector field on the space of de Rham local systems on C.

Thesis Supervisor: Roman Bezrukavnikov
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Contents

Chapter 1

Introduction

Fix an algebraic curve C and a reductive group G. The geometric Langlands correspondence (GLC) is a conjectural equivalence of derived categories between $\mathcal{D}$-modules on the moduli space Bun_G of G-bundles on C and quasi-coherent sheaves on the moduli space Loc of local systems for the Langlands dual group ${}^L G$. It has a classical (commutative) limit which amounts to the derived equivalence of Fourier–Mukai type between Hitchin fibrations for G and ${}^L G$. The latter is a fibration $T^* \mathrm{Bun}_G \to \mathscr{B}$ over an affine space with generic fibers being abelian varieties (or a little more general commutative group stacks).

In [BB], a characteristic p version of GLC is established. Namely, the setup of crystalline (i.e., without divided powers) $\mathcal{D}$-modules in positive characteristic is considered. In this setting, the category of $\mathcal{D}$-modules does not get far from its classical limit: it is described by a $\mathbb{G}_m$-gerbe on the Frobenius twist of the cotangent bundle. So the GLC becomes a twisted version of its classical limit. Based on this reasoning, the GLC is constructed "generically" for the case of general linear group $G = \mathrm{GL}_N$.

In this paper, we apply the same technique to the quantum version of GLC. This deformation of GLC has the same classical limit, but now both sides are "quantized." So in characteristic p we get (generically) a twisted version of the same Fourier–Mukai transform. However, the proof that the twistings on both sides are interchanged by the Fourier–Mukai transform is more complicated than in the case of usual GLC, and contains several new ingredients. Also, we restrict to the case of irrational parameter c because there is a certain degeneration happening at rational c.

First, we need a description of the category of modules for a twisted differential operator (TDO) algebra. The center of a TDO was already described in [BMR], but the description of the corresponding gerbe presented here seems to be new. A convenient language for this turns out to be that of *extended curvature*—an invariant of a line bundle with a (*not necessarily flat*) connection taking values in a canonical coherent sheaf $\mathcal{F}$ on the Frobenius twist of the variety. This is a generalization of the p-curvature of flat connections. Just as with the usual p-curvature, every section of $\mathcal{F}$ defines a gerbe whose splittings correspond to connections with that extended curvature. Now, if $\mathcal{L}$ is a line bundle and $c \in \mathsf{k} \setminus \mathbb{F}_p$ then the gerbe describing $\mathcal{D}_{\mathcal{L}^c}$ corresponds to $c \cdot \alpha_{\mathcal{L}}$ where $\alpha_{\mathcal{L}}$ is the extended curvature of the pullback of $\mathcal{L}$ to the associated twisted cotangent, equipped with the canonical ("universal") connection.

We then apply this to the determinant bundle on Bun whose corresponding twisted cotangent is identified with Loc. So, to construct the desired equivalence, we have to split a gerbe on the fiber product of dual p-Hitchin fibrations $\mathrm{Loc} \to \mathscr{B}^{(1)}$. (Although for GL_N the p-Hitchin fibrations are the same, we use differently scaled projections to the Hitchin base.) This is done by constructing an explicit line bundle with connection on this fiber product. The problem then reduces to proving certain equality involving $\tilde{\theta} = \alpha_{\mathcal{L}}$ for $\mathcal{L}$ being the determinant bundle on Bun, and the torsor structure on the p-Hitchin fibration.

We prove this property for another section $\tilde{\theta}_0$ of $\mathcal{F}_{\mathrm{Loc}}$ and then show that $\tilde{\theta} = \tilde{\theta}_0$. The section $\tilde{\theta}_0$ is constructed from a vector field ξ_0 on Loc lifting the differential of the standard $\mathbb{G}_m$-action on the Hitchin base. This vector field comes from an action of fiberwise dilations of $T^*C^{(1)}$ on the gerbe describing $\mathcal{D}$-modules. Such structure is not unique (it depends on the choice of lifting of C to the 2nd Witt vectors of k), however the corresponding vector fields ξ_0 all differ by Hamiltonian vector fields and give rise to the same $\tilde{\theta}_0$.

This version of the text contains only a sketch of some of the arguments (and some proofs are missing). A more detailed exposition will be presented in the next versions of the paper.

1.1 Quantum geometric Langlands conjecture

Let C be a smooth irreducible projective curve of genus $g > 1$ over an algebraically closed field k of characteristic 0 and G a reductive algebraic group. We denote by $\mathrm{Bun}_G = \mathrm{Bun}_G(C)$ the moduli stack of G-bundles on C. The quantum geometric Langlands correspondence is a conjectural equivalence between certain derived categories of twisted $\mathcal{D}$-modules on Bun_G and $\mathrm{Bun}_{^LG}$ where ^LG denotes the Langlands dual group. The twistings should correspond to invariant bilinear forms on the Lie algebras of G and ^LG that induce dual forms on the Cartan subalgebras (up to the shift by the critical level). When one of the forms tends to 0 the other tends to infinity, which corresponds to degeneration of the TDO algebra into a commutative algebra of functions on a twisted cotangent bundle to Bun_G. This shows that the quantum geometric Langlands is a deformation of the classical geometric Langlands, which is an equivalence between the category of (certain) $\mathcal{D}$-modules on Bun_G and the category of (certain) quasi-coherent sheaves on the stack $\mathrm{Loc}_{^LG}$ of ^LG–local systems on C.

We will be interested in the case $G = \mathrm{GL}_N$—the general linear group. In this case, we think of the quantum Langlands correspondence as the equivalence $\mathcal{D}_{\mathrm{Bun}_N, \mathcal{L}_{\det}^c}$-mod $\sim$ $\mathcal{D}_{\mathrm{Bun}_N, \mathcal{L}_{\det}^{-1/c}}$-mod where $\mathcal{L}_{\det}$ is the determinant line bundle on $\mathrm{Bun}_N = \mathrm{Bun}_G = \mathrm{Bun}_{^LG}$ given by $(\mathcal{L}_{\det})_b = \det \mathrm{R}\Gamma(C, \mathcal{E}_b)$ for any $b \in \mathrm{Bun}_N$ where $\mathcal{E}_b$ denotes the rank N vector bundle corresponding to b. (There is a subtle question of what kind of $\mathcal{D}$-modules one should consider, but we'll ignore it for now.)

1.2 The characteristic p case: classical story

In [BB], R. Bezrukavnikov and A. Braverman established a version of the classical geometric Langlands correspondence for "crystalline" $\mathcal{D}$-modules over a field k of characteristic $p > 0$. Recall that, for a smooth scheme X over k, the sheaf $\mathcal{D}_X$ of *crystalline differential operators* is defined as the universal enveloping algebra of the Lie algebroid $\mathcal{T}_X$ of vector fields on X. The main tool for studying modules over such algebras is their Azumaya property (see [BMR]). Namely, $\mathcal{D}_X$ turns out to be isomorphic to (the pushforward to X of) an Azumaya algebra $\tilde{\mathcal{D}}_X$ on $T^*X^{(1)}$—the cotangent bundle to the Frobenius twist of X. This allows one to identify the category of $\mathcal{D}$-modules on X with the category of coherent sheaves on a $\mathbb{G}_m$-gerbe on $T^*X^{(1)}$.

This observation is generalized in [BB] to the case of (a certain class of) algebraic stacks. Namely, for an irreducible smooth Artin stack $\mathscr{Y}$ over k with $\dim T^*\mathscr{Y} = 2\dim\mathscr{Y}$ (i.e., $\mathscr{Y}$ is good in the sense of [BD]), they construct a sheaf $\tilde{\mathcal{D}}_{\mathscr{Y}}$ of algebras on $T^*\mathscr{Y}^{(1)}$ with properties similar to the Azumaya algebra $\tilde{\mathcal{D}}_X$ described above. The pushforward of $\tilde{\mathcal{D}}_{\mathscr{Y}}$ to $\mathscr{Y}^{(1)}$ is isomorphic to $\mathrm{Fr}_{\mathscr{Y}*}\mathcal{D}_{\mathscr{Y}}$ where $\mathcal{D}_{\mathscr{Y}}$ is the sheaf of differential operators as defined in [BD]. Moreover, the restriction of $\tilde{\mathcal{D}}_{\mathscr{Y}}$ to the maximal open smooth Deligne–Mumford substack of $T^*\mathscr{Y}^{(1)}$ is an Azumaya algebra.[1]

The stack Bun_N is almost "good," namely, it locally looks like product of a good stack and $B\mathbb{G}_m$. So one can apply the above construction to $\mathscr{Y} = \mathrm{Bun}_N$ to get an Azumaya algebra on $T^*\mathrm{Bun}_N^{(1)} = \mathrm{Higgs}^{(1)}$. The latter stack is the total space of the Hitchin fibration $h^{(1)}\colon \mathrm{Higgs}^{(1)} \to \mathscr{B}^{(1)}$ whose generic fibers are Picard stacks of (spectral) curves. On the dual side, one has the "p-Hitchin" map $\mathrm{Loc} \to \mathscr{B}^{(1)}$ given by p-curvature. Generic fibers of this map are torsors over the same Picard stacks, and each point of such a torsor (which corresponds to G–local system on C with given spectral curve) gives a splitting of $\tilde{\mathcal{D}}_{\mathrm{Bun}}$ on the corresponding fiber of $H^{(1)}$. This splitting defines a Hecke eigensheaf corresponding to the local system. The geometric Langlands is thus realized as a twisted version of Fourier–Mukai transform.

[1] In fact, this construction can be strengthened, namely, one can define a $\mathbb{G}_m$-gerbe on *all of* $T^*\mathscr{Y}^{(1)}$, not just its smooth part, as we show in 2.4. This gerbe classifies $\mathcal{D}$-modules on $\mathscr{Y}$, defined in a way similar to [BD, Sect. 1.1]. The "regular" $\mathcal{D}$-module, however, corresponds to a coherent sheaf on this gerbe which is locally free only on the smooth part—that's why $\tilde{\mathcal{D}}_{\mathscr{Y}}$, which is (opposite of) the endomorphism algebra of this coherent sheaf, is an Azumaya algebra only on that smooth locus.

1.3 Quantum story

In this paper, the same ideas are applied to quantum geometric Langlands correspondence. To that end, we generalize the above Azumaya algebra construction to the case of twisted differential operators. The only TDO algebras we will encounter are of the form $\mathcal{D}_{\mathcal{L}^c}$ where $\mathcal{L}$ is a line bundle and $c \in \mathsf{k}$ (and external tensor products of such). In this case, the situation is essentially analogous to the non-twisted case, except that the Azumaya algebra will now live on the twisted cotangent bundle, where the twisting is given by $(c^p - c)$ times the Chern class of $\mathcal{L}^{(1)}$ (cf. [BMR]). We will only consider the case of irrational c (i.e., $c \notin \mathbb{F}_p$): in this case one can identify this twisted cotangent bundle with the Frobenius twist of the space $\tilde{T}^*_{\mathcal{L}}X$ of 0-jets of connections on $\mathcal{L}$. This is discussed in 2.3.

It is not hard to extend it to the stack case using the above-mentioned results from [BB] for usual $\mathcal{D}$-modules on stacks. Thus, for a line bundle $\mathcal{L}$ on a good stack $\mathscr{Y}$, one gets an Azumaya algebra $\tilde{\mathcal{D}}_{\mathscr{Y},\mathcal{L}^c}$ on the smooth part of $(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{(1)}$. (For a discussion of twisted cotangent bundles to stacks, see A.1.)

1.3.1 Main result

We apply this to the determinant bundle $\mathcal{L}_{\det}$ on Bun. One can check (see Appendix A) that the corresponding twisted cotangent is identified with the moduli space $\mathrm{Loc}_{\omega^{1/2}}$ of rank N bundles on C endowed with an action of the TDO algebra $\mathcal{D}_{\omega^{1/2}}$. In fact, we can identify $\mathrm{Loc}_{\omega^{1/2}}$ with Loc by tensoring bundles with $\omega^{\otimes(p-1)/2}$. Thus, both sides of the quantum Langlands are described (again, generically over the Hitchin base) by certain gerbes on $(\mathrm{Loc}^0)^{(1)}$. Here $\mathrm{Loc}^0 = \mathrm{Loc} \times_{\mathscr{B}^{(1)}} (\mathscr{B}^0)^{(1)}$ and $\mathscr{B}^0 \subset \mathscr{B}$ is the open part parametrizing smooth spectral curves. Using the p-Hitchin map as above (this time to $\mathscr{B}^{(2)}$), we see that these gerbes live on two torsors over the relative Picard stack mentioned above. So we get again two "twisted versions" of the derived category of coherent sheaves on this Picard stack. In contrast to the classical (non-quantum) case, however, we have both "torsor" and "gerby" twists on each side. These two kinds of twists are interchanged by Fourier–Mukai duality.

In other words, we prove the following:

Theorem 1. *There is an equivalence between bounded derived categories of modules for* $\mathcal{D}_c = \tilde{\mathcal{D}}_{\mathrm{Bun},\mathcal{L}_{\det}^c}|_{(\mathrm{Loc}^0)^{(1)}}$ *and* $\mathcal{D}_{-1/c} = \tilde{\mathcal{D}}_{\mathrm{Bun},\mathcal{L}_{\det}^{-1/c}}|_{(\mathrm{Loc}^0)^{(1)}}$*. The corresponding kernel is a splitting of* $\mathcal{D}_c \boxtimes \mathcal{D}^{\mathrm{op}}_{-1/c} \sim \mathcal{D}_c \boxtimes \mathcal{D}_{1/c}$ *on the fiber product* $(\mathrm{Loc}^0)^{(1)} \times_{\mathscr{B}^{(2)},c^p} (\mathrm{Loc}^0)^{(1)}$ *where the projection from the second factor to* $\mathscr{B}^{(2)}$ *is modified by the action of* $c^p \in \mathbb{G}_m$*. If we choose, locally on* $\mathscr{B}^{(2)}$*, a trivialization of the torsor* $\mathrm{Loc}^{(1)} \to \mathscr{B}^{(2)}$*, then there are splittings of* $\mathcal{D}_c$ *and* $\mathcal{D}_{-1/c}$ *such that the equivalence is identified with the Fourier–Mukai transform on the Picard stack* $\mathrm{Pic}((\tilde{\mathscr{C}}^0)^{(2)}/\mathscr{B}^{(2)})$*. (Here* $\tilde{\mathscr{C}}^0 \subset T^*C \times \mathscr{B}^0$ *is the universal spectral curve.)*

Note that, although the underlying spaces of the torsors are the same on both sides (namely $(\mathrm{Loc}^0)^{(1)}$), in order to make the duality work, one has to normalize the projection to $\mathscr{B}^{(2)}$ differently. This can also be guessed by considering what happens at rational c (including $c = 0, \infty$).

1.3.2 Extended curvature

So, all we need to check is that the torsors with gerbes corresponding to $\mathcal{D}_c$ and $\mathcal{D}_{-1/c}$ are interchanged by Fourier–Mukai duality. For that purpose we need a description of gerbes attached to TDO algebras. Recall that in the non-twisted case, the splittings of $\tilde{\mathcal{D}}_X$ on an open subset $U^{(1)} \subset T^*X^{(1)}$ correspond to line bundles on U with flat connection of p-curvature equal to the canonical 1-form on $T^*X^{(1)}$.

To extend this description to the TDO case, we introduce a generalization of the notion of p-curvature to non-flat connections. For a line bundle $\mathcal{L}$ with connection ∇ on a smooth variety X, we define in 2.5 a section $\widetilde{\mathrm{curv}}(\mathcal{L},\nabla)$ (called the *extended curvature*) of the quotient sheaf $\mathcal{F}_X$ of Ω^1_X by locally exact forms. This sheaf maps to Ω^2_X via de Rham differential; this map carries $\widetilde{\mathrm{curv}}(\mathcal{L},\nabla)$ to the usual curvature. On the other hand, for flat connections, $\widetilde{\mathrm{curv}}(\mathcal{L},\nabla)$ is a section of closed modulo exact forms, which corresponds to the p-curvature of ∇ under Cartier isomorphism. This construction also allows, starting from a section $\alpha \in \mathcal{F}_X$ (such a section will sometimes be referred to as a *generalized one-form*), to define a $\mathbb{G}_m$-gerbe on $X^{(1)}$:

its splittings correspond to line bundles with connection whose extended curvature is equal to α.

Now, the pullback of any line bundle $\mathcal{L}$ to its associated twisted cotangent $\tilde{T}^*_{\mathcal{L}}X$ acquires a canonical connection. If $\alpha_{\mathcal{L}}$ denotes the extended curvature of this connection, the gerbe on $(\tilde{T}^*_{\mathcal{L}}X)^{(1)}$ corresponding to the Azumaya algebra $\tilde{\mathcal{D}}_{\mathcal{L}^c}$ for $c \in \mathsf{k} \setminus \mathbb{F}_p$ is obtained from the above construction applied to $c\alpha_{\mathcal{L}}$.

1.3.3 The Poincaré bundle

Then we construct an explicit kernel of the equivalence (an analogue of the Poincaré bundle). This is a line bundle with connection on the fiber product of two copies of Loc^0 over the Hitchin base (see formula (4.6)). The construction is similar to that of Poincaré bundle on the square of the Picard stack of a curve:

$$\mathrm{Poincaré}(\mathcal{L}, \mathcal{L}') = \det \mathrm{R}\Gamma(\mathcal{L} \otimes \mathcal{L}') \otimes (\det \mathrm{R}\Gamma(\mathcal{L}) \otimes \det \mathrm{R}\Gamma(\mathcal{L}'))^{\otimes -1}.$$

Namely, the determinant bundle on the Picard stack gets replaced by the determinant bundle on Loc^0 with "tautological" connection (the same one that is used to describe the gerbe on $(\mathrm{Loc}^0)^{(1)}$), while the role of tensor product of line bundles is played by the addition map on the fibers of the p-Hitchin map:

$$\mathrm{Loc}^0_1 \underset{\mathscr{B}^{(1)}}{\times} \mathrm{Loc}^0_c \longrightarrow \mathrm{Loc}^0_{1+c}.$$

Here subscripts indicate scaling of the projection to the Hitchin base. The fiber of Loc^0_c classifies splittings on the spectral curve of the gerbe corresponding to the canonical 1-form on $T^*C^{(1)}$ multiplied by c. This map can then be thought of as "tensoring over the spectral curve."

The main difficulty is then to check that this bundle with connection has the correct p-curvature. This reduces to a certain linear equality on the extended curvatures (formula (4.8)). This formula can be interpreted as a kind of additivity of the generalized one-form $c^{-1}\tilde{\theta}$ on Loc^0_c with respect to the addition maps above, where $\tilde{\theta}$ denotes the extended curvature of the tautological connection on the determinant bundle.

1.3.4 Antiderivative of the symplectic form on Loc

In 4.2, we construct another generalized one-form $\tilde{\theta}_0$ on Loc^0 (actually on the maximal smooth part of Loc) whose image in Ω^2 coincides with that of $\tilde{\theta}$ (both are equal to the symplectic form on Loc^0) but whose behavior with respect to the p-Hitchin map is more controllable. We prove the additivity property for it, and then show that $\tilde{\theta} = \tilde{\theta}_0$. In fact, $\tilde{\theta}_0$ lifts to an actual antiderivative θ_0 of the symplectic form. Such antiderivatives correspond bijectively to Liouville vector fields. We construct such a vector field using an equivariant structure of the gerbe on $T^*C^{(1)}$ under the Euler vector field. Such structures correspond to liftings of C to the 2nd Witt vectors of k. Since $\tilde{\theta} - \tilde{\theta}_0$ is closed, it corresponds by Cartier to a 1-form β_0 on $(\mathrm{Loc}^0)^{(1)}$ and we have to prove that it is 0.

The definition of Loc_c above makes sense for all $c \in \mathsf{k}$; in particular, for $c = 0$ it gives $\mathrm{Loc}_0 = \mathrm{Higgs}^{(1)}$. On $\mathrm{Higgs}^{(1)}$ we have the canonical 1-form $\theta^{(1)}_{\mathrm{Higgs}}$ (as on a cotangent bundle). We prove that both $\tilde{\theta}$ and θ_0 are compatible with $\theta^{(1)}_{\mathrm{Higgs}}$ with respect to the action map

$$\mathrm{Loc}^0_0 \underset{\mathscr{B}^{(1)}}{\times} \mathrm{Loc}^0 \longrightarrow \mathrm{Loc}^0 .$$

In the beginning of Section 4 we prove this for $\tilde{\theta}$. It is enough to prove it on the image of the Abel–Jacobi map in Higgs, which in turn reduces to studying how the determinant bundle (with connection) on Loc^0 changes when we twist the local system by a point of its p-spectral curve. The compatibility of θ_0 with $\theta^{(1)}_{\mathrm{Higgs}}$ is proved as part of the additivity for θ_0. (In fact, the additive family of 1-forms on Loc^0_c constructed in 4.2 specializes to θ_0 for $c = 1$ and to $\theta^{(1)}_{\mathrm{Higgs}}$ for $c = 0$.)

From this we conclude that β_0 descends to the Hitchin base. On the other hand, in 4.3 we study the behavior of β_0 with respect to the projection Loc $\to$ Bun. First, by a degree estimate we show that the restriction to the fibers of this projection have constant coefficients. Then, a global argument shows that in fact this restriction must be 0. The fibers of the two projections $\mathrm{Loc} \to \mathscr{B}^{(1)}$ and $\mathrm{Loc} \to \mathrm{Bun}$ are generically transversal (at least, we know how to prove this for one of the components of Loc assuming C is ordinary), which gives the desired equality $\beta_0 = 0$.

Table of notation

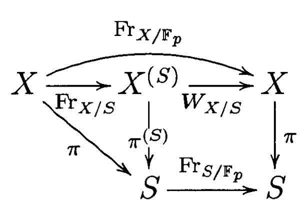
k	– an algebraically closed field of characteristic $p > 0$
G	– the general linear group $\mathrm{GL}(N)$
C	– a complete smooth algebraic curve over k
Bun	– the moduli stack of G-bundles on C
$\mathscr{B}$	– the Hitchin base (the affine space $\bigoplus_{i=1}^{N} H^0(C, \omega_C^{\otimes i})$)
$\mathscr{B}^0 \subset \mathscr{B}$	– the open part classifying smooth spectral curves.
Higgs	– the total space of the Hitchin fibration, the moduli stack of Higgs bundles, $\mathrm{Higgs} = T^* \mathrm{Bun}$
$\tilde{\mathscr{C}}$	– "universal spectral curve," $\tilde{\mathscr{C}} \subset T^*C \times \mathscr{B}$

Chapter 2

Differential operators in positive characteristic

2.1 Frobenius morphisms and twists

For any scheme S of characteristic p (i.e., such that $p\mathcal{O}_S = 0$) the absolute Frobenius $\mathrm{Fr}_{S/\mathbb{F}_p} \colon S \to S$ is defined as id_S on the topological space and $\mathrm{Fr}^{\#}_{S/\mathbb{F}_p}(f) = f^p$ on functions. For any S-scheme $X \xrightarrow{\pi} S$ one constructs a commutative diagram



where the square is Cartesian. We call the S-scheme $X^{(S)} \xrightarrow{\pi^{(S)}} S$ the *relative Frobenius twist* of X over S, and call $\mathrm{Fr}_{X/S}$ the *relative Frobenius morphism*. We will denote by $\bullet^{(S)}$ the pullback by $\mathrm{Fr}_{S/\mathbb{F}_p}$ or $W_{X/S}$. In the case $S = \operatorname{Spec} \mathsf{k}$ we will drop "S" and write Fr_X and $X^{(1)}$ instead of $\mathrm{Fr}_{X/S}$ and $X^{(S)}$. The k'th iterate of $\bullet^{(1)}$ will be denoted $\bullet^{(k)}$.

2.2 λ-connections

Recall that a λ-*connection* on a vector bundle E on a smooth variety X is a k-linear morphism of sheaves $\tilde{\nabla}\colon \mathcal{E} \to \Omega^1 \otimes_{\mathcal{O}} \mathcal{E}$ such that

$$\forall f \in \mathcal{O} \quad \forall s \in \mathcal{E} \qquad \tilde{\nabla}(fs) = f \cdot \tilde{\nabla} s + \lambda \cdot df \otimes s$$

where $\mathcal{E}$ is the sheaf of sections of E.

Define the *curvature* of a λ-connection $\tilde{\nabla}$ to be the section $F_{\tilde{\nabla}}$ of $\Omega^2 \otimes \mathcal{E}nd\,\mathcal{E}$ corresponding to the $\mathcal{O}$-linear map $\tilde{\nabla}^2\colon \mathcal{E} \to \Omega^2 \otimes \mathcal{E}$ where $\tilde{\nabla}$ is extended to $\Omega^\bullet \otimes \mathcal{E}$ by the following "Leibnitz rule":

$$\tilde{\nabla}(\omega \otimes s) = (-1)^{\deg \omega} \omega \wedge \tilde{\nabla} s + \lambda \cdot d\omega \otimes s.$$

An alternative definition of $F_{\tilde{\nabla}}$ is that for any $\xi, \eta \in \mathcal{T}_X$ we must have

$$F_{\tilde{\nabla}}(\xi, \eta) = [\tilde{\nabla}_\xi, \tilde{\nabla}_\eta] - \lambda \cdot \tilde{\nabla}_{[\xi.\eta]}.$$

If $F_{\tilde{\nabla}} = 0$, we say that $\tilde{\nabla}$ is *flat*.

For $\lambda \neq 0$ if $\tilde{\nabla}$ is a λ-connection then $\nabla = \lambda^{-1}\tilde{\nabla}$ is a connection, and vice versa. In this case, the curvature of a λ-connection can be expressed in terms of the ordinary curvature: $F_{\tilde{\nabla}} = \lambda^2 F_{\nabla}$. The case $\lambda = 0$ can be thought of as a limit when $\lambda \to 0$. In particular if $\mathcal{E}$ is trivialized, and $\tilde{\nabla} = \lambda d + \theta$ then $F_{\tilde{\nabla}} = \lambda d\theta + \theta \wedge \theta$

Vector bundles with a flat λ-connection correspond to $\mathcal{O}$-flat $\mathcal{O}$-coherent modules over the algebra $\mathcal{D}_\lambda$ which is the universal enveloping algebra of the Lie algebroid $\mathcal{T}_{X,\lambda} = \mathcal{T}_X$ over $\mathcal{O}$ with rescaled commutator: $[\xi, \eta]_\lambda = \lambda[\xi, \eta]$. There is an inclusion $\mathcal{O}_{T^*X^{(1)}} \hookrightarrow Z(\mathcal{D}_\lambda)$ (the center of $\mathcal{D}_\lambda$) which is an isomorphism for $\lambda \neq 0$. It is given by $f^{(1)} \mapsto f^p$, $\xi^{(1)} \mapsto \hat{\xi}^p - \lambda^{p-1}\hat{\xi}^{[p]}$ where ξ in the LHS is regarded as a fiberwise linear function on T^*X, and $\hat{\xi}$ in the RHS is the corresponding element in $\mathcal{D}_\lambda$. For $\lambda = 0$ the inclusion is just the Frobenius map $\mathrm{Fr}^*\colon \mathcal{O}_{T^*X^{(1)}} \to \mathcal{O}_{T^*X}$.

We can then define the *p-spectral variety* of a λ-connection $\tilde{\nabla}$ on a vector bundle E

as the support of the corresponding $\mathcal{D}_\lambda$-module regarded as an $\mathcal{O}_{T^*X^{(1)}}$-module. By the *p-curvature map* of $\tilde{\nabla}$ we will mean the map $\operatorname{curv}_p(\nabla)\colon \mathcal{E} \to \mathcal{E} \otimes \operatorname{Fr}_X^* \Omega^1_{X^{(1)}}$ coming from the action of $\mathcal{O}_{T^*X^{(1)}}$ on $\mathcal{E}$. For $\lambda \neq 0$ it is related to the ordinary p-curvature by $\operatorname{curv}_p(\nabla) = \lambda^p \operatorname{curv}_p(\lambda^{-1}\nabla)$ (here $\lambda^{-1}\nabla$ is a usual connection).

For a line bundle $\mathcal{L}$ on X and $\lambda \in \mathsf{k}$, define a torsor $\tilde{T}^*_{\mathcal{L}^\lambda}X$ over T^*X whose sections are λ-connections on $\mathcal{L}$.

Remark 1. As a variety, $\tilde{T}^*_{\mathcal{L}^\lambda}X$ is isomorphic to $\tilde{T}^*_{\mathcal{L}}X := \tilde{T}^*_{\mathcal{L}^1}X$ for $\lambda \neq 0$ and to T^*X for $\lambda = 0$.

2.3 Twisted differential operators

If X is a smooth algebraic variety and $\mathcal{L}$ is a line bundle on it, we define a sheaf of algebras $\mathcal{D}_{X,\mathcal{L}^c}$ for any $c \in \mathsf{k}$ as follows. For any local trivialization $\phi\colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}|_U$ of $\mathcal{L}$ on an open set U, we have a canonical isomorphism $\alpha_\phi\colon \mathcal{D}_U \xrightarrow{\sim} \mathcal{D}_{U,\mathcal{L}^c}$, and if ϕ' on U' is another trivialization then the gluing isomorphism $\alpha_{\phi'}^{-1} \circ \alpha_\phi$ is given by

$$\begin{cases} f \mapsto f & \text{for } f \in \mathcal{O}_X, \text{ and} \\ \xi \mapsto \xi - c\xi(h)/h & \text{for } \xi \in \mathcal{T}_X \end{cases} \tag{2.1}$$

where $h \in \mathcal{O}^\times(U \cap U')$ is such that $(\phi')^{-1} \circ \phi\colon \mathcal{O}(U \cap U') \to \mathcal{O}(U \cap U')$ is given by multiplication by h.

Just as the sheaf of usual differential operators, the algebra $\mathcal{D}_{X,\mathcal{L}^c}$ has a filtration with the associated graded isomorphic to (the pushforward of) $\mathcal{O}_{T^*X}$. In particular, taking the first filtered piece gives an extension of coherent sheaves on X:

$$0 \to \mathcal{O}_X \to \mathcal{D}^{\leq 1}_{X,\mathcal{L}^c} \to \mathcal{T}_X \to 0.$$

Moreover, it is not hard to check that the torsor of splittings of this extension is canonically identified with the torsor $\tilde{T}^*_{\mathcal{L}}X$ of c-connections on $\mathcal{L}$.

Now let $\tilde{\nabla}$ be such a c-connection. Then the obstruction for the corresponding map $l_\nabla \colon \mathcal{T}_X \to \mathcal{D}^{\leq 1}_{X,\mathcal{L}^c}$ to be a map of Lie algebras is given by the curvature of ∇:

$$[l_{\tilde\nabla}(\xi), l_{\tilde\nabla}(\eta)] - l_{\tilde\nabla}([\xi,\eta]) = F_{\tilde\nabla}(\xi,\eta),$$

where $F_{\tilde\nabla} \in \Omega^2(X)$ is the curvature of the c-connection ∇. Consequently, a structure of a $\mathcal{D}_{\mathcal{L}^c}$-module on a given quasi-coherent sheaf $\mathcal{E}$ is equivalent to a connection ∇' on $\mathcal{E}$ with the curvature given by

$$F_{\nabla'} = F_{\tilde\nabla} \cdot \mathrm{id}_{\mathcal{E}}\,.$$

Proposition 2. *Denote by Y the relative spectrum of the center of $\mathcal{D}_{\mathcal{L}^c}$: $Y = \mathrm{Spec}_{X^{(1)}} Z(\mathrm{Fr}_{X*}\mathcal{D}_{X,\mathcal{L}^c})$. Then Y is canonically isomorphic to $\tilde{T}^*_{\mathcal{L}^{(1)c^p-c}}X^{(1)}$ (as an $X^{(1)}$-scheme). Moreover, $\mathrm{Fr}_{X*}\mathcal{D}_{\mathcal{L}^c}$ is the pushforward of an Azumaya algebra $\tilde{\mathcal{D}}_{\mathcal{L}^c}$ on Y.*

If we are given a trivialization $\phi \colon \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}$ then the following diagram commutes (where the vertical arrows are induced by ϕ and $\phi^{(1)}$, and the bottom arrow is the standard isomorphism (see e.g., [BMR, Lemma 1.3.2])):

$$\begin{array}{ccc} Y & \xrightarrow{\ \sim\ } & \tilde{T}^*_{\mathcal{L}^{(1)c^p-c}}X^{(1)} \\ {\scriptstyle \phi}\downarrow{\scriptstyle \wr} & & {\scriptstyle \phi^{(1)}}\downarrow{\scriptstyle \wr} \\ \mathrm{Spec}_{X^{(1)}} Z(\mathrm{Fr}_{X*}\mathcal{D}_X) & \xrightarrow{\ \sim\ } & T^*X^{(1)} \end{array}$$

Proof. It is enough to construct the identification for the trivial line bundle $\mathcal{L}$ and then prove that it is independent of the trivialization. But for the trivial line bundle we already have such an identification (referred to as "standard" in the formulation).

To show the independence of the trivialization, suppose we have an automorphism of the trivial line bundle $\mathcal{O}_X$ given by an invertible function h. The corresponding automorphism ψ_h of $\mathcal{D}_X$ is given by (2.1). Combining with the formulas for the identification of $Z(\mathcal{D}_X)$ with the pushforward of $\mathcal{O}_{T^*X^{(1)}}$:

$$\begin{cases} f^{(1)} \mapsto f^p & \text{for } f \in \mathcal{O}_X; \\ \xi^{(1)} \mapsto \xi^p - \xi^{[p]} & \text{for } \xi \in \mathcal{T}_X \end{cases}$$

we see that, for any vector field ξ on (an open subset of) X, the action of ψ_h on the element of $Z(\mathcal{D}_X) \cong \operatorname{Sym} \mathcal{T}_{X^{(1)}}$ corresponding to $\xi^{(1)}$ is given by

$$\psi_h(\xi^p - \xi^{[p]}) = (\xi - c\xi(h)/h)^p - (\xi^{[p]} - c\xi^{[p]}(h)/h).$$

Using the identity (in any associative algebra in characteristic p)

$$(a+b)^p = a^p + b^p + (\operatorname{ad} a)^{p-1}(b) \quad \text{if } [[a,b],b] = 0,$$

the above expression can be rewritten as

$$\psi_h(\xi^p - \xi^{[p]}) = (\xi^p - \xi^{[p]}) - c^p(\xi(h)/h)^p + c\big(\xi^p(h)/h - \xi^{p-1}(\xi(h)/h)\big).$$

Now note that when $c = 1$ we have $\psi_h = \operatorname{Ad} h$. Therefore, since $\xi^p - \xi^{[p]}$ is central, we must have $\psi_h(\xi^p - \xi^{[p]}) = \xi^p - \xi^{[p]}$ in this case. Thus, the above equation gives

$$(\xi(h)/h)^p + \xi^p(h)/h - \xi^{p-1}(\xi(h)/h) = 0,$$

and hence, for arbitrary c, the formula becomes

$$\psi_h(\xi^p - \xi^{[p]}) = (\xi^p - \xi^{[p]}) - (c^p - c)(\xi(h)/h)^p.$$

We see that this formula coincides with the action of h on linear functions on the twisted cotangent $\tilde{T}^*_{\mathcal{L}^{(1)}c^p - c} X^{(1)}$. □

Remark 2. Another way to finish the argument is as follows. Let us observe that the effect of ψ_h on $\mathcal{D}_X$-modules is equivalent to tensoring by the line bundle with connection $(\mathcal{O}, d + c \cdot d\log h)$. Therefore its effect on the p-support is given by the shift by the p-curvature of $d+c\cdot d\log h$. This p-curvature equals $(c\cdot d\log h)^{(1)}-\mathsf{C}(c\cdot d\log h) = (c^p - c)\cdot d\log h^{(1)}$ (where C is the Cartier operator), hence we get the desired formula.

2.3.1 TDO with "Planck's constant"

Suppose $X, \mathcal{L}$ are as in 2.3. Define the $\mathsf{k}[c,\hbar]$-algebra $\mathcal{D}^{\mathcal{L}}_{c,\hbar}(X)$ as follows. Let $\pi\colon \tilde{X} \to X$ be the principal $\mathbb{G}_m$-bundle corresponding to $\mathcal{L}$. Denote by $\mathcal{D}_\hbar(\tilde{X})$ the algebra of "differential operators with parameter," that is, the algebra

$$\mathcal{D}_\hbar(\tilde{X}) := \bigoplus_{n\geq 0} \mathcal{D}^{\leq n}(\tilde{X})$$

over $\mathsf{k}[\hbar]$ where the inclusion $\mathsf{k}[\hbar] \hookrightarrow \mathcal{D}_\hbar(\tilde{X})$ is given by $\hbar \mapsto 1 \in \mathcal{D}^{\leq 1}(\tilde{X})$. Here we introduce its TDO analog. For $\xi \in \mathcal{T}_{\tilde{X}}$ let $\hat{\xi}$ be the corresponding element in $\mathcal{D}^{\leq 1} \subset \mathcal{D}_\hbar$. Let Eu be the Euler vector field on $\tilde{X}$ (the differential of the $\mathbb{G}_m$-action). Now set

$$\mathcal{D}^{\mathcal{L}}_{c,\hbar}(X) := (\pi_*\mathcal{D}_\hbar(\tilde{X}))^{\mathbb{G}_m}.$$

This is a $\mathsf{k}[c,\hbar]$-algebra via $\hbar \mapsto \hbar \in \mathcal{D}_\hbar(\tilde{X})$, $c \mapsto \widehat{\mathrm{Eu}}$. Note also that $\pi_*\mathcal{D}_\hbar(\tilde{X})$ has two gradings: one comes from the definition of $\mathcal{D}_\hbar$ as a direct sum, and the other comes from the $\mathbb{G}_m$-action on $\tilde{X}$. But on the $\mathbb{G}_m$-invariant part, we have only one grading (the first one), and with respect to this grading $\deg c = \deg \hbar = 1$. The algebra $\mathcal{D}^{\mathcal{L}}_{c,\hbar}$ being graded implies that it carries an action of $\mathbb{G}_m$ and, in particular, if $\mathcal{D}^{\mathcal{L}}_{c_0,\hbar_0}$ denotes the specialization $c \mapsto c_0$, $\hbar \mapsto \hbar_0$ of the algebra $\mathcal{D}^{\mathcal{L}}_{c,\hbar}$ (where $c_0, \hbar_0 \in \mathsf{k}$) then

$$\mathcal{D}^{\mathcal{L}}_{c_0,\hbar_0} \cong \mathcal{D}^{\mathcal{L}}_{\lambda c_0,\lambda\hbar_0} \tag{2.2}$$

for any $\lambda \in \mathsf{k}^\times$.

The specialization $c \mapsto 0$ gives the algebra $\mathcal{D}_\hbar$ defined above, and in particular, $\mathcal{D}^{\mathcal{L}}_{0,0} = (\mathrm{pr}_X)_*\mathcal{O}_{T^*X}$ where $\mathrm{pr}_X\colon T^*X \to X$. Furthermore, it is not hard to show that $\mathcal{D}^{\mathcal{L}}_{c,0} = (\mathrm{pr}'_X)_*\mathcal{O}_{\tilde{T}^*_{\mathcal{L}^c}X}$ (where pr'_X is again the appropriate projection to X). One can also check that specialization $\hbar \mapsto 1$ recovers the algebra $\mathcal{D}_{\mathcal{L}^c}$ from 2.3. Taking the isomorphism (2.2) into account, we can summarize:

$$\mathcal{D}^{\mathcal{L}}_{c_0,\hbar_0} \cong \begin{cases} \mathrm{pr}_* \mathcal{O}_{T^*X} & \text{if } c_0 = \hbar_0 = 0; \\ \mathrm{pr}'_* \mathcal{O}_{\tilde{T}_{\mathcal{L}}X} & \text{if } c_0 \neq 0,\ \hbar_0 = 0; \\ \mathcal{D}_{\mathcal{L}^{c_0/\hbar_0}} & \text{if } \hbar_0 \neq 0. \end{cases}$$

The following theorem is a generalization of Proposition 2:

Proposition 2′.

1. *The center of the algebra* $\mathcal{D}^{\mathcal{L}}_{c,\hbar}(X)$ *is canonically isomorphic to* $\mathcal{O}_{\mathscr{X}}$ *where* $\mathscr{X} = \tilde{T}^*_{\mathcal{L}^{(1)c^p - c\hbar^{p-1}}}X^{(1)}$.[1]

2. *Moreover, if* $c_0, \hbar_0 \in \mathsf{k}$*, the specialization* $c \mapsto c_0,\ \hbar \mapsto \hbar_0$ *induces a map* $\mathcal{O}_{\mathscr{X}_{c_0,\hbar_0}} \to Z(\mathcal{D}^{\mathcal{L}}_{c_0,\hbar_0})$ *which is an isomorphism if and only if* $\hbar_0 \neq 0$*, in which case* $\mathcal{D}^{\mathcal{L}}_{c_0,\hbar_0}$ *is an Azumaya algebra over* $\mathscr{X}_{c_0,\hbar_0}$.

3. *The isomorphism* (2.2) *is compatible with the* $\mathbb{G}_m$*-action on* $\mathscr{X}$ *given by scaling connections by* λ^p.

2.3.2 Central reductions

Suppose $X, \mathcal{L}$ are as above, $c \in \mathsf{k}$. Then any $(c^p - c)$-connection $\tilde{\nabla}_0$ on $\mathcal{L}^{(1)}$ gives a section of the bundle $\tilde{T}^*_{\mathcal{L}^{(1)c^p-c}}X^{(1)}$ over $X^{(1)}$ defined above (we've changed notation from c_0 to c). Denote by $\mathcal{D}_{\mathcal{L}^c,\tilde{\nabla}_0}$ the pullback of $\tilde{\mathcal{D}}_{\mathcal{L}^c}$ to this section. It is an Azumaya algebra on $X^{(1)}$. If $c \notin \mathbb{F}_p$ then $(c^p - c)$-connections on $\mathcal{L}^{(1)}$ correspond bijectively to ordinary connections on $\mathcal{L}^{(1)}$ by multiplication by $c^p - c$, and, by a slight abuse of notation, we will sometimes denote the algebra $\mathcal{D}_{\mathcal{L}^c,\tilde{\nabla}_0}$ by $\mathcal{D}_{\mathcal{L}^c,\nabla_0}$ where ∇_0 is the connection on $\mathcal{L}^{(1)}$ for which $\tilde{\nabla}_0 = (c^p - c)\nabla_0$.. One can check that in this case then $\mathcal{O}$-coherent modules over $\mathcal{D}_{\mathcal{L}^c,\tilde{\nabla}_0}$ correspond to pairs $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a coherent

[1] $\mathscr{X}$ is a scheme over $\mathsf{k}[c, \hbar]$. One should extend the definition of $\tilde{T}^*_{\mathcal{L}^\lambda}X$ to the case of families over an arbitrary scheme in order to make sense of the definition of $\mathscr{X}$.

sheaf on X and ∇ is a connection on $\mathcal{E}$ such that

$$F_{\nabla} = c \cdot F_{\tilde{\nabla}'_0};$$
$$\nabla_\xi^p - \nabla_{\xi^{[p]}} = c \cdot \left((\nabla'_0)_\xi^p - (\nabla'_0)_{\xi^{[p]}}\right)$$

for any $\xi \in \mathcal{T}_X$, where ∇'_0 is a (usual) connection on X such that $\nabla_0 = \nabla_0'^{(1)}$. (The operators in the RHS are always multiplication by a function[2] (resp. a two-form), and we want the LHS to be multiplication by the same function (resp. two-form), though on a different bundle.) We will give another interpretation of these conditions in 2.5.

Now assume for simplicity that $c \notin \mathbb{F}_p$. Let $\tilde{T}^*_{\mathcal{L}}X$ be the twisted cotangent bundle associated to $\mathcal{L}$. The pullback $\mathcal{L}'$ of $\mathcal{L}$ to $\tilde{T}^*_{\mathcal{L}}X$ has a canonical "universal" connection which we will denote by $\nabla^{\mathcal{L}}_{\text{can}}$.

Proposition 3. *There is a canonical Morita equivalence of Azumaya algebras on* $(\tilde{T}^*_{\mathcal{L}}X)^{(1)}$*:*

$$\tilde{\mathcal{D}}_{\mathcal{L}^c} \sim \mathcal{D}_{\mathcal{L}'^c,(\nabla^{\mathcal{L}}_{\text{can}})^{(1)}}.$$

Proof. First we will construct a functor $\tilde{\mathcal{D}}_{\mathcal{L}^c}$-mod $\to \mathcal{D}_{\mathcal{L}'^c,\nabla^{\mathcal{L}}_{\text{can}}}$-mod and then explain why it is an equivalence. Let π denote the projection $\tilde{T}^*_{\mathcal{L}}X \to X$. We define the desired functor as the composition of the following functors:

$$\tilde{\mathcal{D}}_{\mathcal{L}^c}\text{-mod} \xrightarrow{\sim} \mathcal{D}_{\mathcal{L}^c}\text{-mod} \xrightarrow{\pi^!} \mathcal{D}_{\mathcal{L}'^c}\text{-mod} \to \mathcal{D}_{\mathcal{L}'^c,\nabla^{\mathcal{L}}_{\text{can}}}\text{-mod}.$$

Here $\pi^!$ is the usual pullback for twisted $\mathcal{D}$-modules (given by the $\mathcal{O}$-module pullback of the underlying quasi-coherent sheaves), and the last functor is given by induction (i.e., central reduction).

In order to check that this functor is $\mathcal{O}_{\tilde{T}^*_{\mathcal{L}}X}$-linear and is an equivalence, it is enough to consider the case when $\mathcal{L}$ is trivial, which reduces to the analogous statement for non-twisted $\mathcal{D}$-modules. This was proved in [BB]. □

[2]which is a pullback from $X^{(1)}$

2.4 TDOs on stacks

In this section we are going to generalize the above results to the case of stacks. So let $\mathscr{Y}$ be a smooth pure-dimensional Artin stack over k, $\mathcal{L}$ a line bundle on $\mathscr{Y}$, and $c \in \mathsf{k}$. The $\mathcal{D}$-modules and twisted $\mathcal{D}$-modules on stacks are discussed in [BD, Sect. 1.1] for the characteristic 0 case. For the non-twisted case in characteristic p, see [BB, Sect. 3.13].

Recall that a quasi-coherent sheaf, resp. $\mathcal{D}$-module, on $\mathscr{Y}$ is defined as a datum, for every smooth morphism $S \to \mathscr{Y}$ from a scheme S, of a quasi-coherent sheaf, resp. $\mathcal{D}$-module, $\mathcal{F}_S$ on S, together with identifications $f^*\mathcal{F}_S \xrightarrow{\sim} \mathcal{F}_{S'}$ for every morphism $f\colon S' \to S$ of smooth schemes over $\mathscr{Y}$ (i.e., S runs over the smooth site $\mathscr{Y}_{\mathrm{sm}}$ of $\mathscr{Y}$). These identifications are required to satisfy the cocycle condition for compositions. Similarly, one can define the category of $\mathcal{L}^c$-twisted $\mathcal{D}$-modules on $\mathscr{Y}$. In this subsection we will construct a certain $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$ on $\tilde{T}^*_{\mathcal{L}^{(1)c^p-c}}\mathscr{Y}^{(1)}$ such that modules over it are equivalent to $\mathcal{L}^c$-twisted $\mathcal{D}$-modules on $\mathscr{Y}$. For the sake of brevity of notation, we'll denote the latter twisted cotangent by $\tilde{T}^*\mathscr{Y}^{(1)}$, and for $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$, denote by $\tilde{T}^*S^{(1)}$ the twisted cotangent bundle for $(\pi^*\mathcal{L})^{(1)c^p-c}$.

First of all, in order to define a $\mathbb{G}_m$-gerbe on $\tilde{T}^*\mathscr{Y}^{(1)}$, it's enough to supply a $\mathbb{G}_m$-gerbe on $S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$ for every $S \in \mathscr{Y}_{\mathrm{sm}}$ (together with compatibility isomorphisms). But we have a map $S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)} \to \tilde{T}^*S^{(1)}$, and we have a gerbe $\mathcal{G}_{S,\pi^*\mathcal{L}^c}$ on $\tilde{T}^*S^{(1)}$ corresponding to the Azumaya algebra $\tilde{\mathcal{D}}_{S,\pi^*\mathcal{L}^c}$, so we can let $(\mathcal{G}_{\mathscr{Y},\mathcal{L}^c})_{S^{(1)}}$ be the pullback of $\mathcal{G}_{S,\pi^*\mathcal{L}^c}$ under this map. These gerbes are compatible with pullbacks: the equivalence is given by the TDO analog of the $\mathcal{D}$-module $\mathcal{D}_{S'\to S}$ from [BB]. The compatibility of these equivalences with compositions follows from the isomorphism $\mathcal{D}_{S''\to S'} \star \mathcal{D}_{S'\to S} \cong \mathcal{D}_{S''\to S}$ for a pair of morphisms $S'' \xrightarrow{g} S' \xrightarrow{f} S$, which in turn follows from the isomorphism $(fg)^! = g^! f^!$ for $\mathcal{D}$-module pullbacks, and similarly for the cocycle condition.

For a $\mathbb{G}_m$-gerbe $\mathcal{G}$, we denote $\mathcal{G}$-mod the category of "$\mathcal{G}$-twisted quasi-coherent sheaves," i.e., the 1st component of the category of quasi-coherent sheaves on the "total space" of $\mathcal{G}$.

Theorem 4. *There is a natural equivalence of categories*

$$\mathcal{D}\text{-mod}_{\mathcal{L}^c}(\mathscr{Y}) \sim \mathcal{G}_{\mathscr{Y},\mathcal{L}^c}\text{-mod}.$$

Proof. It is easy to construct a functor in one direction. Namely, given an object $\tilde{\mathcal{M}} \in \mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$-mod, we can construct an $\mathcal{L}^c$-twisted $\mathcal{D}$-module $\mathcal{M}$ on $\mathscr{Y}$ as follows. Given any $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$, we have maps $(\widetilde{d\pi})^{(1)}\colon S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)} \to \tilde{T}^*S^{(1)}$ and $\tilde{\pi}^{(1)}\colon S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)} \to \tilde{T}^*\mathscr{Y}^{(1)}$. So we can consider the $\mathcal{G}_{S,\mathcal{L}^c}$-module given by $(\widetilde{d\pi})^{(1)}_*\tilde{\pi}^{(1)*}\tilde{\mathcal{M}}$; this makes sense because by definition we have the canonical equivalence $(\widetilde{d\pi})^{(1)*}\mathcal{G}_{S,\pi^*\mathcal{L}^c} \cong \tilde{\pi}^{(1)*}\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$. We let $\mathcal{M}_S$ be the $\mathcal{D}_{S,\pi^*\mathcal{L}^c}$-module corresponding to this $\mathcal{G}_{S,\mathcal{L}^c}$-module. It is clear that for a map $f\colon S' \to S$ in $\mathscr{Y}_{\mathrm{sm}}$ we have an isomorphism $\mathcal{M}_{S'} \cong f^!\mathcal{M}_S$ and these isomorphisms are compatible with compositions, so the modules $\mathcal{M}_S$ for every $S \in \mathscr{Y}_{\mathrm{sm}}$ define an $\mathcal{L}^c$-twisted $\mathcal{D}$-module $\mathcal{M}$ on $\mathscr{Y}$. Moreover the assignment $\tilde{\mathcal{M}} \mapsto \mathcal{M}$ gives a functor $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}\text{-mod} \to \mathcal{D}\text{-mod}_{\mathcal{L}^c}(\mathscr{Y})$, which we (temporarily for this proof) denote by Φ.

Note that for any $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$, since π is smooth by assumption, the map $\widetilde{d\pi}$ (and hence $(\widetilde{d\pi})^{(1)}$) is a closed embedding. Therefore the functor $(\widetilde{d\pi})^{(1)}_*$ is fully faithful. Being by definition a limit of such functors, our functor Φ is thus fully faithful as well. It remains to show that it is essentially surjective. Fix an object $\mathcal{M} \in \mathcal{D}\text{-mod}_{\mathcal{L}^c}(\mathscr{Y})$. We see that $\mathcal{M}$ is in the essential image of Φ if and only if, for every $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$, the $\mathcal{D}_{S,\pi^*\mathcal{L}^c}$-module $\pi^!\mathcal{M} = \mathcal{M}_S$ has p-support inside $S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)} \subset \tilde{T}^*S^{(1)}$.

To see that the p-support of $\mathcal{M}_S$ is indeed contained in this closed subscheme, consider the fiber square $Q = S \times_{\mathscr{Y}} S$ and denote by $\mathrm{pr}_{1,2}\colon Q \rightrightarrows S$ the two projections. We have an isomorphism $\mathrm{pr}_1^!\mathcal{M}_S \cong \mathcal{M}_Q \cong \mathrm{pr}_2^!\mathcal{M}_S$ of twisted $\mathcal{D}$-modules on Q. But note that the p-support of $\mathrm{pr}_i^!\mathcal{M}_S$ is inside $Q^{(1)} \times_{\mathrm{pr}_i^{(1)},S^{(1)}} \tilde{T}^*S^{(1)}$, and the above isomorphism forces this p-support to lie in the intersection of these two closed subschemes inside $\tilde{T}^*Q^{(1)}$. This intersection coincides with $Q^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$. Since pr_i are surjective, the equality $\mathrm{supp}_p\,\mathrm{pr}_i^!\mathcal{M}_S = Q^{(1)} \times_{S^{(1)}} \mathrm{supp}_p\,\mathcal{M}_S$ implies that $\mathrm{supp}_p\,\mathcal{M}_S \subset S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$ as desired. □

Now we will prove an analog of Proposition 3 for the stack case. For a smooth stack $\mathscr{Y}$, a line bundle $\mathcal{L}$ on $\mathscr{Y}$, a scalar $c \in \mathsf{k} \setminus \mathbb{F}_p$, and a connection ∇ on $\mathcal{L}^{(1)}$, define a $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c,\nabla}$ on $\mathscr{Y}^{(1)}$ as follows. By definition, the connection ∇ defines a section s_∇ of the projection $(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{(1)} \to \mathscr{Y}^{(1)}$. Let $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c,\nabla}$ be the pullback of $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$ by this section.

Proposition 5. *For $\mathscr{Y}$, $\mathcal{L}$ and c as above, we have a canonical equivalence of gerbes over $((\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{(1)})^{sm}$:*

$$\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}|_{((\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{(1)})^{sm}} \sim \mathcal{G}_{(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm},\mathcal{L}'^c,\nabla^{(1)}_{\mathrm{can}}}$$

where $(\mathcal{L}', \nabla_{\mathrm{can}})$ is the pullback of $\mathcal{L}$ to $(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm}$ equipped with the canonical connection.

The proposition will follow from the following lemma:

Lemma 6. *Let $f\colon \mathscr{Y} \to \mathscr{X}$ be a map of smooth stacks and $\mathcal{L}$ a line bundle on $\mathscr{X}$. Let $\tilde{f}^{(1)}\colon \mathscr{Y}^{(1)} \times_{\mathscr{X}^{(1)}} \tilde{T}^*_{\mathcal{L}^{(1)}}\mathscr{X}^{(1)} \to \tilde{T}^*_{\mathcal{L}^{(1)}}\mathscr{X}^{(1)}$ and $\widetilde{df}^{(1)}\colon \mathscr{Y}^{(1)} \times_{\mathscr{X}^{(1)}} \tilde{T}^*_{\mathcal{L}^{(1)}}\mathscr{X}^{(1)} \to \tilde{T}^*_{f^*\mathcal{L}^{(1)}}\mathscr{Y}^{(1)}$ be the natural projections. Then we have an equivalence*

$$\tilde{f}^{(1)*}\mathcal{G}_{\mathscr{X},\mathcal{L}^c} \sim \widetilde{df}^{(1)*}\mathcal{G}_{\mathscr{Y},f^*\mathcal{L}^c}.$$

Proof. We have already mentioned this fact in the case when $\mathscr{Y}$ and $\mathscr{X}$ are smooth varieties. In this case, the statement follows from the fact that there is a splitting of $\tilde{\mathcal{D}}_{\mathscr{Y},f^*\mathcal{L}^c} \boxtimes \tilde{\mathcal{D}}^{\mathrm{op}}_{\mathscr{X},\mathcal{L}^c}$ given by $\mathcal{D}_{\mathscr{Y}\to\mathscr{X},\mathcal{L}^c}$ (see [BB, Proposition 3.7] for the non-twisted case; the twisted case in completely analogous).

The stack case can easily be obtained by descent. □

Proof of Proposition 5. We apply Lemma 6 to the map $f\colon (T^*_{\mathcal{L}}\mathscr{Y})^{sm} \to \mathscr{Y}$ and the line bundle $\mathcal{L}$. Consider the diagonal map $\Delta\colon (\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm} \to (\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm} \times_{\mathscr{Y}} \tilde{T}^*_{\mathcal{L}}\mathscr{Y}$. It is easy to see that the map $\tilde{f} \circ \Delta$ is the inclusion $(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm} \hookrightarrow \tilde{T}^*_{\mathcal{L}}\mathscr{Y}$ and the map $\widetilde{df} \circ \Delta\colon (\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm} \to \tilde{T}^*_{\mathcal{L}'}(\tilde{T}^*_{\mathcal{L}}\mathscr{Y})^{sm}$ corresponds to the connection ∇_{can} on $\mathcal{L}'$. Thus, pulling back the equivalence in Lemma 6 by the map Δ, we get the desired statement. □

2.4.1 The regular $\mathcal{D}$-module and its endomorphism algebra

It is clear that the forgetful functor $\mathcal{D}\text{-mod}_{\mathcal{L}^c}(\mathscr{Y}) \to \mathrm{QCoh}(\mathscr{Y})$ has a left adjoint. Denote by $\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$ the image of $\mathcal{O}_\mathscr{Y}$ under this left adjoint; this is a twisted $\mathcal{D}$-module (in particular, a quasi-coherent sheaf) on $\mathscr{Y}$. In other words, this is the object (co)representing the functor of global sections $\Gamma\colon \mathcal{D}\text{-mod}_{\mathcal{L}^c}(\mathscr{Y}) \to \mathrm{QCoh}_\mathscr{Y} \to \mathsf{k}\text{-Vect}$. Let also $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c}$ be the opposite to the sheaf (on $\mathscr{Y}_{\mathrm{sn}}$) of endomorphism algebras of $\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$ as a twisted $\mathcal{D}$-module, i.e., $(\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c})_S = \mathcal{E}nd_{\mathcal{D}_{S,\pi^*\mathcal{L}^c}}(\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$. Since $\mathcal{O}_{S^{(1)}} \subset Z(\mathcal{D}_{S,\pi^*\mathcal{L}^c})$ for every $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$, we can regard $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c}$ as an $\mathcal{O}_{\mathscr{Y}^{(1)}}$-module. One can check that for any map $f\colon S' \to S$ in $\mathscr{Y}_{\mathrm{sm}}$, we have $(\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c})_{S'} = f^{(1)*}(\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c})_S$, so $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c}$ is a quasi-coherent sheaf on $\mathscr{Y}^{(1)}$.

To describe these sheaves in more concrete terms, fix $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$ and consider the left ideal $\mathcal{I}$ in $\mathcal{D}_{S,\pi^*\mathcal{L}^c}$ generated by the image of the map $\mathcal{T}_{S/\mathscr{Y}} \to \mathcal{D}^{\leq 1}_{S,\pi^*\mathcal{L}^c}$ (this is the canonical lift of the map $\mathcal{T}_{S/\mathscr{Y}} \to \mathcal{T}_S$ which comes from the fact that the vector fields on S coming from $\mathcal{T}_{S/\mathscr{Y}}$ have canonical lifting to the total space of $\pi^*\mathcal{L}$), and let $N(\mathcal{I})$ be its normalizer. Then $(\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S = \mathcal{D}_{S,\pi^*\mathcal{L}^c}/\mathcal{I}$ and $(\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c})_S = N(\mathcal{I})/\mathcal{I}$.

Remark 3. In the non-twisted case, the sheaves $\mathcal{D}^\sharp_\mathscr{Y}$ and $\mathcal{D}^\flat_\mathscr{Y}$ are what is denoted by $\mathcal{D}_\mathcal{Y}$ and $D_\mathcal{Y}$ respectively in [BD] and by $D^\sharp_Y$ and D_Y respectively in [BB]. In [BB], the authors state that the definition of $\mathcal{D}^\flat_\mathscr{Y}$ should be modified in characteristic p because they want $(\mathcal{D}^\flat_\mathscr{Y})_{S'}$ to be the Zariski (or étale) sheaf-theoretic inverse image of $(\mathcal{D}^\flat_\mathscr{Y})_S$ for any morphism $S' \to S$ in $\mathscr{Y}_{\mathrm{sm}}$, as it happens in the characteristic 0 case. Although this property doesn't hold in characteristic p, the fact that we have a quasi-coherent sheaf on $\mathscr{Y}^{(1)}$ can be regarded as a substitute.

By definition we have that $\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$ is a right module over $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c}$. Actually, $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c}$ is (the opposite of) the endomorphism sheaf of $\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$. We also have a canonical "unit" global section of $\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$. Thus there is a map $u\colon \mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c} \to \mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$ of sheaves on $\mathscr{Y}_{\mathrm{sm}}$.

Claim 7. *The map u induces an identification $\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c} \xrightarrow{\sim} \mathrm{Fr}_{\mathscr{Y}*}\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c}$.*

Note that for any $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$ the quasi-coherent sheaves $(\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$ and $(\mathcal{D}^\flat_{\mathscr{Y},\mathcal{L}^c})_S$ on $S^{(1)}$ are push-forwards from $\tilde{T}^*S^{(1)}$, since they are acted on by the center of $\mathcal{D}_{S,\pi^*\mathcal{L}^c}$. Denote by $(\tilde{\mathcal{D}}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$ and $(\tilde{\mathcal{D}}^\flat_{\mathscr{Y},\mathcal{L}^c})_S$ the corresponding sheaves on $\tilde{T}^*S^{(1)}$. As we saw in the proof of Theorem 4, these sheaves are actually supported on $S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*S^{(1)}$. Moreover, the $\mathcal{O}_S$-module structure on $(\mathcal{D}^{\sharp}_{\mathscr{Y},\mathcal{L}^c})_S$ allows us to view $(\tilde{\mathcal{D}}^{\sharp}_{\mathscr{Y},\mathcal{L}^c})_S$ as a quasi-coherent sheaf on $S \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$. Thus we see that the collection of $(\tilde{\mathcal{D}}^{\sharp}_{\mathscr{Y},\mathcal{L}^c})_S$ for all $S \in \mathscr{Y}_{\mathrm{sm}}$ defines a quasi-coherent (actually, coherent) sheaf $\tilde{\mathcal{D}}^{\sharp}_{\mathscr{Y},\mathcal{L}^c}$ on $\mathscr{Y} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$, whereas $(\tilde{\mathcal{D}}^{\flat}_{\mathscr{Y},\mathcal{L}^c})_S$ define a coherent sheaf $\tilde{\mathcal{D}}^{\flat}_{\mathscr{Y},\mathcal{L}^c}$ on $\mathscr{Y}^{(1)}$. It is clear from the above claim that $\tilde{\mathcal{D}}^{\sharp}_{\mathscr{Y},\mathcal{L}^c}$ is the pushforward of $\tilde{\mathcal{D}}^{\sharp\prime}_{\mathscr{Y},\mathcal{L}^c}$ along the map $\mathscr{Y} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)} \to \tilde{T}^*\mathscr{Y}^{(1)}$.

Proposition 8. *Assume that $\mathscr{Y}$ is "good" in the sense of [BD, Sect. 1.1.1], i.e., its cotangent stack has the expected dimension: $\dim T^*\mathscr{Y} = 2\dim\mathscr{Y}$. Denote by $(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}$ the maximal smooth open substack of $\tilde{T}^*\mathscr{Y}^{(1)}$.*[3] *Then*

a. *The restriction of the coherent sheaf $\mathcal{F}$ on $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$ corresponding to $\mathcal{D}^{\sharp}_{\mathscr{Y},\mathcal{L}^c}$ to $(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}$ is locally free of rank $p^{\dim\mathscr{Y}}$.*

b. *The restriction of the algebra $\tilde{\mathcal{D}}^{\flat}_{\mathscr{Y},\mathcal{L}^c}$ to $(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}$ is an Azumaya algebra of rank $p^{2\dim\mathscr{Y}}$*

c. *The gerbe $\mathcal{G}_{\mathscr{Y},\mathcal{L}^c}|_{(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}}$ classifies splittings of $\tilde{\mathcal{D}}^{\flat}_{\mathscr{Y},\mathcal{L}^c}|_{(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}}$.*

Proof. We first note that (b) and (c) are direct consequences of (a). Indeed, (b) follows since $\tilde{\mathcal{D}}^{\flat}_{\mathscr{Y},\mathcal{L}^c}$ is the endomorphism sheaf of $\mathcal{F}$, and the identification in (c) is induced by $\mathcal{F}$.

Now, to prove (a), it is enough to show that for any $(S,\pi) \in \mathscr{Y}_{\mathrm{sm}}$ the sheaf $\mathcal{F}_S$ on $S^{(1)} \times_{\mathscr{Y}^{(1)}} \mathcal{G}_{\mathscr{Y},\mathcal{L}^c}$ given by pullback of $\mathcal{F}$ is locally free on $S^{(1)} \times_{\mathscr{Y}^{(1)}} (\tilde{T}^*\mathscr{Y}^{(1)})^{sm} \subset S^{(1)} \times_{\mathscr{Y}^{(1)}} \tilde{T}^*\mathscr{Y}^{(1)}$ of rank $p^{\dim\mathscr{Y}}$. We will achieve that by showing that the twisted $\mathcal{D}$-module $(\mathcal{D}^{\sharp}_{\mathscr{Y},\mathcal{L}^c})_S$ corresponding to $\mathcal{F}_S$ is Cohen–Macaulay of depth $k := \dim_{\mathscr{Y}} S = \dim S - \dim\mathscr{Y}$. (We assume that this dimension is constant along S, for example, it is true if S is connected.)

Denote $\mathcal{D}'_S = \mathcal{D}_{S,\pi^*\mathcal{L}^c}$ for short. We can express $(\mathcal{D}^{\sharp}_{\mathscr{Y},\mathcal{L}^c})_S$ as the homology in the last term (which we put in degree 0) of the complex

$$\mathcal{D}'_S \underset{\mathcal{O}_S}{\otimes} \Lambda^k \mathcal{T}_{S/\mathscr{Y}} \to \mathcal{D}'_S \underset{\mathcal{O}_S}{\otimes} \Lambda^{k-1}\mathcal{T}_{S/\mathscr{Y}} \to \cdots \to \mathcal{D}'_S \underset{\mathcal{O}_S}{\otimes} \mathcal{T}_{S/\mathscr{Y}} \to \mathcal{D}'_S \tag{2.3}$$

[3]It is easy to see that $(\tilde{T}^*\mathscr{Y}^{(1)})^{sm}$ is a smooth Deligne–Mumford stack.

where the rightmost map is constructed as in the first paragraph in this subsection, and the other maps are obtained from it by the Leibnitz rule. There is a natural filtration by degree on this complex, and the associated graded is isomorphic to the Koszul complex for $\operatorname{Sym} \mathcal{T}_S \otimes^{\mathrm{L}}_{\operatorname{Sym} \mathcal{T}_{S/\mathscr{Y}}} \mathcal{O}_S$. The goodness condition on $\mathscr{Y}$ implies that this Koszul complex has cohomology in degree 0. (This complex can be thought of as functions on $S \times_{\mathscr{Y}} T^*\mathscr{Y}$ where $T^*\mathscr{Y}$ is understood in the derived sense, and the goodness condition guarantees that $T^*\mathscr{Y}$ is actually non-derived.) Therefore the same is true for the complex (2.3). So (2.3) is a locally free resolution of $(\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$.

Now, it is easy to see that applying Verdier duality to (2.3) gives the same complex up to a twist by $\pi^*\omega_{\mathscr{Y}}[\dim \mathscr{Y}]$. Since the Verdier duality for $\mathcal{D}$-modules agrees with the Serre duality for modules over the gerbe $\mathcal{G}_{S,\pi^*\mathcal{L}^c}$ up to a twist by line bundle and the identification of $\tilde{T}^*_{\pi^*(\mathcal{L}^{(1)})^{c^p-c}} S^{(1)}$ with $\tilde{T}^*_{\pi^*(\mathcal{L}^{(1)})^{-(c^p-c)}} S^{(1)}$ via multiplication by -1, we see that $(\mathcal{D}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$ is Cohen–Macaulay of depth k, as desired.

To see that the generic rank of $\mathcal{F}_S$ is equal to $p^{\dim \mathscr{Y}}$, we have to show that the generic rank of $(\tilde{\mathcal{D}}^\sharp_{\mathscr{Y},\mathcal{L}^c})_S$ as an $\mathcal{O}_{\tilde{T}^*S^{(1)}}$-module along its support is $p^{\dim \mathscr{Y}} \cdot \sqrt{\operatorname{rk} \tilde{\mathcal{D}}_{S,\pi^*\mathcal{L}^c}} = p^{\dim \mathscr{Y} + \dim S}$. But this can be checked on the level of the associated graded module, which reduces to the fact that the pushforward of $\mathcal{O}_{S \times_{\mathscr{Y}} T^*\mathscr{Y}}$ under the Frobenius map has that generic rank, which is true since $\dim(S \times_{\mathscr{Y}} T^*\mathscr{Y}) = \dim \mathscr{Y} + \dim S$ by goodness assumption. □

2.5 Extended curvature

Let X be a smooth variety. Define a (coherent) sheaf $\mathfrak{F}_X$ of $\mathcal{O}_X^p$-modules (= coherent sheaf on $X^{(1)}$) by the exact sequence

$$0 \to \mathcal{O}_{X^{(1)}} \xrightarrow{\mathrm{Fr}^*} \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{\delta} \mathfrak{F}_X \to 0 \,. \tag{2.4}$$

Then we also have an exact sequence

$$0 \to \Omega^1_{X^{(1)}} \xrightarrow{\mathsf{P}} \mathfrak{F}_X \xrightarrow{\mathsf{Q}} \Omega^2_{X,cl} \xrightarrow{\mathsf{C}} \Omega^2_{X^{(1)}} \to 0 \tag{2.5}$$

where P is induced by (the inverse of) Cartier isomorphism $\mathrm{Coker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_{X,cl}) \cong \Omega^1_{X^{(1)}}$, Q is induced by $d\colon \Omega^1_X \to \Omega^2_X$ and C is the Cartier operation. It is immediate from the definition that

$$\delta(\omega) = \mathsf{P}(\mathsf{C}(\omega)) \quad \text{for } \omega \in \Omega^1_{X,cl}. \tag{2.6}$$

If we define $\kappa\colon \Omega^1_X \to \mathfrak{F}_X$ by setting

$$\kappa(\omega) = \mathsf{P}(\omega^{(1)}) - \delta(\omega) \tag{2.7}$$

then we will have an exact sequence

$$0 \to (\mathcal{O}_X^\times)^p \to \mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_X \xrightarrow{\kappa} \mathfrak{F}_X \to 0\,. \tag{2.8}$$

(Unlike (2.4) and (2.5), (2.8) is not k-linear.)

Now, if $(\mathcal{L},\nabla)$ is a line bundle with connection, we define its *extended curvature* $\widetilde{\mathrm{curv}}(\mathcal{L},\nabla) \in \Gamma(X,\mathfrak{F}_X)$ to be locally given by

$$\widetilde{\mathrm{curv}}(\mathcal{L},\nabla) = \kappa(\omega)$$

if $(\mathcal{L},\nabla) \cong (\mathcal{O}, d+\omega)$. It is clear from (2.8) that this is independent of the trivialization, and that $\mathsf{Q}(\widetilde{\mathrm{curv}}(\mathcal{L},\nabla)) = F_\nabla$ — the usual curvature of ∇.

Proposition 9. *Suppose given a line bundle $\mathcal{L}$, a connection ∇ on it, and $c \in \mathsf{k}\setminus\mathbb{F}_p$. Then splittings of the algebra $\mathcal{D}_{\mathcal{L}^c,\nabla^{(1)}}$ defined in 2.3.2 correspond canonically to line bundles on X with connection $(\mathcal{L}',\nabla')$ such that*

$$\widetilde{\mathrm{curv}}(\mathcal{L}',\nabla') = c\cdot\widetilde{\mathrm{curv}}(\mathcal{L},\nabla)\,. \tag{2.9}$$

Proof. As we saw in 2.3, the connection ∇ on $\mathcal{L}$ gives an identification of the category of $\mathcal{D}_{\mathcal{L}^c}$-modules $\mathcal{M}$ with the category of quasi-coherent sheaves with connection $(\mathcal{F},\nabla')$ such that ∇' is projectively flat of curvature cF_∇. Moreover, $\mathcal{F}$ is a line bundle if and only if $\mathcal{M}$ is locally free of rank 1 over $\mathcal{O}_X$, which is equivalent to $\mathrm{Fr}_{X*}\mathcal{M}$ being locally free of rank $p^{\dim X}$ over $\mathcal{O}_{X^{(1)}}$. Now denote by $\tilde{\mathcal{M}}$ the $\tilde{\mathcal{D}}_{\mathcal{L}^c}$-module corresponding to $\mathcal{M}$, and let $\pi\colon (\tilde{T}^*_{\mathcal{L}}X)^{(1)} \to X^{(1)}$ be the projection. Since $\mathrm{Fr}_{X*}\mathcal{M} = \pi_*\tilde{\mathcal{M}}$ and $\tilde{\mathcal{D}}_{\mathcal{L}^c}$ is an Azumaya algebra of rank $p^{2\dim X}$, the above condition means that $\tilde{\mathcal{M}}$ is a splitting of $\tilde{\mathcal{D}}_{\mathcal{L}^c}$ on a section of the map π. So all that remains to check is that this section corresponds to $\nabla^{(1)}$ if and only if (2.9) is satisfied with $\mathcal{L}' = \mathcal{F}$.

Since this question is local, we can assume that $\mathcal{L} = \mathcal{L}' = \mathcal{O}$. The trivialization of $\mathcal{L}$ gives rise to an equivalence $\mathcal{D}_{X,\mathcal{L}^c}$-mod $\sim \mathcal{D}_X$-mod. Under this equivalence, the $\mathcal{D}_{\mathcal{L}^c}$-module $\mathcal{M}$ corresponds to the following line bundle with connection:

$$(\mathcal{L}', \nabla' - c\alpha) = (\mathcal{O}, d + \beta - c\alpha),$$

where α, β are the forms of the connections ∇, ∇' in the chosen trivializations: $(\mathcal{L}, \nabla) = (\mathcal{O}, d + \alpha)$, $(\mathcal{L}', \nabla') = (\mathcal{O}, d + \beta)$.

Now, if we identify $\tilde{T}^*_{\mathcal{L}}X$ with T^*X using our trivialization of $\mathcal{L}$ then using the diagram

$$\begin{array}{ccccc}
\mathrm{Spec}_{X^{(1)}} Z(\mathcal{D}_{\mathcal{L}^c}) & \xrightarrow{\ \sim\ } & \tilde{T}^*_{(\mathcal{L}^{(1)})^{c^p - c}}X^{(1)} & \xrightarrow[(c^p-c)^{-1}]{\ \sim\ } & \tilde{T}^*_{\mathcal{L}^{(1)}}X^{(1)} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathrm{Spec}_{X^{(1)}} Z(\mathcal{D}_X) & \xrightarrow{\ \sim\ } & T^*X^{(1)} & \xrightarrow[(c^p-c)^{-1}]{\ \sim\ } & T^*X^{(1)}
\end{array}$$

we see that the connection on $\mathcal{L}^{(1)}$ corresponding to the support of $\tilde{\mathcal{M}}$ is given by $d + (c^p - c)^{-1}((\beta - c\alpha)^{(1)} - \mathsf{C}(\beta - c\alpha))$. (Note that since $d\beta = F_{\nabla'} = cF_{\nabla} = c\,d\alpha$, the form $\beta - c\alpha$ is closed, so it makes sense to apply C to it.) So our condition now takes the following form:

$$(c^p - c)^{-1}((\beta - c\alpha)^{(1)} - \mathsf{C}(\beta - c\alpha)) = -\alpha^{(1)},$$

which can be rewritten (after multiplication by $c^p - c$ and canceling $c^p\alpha^{(1)} = (c\alpha)^{(1)}$) as

$$\beta^{(1)} - c\alpha^{(1)} = \mathsf{C}(\beta - c\alpha).$$

Now, applying P (which is injective) to both sides, and using (2.6) and (2.7), we see that the above equation is equivalent to

$$\kappa(\beta) = c\kappa(\alpha)$$

which is the same as (2.9) by definition of the extended curvature. $\square$

Remark 4. It is clear (by descent) that this proposition extends to smooth Artin stacks, in particular, it can be applied to $X = (\tilde{T}^*_{\mathcal{L}_{\mathrm{det}}} \mathrm{Bun})^{sm}$.

Chapter 3

$\mathcal{D}$-modules on Bun_N and the Hitchin fibration

In this section we will recall some constructions from [BB] and extend them to twisted $\mathcal{D}$-modules.

3.1 Hitchin fibration

Let C be a smooth connected projective curve over k of genus greater than 1, and fix an integer $N > 1$. Denote by Bun_N the moduli stack of rank N bundles on C. By definition, for a scheme S, the groupoid of maps $S \to \mathrm{Bun}_N$ is equivalent to the groupoid of rank N vector bundles on $C \times S$. It is classical that the cotangent bundle to Bun_N is identified with the stack Higgs of Higgs bundles. Recall that for a vector bundle $\mathcal{E}$ on C, a structure of *Higgs bundle* on it, also known as a *Higgs field*, is an $\mathcal{O}$-linear map $\mathcal{E} \to \mathcal{E} \otimes \omega_C = \mathcal{E} \otimes \Omega^1_C$. Denote by $\mathscr{B}$ the scheme which the affine space corresponding to the vector space

$$\bigoplus_{i=1}^{N} \Gamma(C, \omega_C^{\otimes i}).$$

Define the Hitchin map H: Higgs $\to \mathscr{B}$ as follows. For a k-point y of Higgs corresponding to a Higgs bundle $(\mathcal{E}, a)$, we define $H(y)$ to be the point of $\mathscr{B}$ given by

$(\operatorname{tr} a, \operatorname{tr} \Lambda^2 a, \dots, \operatorname{tr} \Lambda^N a = \det a)$ (one can extend this to S-points in a straightforward way). Here is another interpretation of the Hitchin map. Note that a Higgs field on a given vector bundle $\mathcal{E}$ is equivalent to a map $\mathcal{T}_C \otimes \mathcal{E} \to \mathcal{E}$, and therefore to an action of $\operatorname{Sym} \mathcal{T}_C$ on $\mathcal{E}$. In other words, a Higgs bundle of rank N is equivalent to a coherent sheaf $\tilde{\mathcal{E}}$ on T^*C whose pushforward to C is a rank N vector bundle. Now define a divisor $D \subset T^*C$ as the "support with multiplicities" of $\tilde{\mathcal{E}}$ (i.e., each irreducible component of $\operatorname{supp} \tilde{\mathcal{E}}$ is taken with multiplicity equal to the length of the stalk of $\tilde{\mathcal{E}}$ at the generic point of that component).

It is clear that the divisor D is finite of degree N over C. We claim that such divisors are naturally parametrized by $\mathscr{B}$. Indeed, let $\pi \colon T^*C \to C$ be the projection, and let s be the canonical section of $\pi^*\omega_C$. Then any S-point b of $\mathscr{B}$ given by $(\tau_1, \dots, \tau_N)$ where $\tau_i \in \Gamma(C \times S, \omega_C^{\otimes i} \boxtimes \mathcal{O}_S)$ defines a section t_b of $\pi^*\omega_C^{\otimes N} \boxtimes \mathcal{O}_S$ by the formula

$$t_b = s^{\otimes N} \boxtimes 1 + \sum_{i=1}^{N} (-1)^i (\pi \times \mathrm{id}_S)^* \tau_i \otimes (s^{\otimes N-i} \boxtimes 1).$$

The divisor $\tilde{C}_b$ of zeroes of t_b is finite of degree N over $C \times S$, and it is easy to see that $b \mapsto \tilde{C}_b$ defines a one-to-one correspondence between maps $S \to \mathscr{B}$ and divisors in $C \times S$ finite of degree N over S. Moreover, in the situation of the previous paragraph, the point $H(y)$ corresponds to the divisor D: $D = \tilde{C}_{H(y)}$. (This is essentially because the support divisor of $\tilde{\mathcal{E}}$ can be computed using the characteristic polynomial of a.) Also, we will need the universal spectral curve $\tilde{\mathscr{C}} \subset T^*C \times \mathscr{B} \to C \times \mathscr{B}$ (so that, in the above notation, $\tilde{C}_b = \tilde{\mathscr{C}} \times_{\mathscr{B}} \{b\}$).

We will be interested in the open subset of $\mathscr{B}$ parametrizing smooth spectral curves, that is, the maximal open subset $\mathscr{B}^0 \subset \mathscr{B}$ for which the map $\tilde{\mathscr{C}}^0 := \tilde{\mathscr{C}} \times_{\mathscr{B}} \mathscr{B}^0 \to \mathscr{B}^0$ is smooth. One can show that $\mathscr{B}^0$ is non-empty, and that fibers of $\tilde{\mathscr{C}}^0 \to \mathscr{B}^0$ are irreducible (and smooth). Denote also by Higgs^0 the preimage of $\mathscr{B}^0$ under the Hitchin map H:

$$\mathrm{Higgs}^0 := \mathrm{Higgs} \underset{\mathscr{B}}{\times} \mathscr{B}^0.$$

We claim that there is a natural identification $\mathrm{Higgs}^0 \cong \operatorname{Pic}(\tilde{\mathscr{C}}^0/\mathscr{B}^0)$. Indeed, for an S-point y of Higgs^0, one can define a line bundle on $\tilde{C}_{H(y)} := \tilde{\mathscr{C}} \times_{\mathscr{B}, H \circ y} S$ as follows.

Let $(\mathcal{E}, a)$ be the S-family of Higgs bundles corresponding to y. As discussed above, this is the same as a coherent sheaf $\tilde{\mathcal{E}}$ on $T^*C \times S$, and the support of this sheaf is $\tilde{C}_{H(y)}$. Moreover, since $H(y) \in \mathscr{B}^0(S)$, the spectral curve $\tilde{C}_{H(y)}$ is smooth over S, and $\tilde{\mathcal{E}}$ must be the pushforward of a line bundle on $\tilde{C}_{H(y)}$. This is the desired line bundle, which gives an S-point of $\mathrm{Pic}(\tilde{\mathscr{C}}^0/\mathscr{B}^0)$.

3.2 The p-Hitchin fibration

In this subsection, we will present a description of the stack $\mathrm{Loc} = \mathrm{Loc}_N$ of de Rham local systems of rank N on C, analogous to the one given above for Higgs. Recall that by "de Rham local system" we just mean a vector bundle with a flat connection (since C is one-dimensional, all connections on it are automatically flat), so for a given test scheme S, the groupoid $\mathrm{Loc}_N(S)$ is defined as that of rank N vector bundles on $C \times S$ equipped with a connection in the C-direction.

The construction of the p-Hitchin map is similar to that of the ordinary Hitchin map, but uses the notion of p-support, and so exists only in positive characteristic. It is a map $\chi\colon \mathrm{Loc} \to \mathscr{B}^{(1)}$ defined as follows. Suppose we are given an S-point of Loc defined by an S-family of local systems, i.e., a vector bundle $\mathcal{E}$ on $C \times S$ of rank N with a connection ∇ relative to S. We can think of $(\mathcal{E}, \nabla)$ as an S-family of $\mathcal{D}$-modules on C; in particular, similarly to the Higgs field case, we can define its p-support with multiplicities—this is a divisor in $T^*C^{(1)} \times S$ finite of degree N over $C^{(1)} \times S$. The corresponding S-point of $\mathscr{B}^{(1)}$ is by definition the value of χ on $(\mathcal{E}, \nabla)$. Again, another way to define it is to apply the invariant polynomials to the p-curvature map $\mathrm{curv}_p(\nabla)\colon \mathcal{E} \to \mathcal{E} \otimes (\mathrm{Fr}_C^* \Omega^1_{C^{(1)}} \boxtimes \mathcal{O}_S)$.

One can show that, étale locally over $\mathscr{B}^{(1)}$, the p-Hitchin fibration $\chi\colon \mathrm{Loc} \to \mathscr{B}^{(1)}$ looks like the (Frobenius twisted) usual Hitchin fibration $H^{(1)}\colon \mathrm{Higgs}^{(1)} \to \mathscr{B}^{(1)}$ (see [Gr]). The identification of formal neighborhoods of fibers over a given point of $\mathscr{B}^{(1)}$ can be constructed using a splitting of the Azumaya algebra $\tilde{\mathcal{D}}_C$ on the formal neighborhood of the corresponding spectral curve. Similarly, an étale local identification near a given point $b \in \mathscr{B}^{(1)}$ can be obtained from a splitting of the pullback of $\tilde{\mathcal{D}}_C$ to $\tilde{\mathscr{C}}^{(1)} \times_{\mathscr{B}^{(1)}} U$ where $U \to \mathscr{B}^{(1)}$ is an étale neighborhood of b in $\mathscr{B}^{(1)}$. (See *ibid.* for the proof of the existence of such a splitting.) The identification is canonical up to the action of a section of the group stack $\mathrm{Pic}(\tilde{\mathscr{C}}^{(1)}/\mathscr{B}^{(1)})$ on $\mathrm{Higgs}^{(1)}$, where the latter stack is identified with the "compactified" relative Picard stack of $\tilde{\mathscr{C}}^{(1)}$ over $\mathscr{B}^{(1)}$.

We will be mostly concerned with the part of Loc lying over $(\mathscr{B}^0)^{(1)}$ which we will denote by

$$\mathrm{Loc}^0 := \mathrm{Loc} \underset{\mathscr{B}^{(1)}}{\times} (\mathscr{B}^0)^{(1)}.$$

As explained above, we have an identification $\mathrm{Higgs}^0 \cong \mathrm{Pic}(\tilde{\mathscr{C}}^0/\mathscr{B}^0)$, and hence $(\mathrm{Higgs}^0)^{(1)} \cong \mathrm{Pic}((\tilde{\mathscr{C}}^0)^{(1)}/(\mathscr{B}^0)^{(1)})$. Moreover, these identifications are compatible with the action of the corresponding Picard stacks. Therefore, from the results discussed in the previous paragraph, we see that the stack Loc^0 carries a natural structure of $\mathrm{Pic}((\tilde{\mathscr{C}}^0)^{(1)}/(\mathscr{B}^0)^{(1)})$-torsor. This torsor can be described as that of fiberwise (along fibers of $(\tilde{\mathscr{C}}^0)^{(1)} \to (\mathscr{B}^0)^{(1)}$) splittings of the Azumaya algebra $(\mathrm{pr}^{\tilde{\mathscr{C}}^0}_{T^*C})^{(1)*}\tilde{\mathcal{D}}_C$ where $\mathrm{pr}^{\tilde{\mathscr{C}}^0}_{T^*C}$ is the natural projection $\tilde{\mathscr{C}}^0 \to T^*C$ (obtained by restriction from the projection $\tilde{\mathscr{C}} \to T^*C$).

3.3 $\mathcal{D}$-modules on Bun

Now we apply the above results to the main objects of study in this paper—twisted $\mathcal{D}$-modules on Bun. Let us begin by recalling the non-twisted case. According to Theorem 4 (with $\mathcal{L} = \mathcal{O}$), $\mathcal{D}$-modules on Bun are classified by a certain gerbe on $\mathrm{Higgs}^{(1)} = T^*\,\mathrm{Bun}^{(1)}$. The class of this gerbe on the smooth part $(\mathrm{Higgs}^{sm})^{(1)}$ of $\mathrm{Higgs}^{(1)}$ (in particular, on $(\mathrm{Higgs}^0)^{(1)}$) corresponds to the canonical 1-form on $(\mathrm{Higgs}^{sm})^{(1)}$ as on (the smooth part of) a cotangent bundle.

Now we turn to the twisted case. The twists that we will consider are of the form $\mathcal{L}^c_{\det}$. Here $\mathcal{L}_{\det}$ denotes the determinant bundle on Bun: for any vector bundle $\mathcal{E}$ on C of rank N, the fiber of $\mathcal{L}_{\det}$ at the corresponding point of Bun is given by $\det \mathrm{R}\Gamma(\mathcal{E})$ (and similarly for families). In Appendix A, we prove that the corresponding twisted cotangent bundle is isomorphic to the moduli space $\mathrm{Loc}_{\omega^{1/2}}$ of $\omega_C^{\otimes 1/2}$-twisted de Rham local systems of rank N on C. Therefore, according to Theorem 4, for any $c \in \mathsf{k} \setminus \mathbb{F}_p$, the $\mathcal{L}_{\det}^c$-twisted $\mathcal{D}$-modules on Bun are classified by a $\mathbb{G}_m$-gerbe on $\mathrm{Loc}_{\omega^{1/2}}^{(1)}$.

3.4 Connections on determinant bundles

We already said before that we identify the twisted cotangent bundle corresponding to the determinant line bundle on Bun_G. For that (and for later use) we'll need some facts about connections on determinants. So let $\pi\colon X \to S$ be a smooth projective morphism of relative dimension 1, and let $\mathcal{E}$ be a vector bundle (or an S-flat coherent sheaf) on X. We are interested in the line bundle $\det R\pi_*\mathcal{E}$.

Define the sheaf of relative differentials $\Omega^1_{X/S} = \omega_{X/S}$ as usual. By an "*S-relative connection*" on $\mathcal{E}$ we mean a map $\mathcal{E} \to \Omega^1_{X/S} \otimes \mathcal{E}$ satisfying the usual Leibnitz rule.

Proposition 10. *Suppose X is a trivial family, i.e., $X = X_0 \times S$ for some curve X_0.*

1. *Let $\mathcal{E}$ be as above, and ∇ an S-relative connection on $\mathcal{E} \otimes \omega^{1/2}_{X/S}$. Then there is a canonical connection on the line bundle* $\det R\pi_*\mathcal{E}$.

2. *Let $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ be three sheaves equipped with the data of point 1, and we have an exact sequence*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

compatible with connections. Then the corresponding isomorphism of line bundles $\det R\pi_*\mathcal{E} \cong \det R\pi_*\mathcal{E}' \otimes \det R\pi_*\mathcal{E}''$ *is also compatible with connections.*

Proof. See Appendix A. □

Chapter 4

Proof of Theorem 1

4.1 Main part of the argument

We begin by stating the following proposition:

Proposition 11 (cf. [A]). *Suppose X is a variety (or a stack) and $\pi\colon A \to X$, $\pi^\vee\colon A^\vee \to X$ are dual families of abelian varieties. Suppose we have a torsor $\mathscr{T} \to X$ for $A \to X$ and a gerbe $\mathcal{G}$ on $\mathscr{T}$ with a fixed degree of the splitting. Then there is a canonical dual torsor $\mathscr{T}^\vee$ over $A^\vee$ and a gerbe $\mathcal{G}^\vee$ on it such that the derived categories of coherent modules over $\mathcal{G}$ and $\mathcal{G}^\vee$ are equivalent.*

Idea of proof. When the torsor $\mathscr{T}$ and the gerbe $\mathcal{G}$ are trivial, we can take $\mathscr{T}^\vee$ and $\mathcal{G}^\vee$ to be trivial, and use (the in-families version of) the usual Fourier–Mukai transform. In general choose a (say, étale) cover $\tilde{X}$ of X on which $\mathscr{T}$ and $\mathcal{G}$ trivialize so that we can apply the trivial case to get an equivalence over $\tilde{X}$, and then use properties of the Fourier–Mukai transform (namely, that it interchanges shifts along A with twists by line bundles) to descend it to X. □

We will apply the above proposition to $X = (\mathscr{B}^0)^{(2)}$, $A = A^\vee = (\mathrm{Higgs}^0)^{(2)}$, $\mathscr{T} = \tilde{T}^*_{\mathcal{L}_{\det}^{(1)c^p - c}} \mathrm{Bun}^{(1)} = (\tilde{T}^*_{\mathcal{L}_{\det}} \mathrm{Bun})^{(1)} = (\mathrm{Loc}^0_{\omega^{1/2}})^{(1)}$ (since $c \notin \mathbb{F}_p$) and the gerbe $\mathcal{G}$ corresponding to the Azumaya algebra $\tilde{\mathcal{D}}_{\mathcal{L}^c_{\det}}(\mathrm{Bun})$. We need to prove that $\mathscr{T}^\vee$ is also isomorphic to $\mathrm{Loc}^0_{\omega^{1/2}}$ and $\mathcal{G}^\vee$ corresponds to the Azumaya algebra $\mathcal{D}_{\mathcal{L}_{\det}^{-1/c}}(\mathrm{Bun})$.

First, we prove that $\mathscr{T}^\vee \cong (\mathrm{Loc}^0_{\omega^{1/2}})^{(1)}$. This is equivalent to the existence of an "action" on the gerbe $\mathcal{G}$ of the gerbe $\mathcal{G}_1$ on $A = (\mathrm{Higgs}^0)^{(2)}$ where $\mathcal{G}_1$ is dual to the torsor $\tilde{\mathscr{T}}^\vee \cong (\mathrm{Loc}^0_{\omega^{1/2}})^{(1)} \to (\mathscr{B}^0)^{(2)}$ with the projection rescaled by c^p. To describe what we mean by an action of $\mathcal{G}_1$ on $\mathcal{G}$, consider the graph of action of the group scheme $(\mathrm{Higgs}^0)^{(1)}$ (over $(\mathscr{B}^0)^{(1)}$) on $\mathrm{Loc}^0_{\omega^{1/2}}$ — it is a closed substack Γ in $\mathrm{Loc}^0_{\omega^{1/2}} \times_{\mathscr{B}^0} \mathrm{Loc}^0_{\omega^{1/2}} \times_{\mathscr{B}^0} (\mathrm{Higgs}^0)^{(1)}$ isomorphic to $\mathrm{Loc}^0_{\omega^{1/2}} \times_{\mathscr{B}} (\mathrm{Higgs}^0)^{(1)}$. From this graph we have three projections $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3$ to the factors. An action of $\mathcal{G}_1$ on $\mathcal{G}$ is by definition a splitting of $\mathrm{pr}_1^{(1)*}\mathcal{G} \otimes \mathrm{pr}_2^{(1)*}\mathcal{G}^{-1} \otimes \mathrm{pr}_3^{(1)*}\mathcal{G}_1$ satisfying a cocycle condition. Now, according to Proposition 5, the gerbe $\mathcal{G}$ corresponds to the algebra $\mathcal{D}_{\mathcal{L}'^c_{\mathrm{det}},\nabla}$ where $\mathcal{L}'_{\mathrm{det}}$ is the pullback of the determinant line bundle, and ∇ is the canonical ("universal") connection on this pullback. One can also show that $\mathcal{G}_1$ is given by the Azumaya algebra $\mathcal{D}_{c^p\theta}(\mathrm{Higgs}^0)$. Now the statement follows from the results of 2.3.1 and the following proposition (whose statement does not depend on c).

Proposition 12. *The line bundle with connection*

$$(\mathcal{S}, \nabla_{\mathcal{S}}) := \mathrm{pr}_1^*(\mathcal{L}'_{\mathrm{det}}, \nabla) \otimes \mathrm{pr}_2^*(\mathcal{L}'_{\mathrm{det}}, \nabla)^{\otimes -1} \otimes \mathrm{pr}_3^*(\mathcal{O}, d + \theta^{(1)})$$

on Γ *is flat and has* p*-curvature* $\mathrm{pr}_3^*\,\theta^{(2)}$.

If we define $\tilde{\theta} = \widetilde{\mathrm{curv}}(\mathcal{L}'_{\mathrm{det}}, \nabla)$ then Proposition 12 implies the following identity:

$$\mathrm{pr}_1^*\,\tilde{\theta} - \mathrm{pr}_2^*\,\tilde{\theta} = \mathsf{P}(\mathrm{pr}_3^*\,\theta^{(2)}) - \kappa(\mathrm{pr}_3^*\,\theta^{(1)}) = \delta(\mathrm{pr}_3^*\,\theta^{(1)}) \tag{4.1}$$

where $\mathsf{P}, \kappa, \delta$ are defined in 2.5.

For the proof of this proposition we will need the following lemma which will also be useful later.

Lemma 13. *Let* $\mathrm{Pic}(\tilde{C})$ *denote the Picard stack of a smooth projective curve* $\tilde{C}$. *Let* $\mathrm{pr}_1, a\colon \mathrm{Pic}(\tilde{C}) \times \tilde{C} \to \mathrm{Pic}(\tilde{C})$ *be the projection and addition maps, respectively (*a *is obtained from the Abel–Jacobi map). If* $\mathcal{L}$ *is a line bundle on* $\mathrm{Pic}(\tilde{C})$ *such that* $\mathrm{pr}_1^*\,\mathcal{L} \cong a^*\mathcal{L}$ *then* $\mathcal{L}$ *is trivial.*

We will prove that the restriction of the line bundle from Proposition 12 to the preimage Γ_{AJ} in Γ of the image of Abel–Jacobi map in $\mathrm{Higgs}^0 = \mathrm{Pic}(\tilde{\mathscr{C}}^0/\mathscr{B}^0)$ under pr_3 is flat and has p-curvature $\mathrm{pr}_3^* \theta^{(2)}$. This will follow from an alternative description of this bundle with connection. Namely, we have an equivalence $\mathcal{D}_C$-mod $\sim \mathcal{D}_{T^*C,\theta}$-mod ($\theta$ is the canonical form on T^*C). Moreover, the "in-families" version of this is true. So if $\mathscr{X}$ is any stack then the category of "$\mathscr{X}^{(1)}$-families of $\mathcal{D}_C$-modules" (i.e., quasi-coherent sheaves on $C \times \mathscr{X}^{(1)}$ equipped with a connection along C) is equivalent to the category of $\mathcal{D}_{T^*C\times\mathscr{X},\mathrm{pr}_1^*\theta}$-modules. If we replace C by $C^{(1)}$ and $\mathscr{X}$ by Loc^0 then we have the universal bundle with connection (along $C^{(1)}$) on $C^{(1)} \times (\mathrm{Loc}^0)^{(1)}$, so applying the equivalence gives a $\mathcal{D}$-module on $T^*C^{(1)} \times \mathrm{Loc}^0$ with p-curvature $\mathrm{pr}_1^* \theta^{(1)}$. It is supported on $(\tilde{\mathscr{C}}^0)^{(1)} \times_{(\mathscr{B}^0)^{(1)}} \mathrm{Loc}^0 \cong \Gamma_{\mathrm{AJ}}$ and therefore corresponds to a $\mathcal{D}_{\Gamma_{\mathrm{AJ}},\mathrm{pr}_3^*\theta^{(2)}}$-module which we denote by $(\mathcal{L}, \nabla)_{\mathrm{univ}}$.

Lemma 14. *The restriction to Γ_{AJ} of the line bundle with connection $(\mathcal{S}, \nabla_{\mathcal{S}})$ from Proposition 12 is isomorphic to $(\mathcal{L}, \nabla)_{\mathrm{univ}}$ where we identify* Loc *with* $\mathrm{Loc}^0_{\omega^{1/2}}$ *via twisting by $\omega_C^{(p-1)/2}$.*

Proof. We will first construct an isomorphism of line bundles, and then prove that it is compatible with connections. From the definition of Γ_{AJ} it is easy to see that a point $\gamma \in \Gamma_{\mathrm{AJ}}$ corresponds to a pair of rank N bundles with $\omega^{1/2}$-connections $(\mathcal{E}_1, \nabla_1)$, $(\mathcal{E}_2, \nabla_2)$ that fit into a short exact sequence of $\mathcal{D}_{C,\omega^{1/2}}$-modules

$$0 \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{F} \to 0 \tag{4.2}$$

where $\mathcal{F} \cong \delta_{x,\xi}$ is an irreducible $\mathcal{D}_{C,\omega^{1/2}}$-module corresponding to a point $(x,\xi) \in T^*C^{(1)}$. Now recall that the fiber of $\mathcal{S}$ at γ is given by $\det \mathrm{R}\Gamma(\mathcal{E}_1) \otimes (\det \mathrm{R}\Gamma(\mathcal{E}_2))^{\otimes -1} \cong \det \mathrm{R}\Gamma(\mathcal{F})$. If $(\bar{x}, \bar{\xi}) \in T^*C$ is such that $(x,\xi) = (\bar{x},\bar{\xi})^{(1)}$ then $\mathcal{F}$ has a filtration for which $\mathrm{gr}\,\mathcal{F} \cong \mathcal{F}_{\bar{x}} \otimes \bigoplus_{i=0}^{p-1} \omega_{C,\bar{x}}^{\otimes i}$. Therefore $\det \mathrm{R}\Gamma(\mathcal{F}) = \mathcal{F}_{\bar{x}}^{\otimes p} \otimes \omega_{\bar{x}}^{\otimes \frac{p(p-1)}{2}}$. Let $\mathcal{F}' = \mathcal{F} \otimes_{\mathcal{O}_C} \omega_C^{(1-p)/2}$ — it is an irreducible $\mathcal{D}_C$-module supported at $\bar{x}$. Then we can rewrite $\mathcal{S}_\gamma = \det \mathrm{R}\Gamma(\mathcal{F}) = (\mathcal{F}')_{\bar{x}}^{\otimes p}$. Let $\tilde{C} \subset T^*C^{(1)}$ be the spectral curve of $\mathcal{E}_1$ (same as that of $\mathcal{E}_2$). So $\tilde{C} = \tilde{\mathscr{C}} \times_{\mathscr{B}^{(1)}} \{\mathcal{E}_i\}$. Let $\mathcal{L}_i$ ($i = 1,2$) be $\mathcal{D}_{T^*C,\theta}$-modules corresponding to $\mathcal{E}_i'$. They are supported on the Frobenius neighborhood of $\tilde{C}$ in T^*C, and $\mathcal{L}_1/\mathcal{L}_2$ is the irreducible corresponding to $\mathcal{F}'$, which is supported at $(\bar{x}, \bar{\xi})$. From the definition of the equivalence $\mathcal{D}_C$-mod $\sim$ $\mathcal{D}_{T^*C,\theta}$-mod we see that $\mathcal{F}'_{\bar{x}} = (\mathcal{L}_1/\mathcal{L}_2)_{(\bar{x},\bar{\xi})} = (\mathcal{L}_1)_{(\bar{x},\bar{\xi})}$. Unraveling the definition of $\mathcal{L}_{\text{univ}}$, one can see that its fiber at γ is identified with

$$(\mathcal{L}_{\text{univ}})_\gamma = (\mathcal{L}_1^{(1)})_{(x,\xi)} = (\mathcal{L}_1)^{\otimes p}_{(\bar{x},\bar{\xi})} = (\mathcal{F}')^{\otimes p}_{\bar{x}} = \mathcal{S}_\gamma\,.$$

Saying this for γ being an S-point for arbitrary scheme S, we can thus prove that $\mathcal{L}_{\text{univ}}$ and $\mathcal{S}$ become isomorphic after pullback to $\Gamma_{\text{AJ}} \times_{C^{(1)}} C$. To prove that they are isomorphic on Γ_{AJ}, we'll use the following

Claim 15. *Let $\pi\colon T \to S$ be a smooth morphism of relative dimension 1. Let $\phi\colon T \to \tilde{T} := T^{(1)} \times_{S^{(1)}} S$ be the relative Frobenius map, and $\mathcal{L}$ a line bundle on T. Then there is a canonical isomorphism*

$$\det R\phi_*\mathcal{L} \cong \mathcal{L}^{(S)} \otimes (\Omega^1_{\tilde{T}/S})^{\otimes(p-1)/2} \tag{4.3}$$

where $\mathcal{L}^{(S)}$ is the "relative Frobenius twist" of $\mathcal{L}$: it is the pullback of $\mathcal{L}^{(1)}$ along the map $\tilde{T} \to T^{(1)}$.

Now suppose we are given an S-point γ of Γ. It corresponds to a sequence (4.2) of $(\mathcal{O}_S \boxtimes \mathcal{D}_{C,\omega^{1/2}})$-modules where $\mathcal{E}_1, \mathcal{E}_2$ are rank N vector bundles on $S \times C$, and $\mathcal{F}$ is a line bundle on $\operatorname{supp}\mathcal{F} = S \times_{C^{(1)}} C \subset S \times C$ for a certain $x\colon S \to C^{(1)}$. Just as before, we have $\mathcal{S}_\gamma = \det R\pi_*\mathcal{F}$ where π is the projection $S \times C \to S$. Now note that $\mathcal{F} = \mathcal{E}_1|_{\operatorname{supp}\mathcal{F}}$. Let $\Gamma_x \subset S \times C^{(1)}$ be the graph of x. Then $R\pi_*\mathcal{F} = ((\mathrm{id}_S \times \mathrm{Fr}_C)_*\mathcal{E}_1)|_{\Gamma_x}$. Now, Claim 15 gives

$$\begin{aligned}\mathcal{S}_\gamma = \det R\pi_*\mathcal{F} &= (\det(\mathrm{id}_S \times \mathrm{Fr}_C)_*\mathcal{E}_1)|_{\Gamma_x} \\ &= \left((\mathcal{E}_1)^{(S)} \otimes (\mathcal{O}_S \boxtimes \omega_{C^{(1)}}^{\otimes(p-1)/2})\right)\Big|_{\Gamma_x} = (\mathcal{F}^{(S)})_x \underset{\mathcal{O}_S}{\otimes} x^*\omega_{C^{(1)}}^{\otimes(p-1)/2}\,.\end{aligned}$$

We claim that the right-hand side is canonically identified with $(\mathcal{L}_{\text{univ}})_\gamma$.

To prove that this isomorphism is compatible with connections, we will need the following statement:

Claim 16 (cf. Proposition 35). *Suppose that* $T = C \times S \xrightarrow{\pi} S$ *is a trivial family of smooth curves and* $s\colon S \to T^{(S)}$ *is a section of* $T^{(S)} \to S$. *Let* $(\mathcal{F}, \nabla_{\mathcal{F}})$ *be a coherent sheaf on* T *with an* S*-relative* $\omega_{T/S}^{\otimes 1/2}$*-connection and assume that* $\mathcal{F}$ *is a pushforward of a line bundle* $\mathcal{L}$ *on the closed subscheme* $Z = S \times_{s,T^{(S)}} T$. *Let* $\nabla_{\det}$ *be the connection on* $\mathcal{L}' := \det R\pi_*\mathcal{F}$ *given by Proposition 10. Let* $w\colon S \to Z$ *be the (*k*-semilinear) map such chat* $\iota_Z \circ w = W_{T/S} \circ s$. *Then* $\nabla_{\mathcal{F}}$ *gives rise to a connection* ∇_w *on* $\mathcal{L}'' := w^*(\mathcal{L} \otimes \omega_{Z/S}^{(p-1)/2})$.

The two line bundles with connection are related by

$$(\mathcal{L}', \nabla_{\det}) \cong (\mathcal{L}'', \nabla_w - \tilde{s}^*\theta) \tag{4.4}$$

where $\tilde{s}\colon T^*(T^{(S)}/S) = T^*C^{(1)} \times S$ *is the section of* $T^*(T^{(S)}/S) \to S$ *whose image is the* p*-support of* $(\mathcal{F}, \nabla_{\mathcal{F}})$ *and* θ *is the pullback of the canonical* 1*-form on* $T^*C^{(1)}$ *under the projection* $T^*C^{(1)} \times S \to T^*C^{(1)}$.

□

Proposition 12 (at least, restricted to Γ_{AJ}) is an immediate consequence of Lemma 14.

Denote by $\mathscr{T}_c \to (\mathscr{B}^0)^{(1)}$ the torsor over $(\mathrm{Higgs}^0)^{(1)} \to (\mathscr{B}^0)^{(1)}$ pulled back from the standard torsor $\mathrm{Loc}^0_{\omega^{1/2}} \to (\mathscr{B}^0)^{(1)}$ under the action of $c \in \mathsf{k}^\times$ on $(\mathscr{B}^0)^{(1)}$.

Now, to prove Theorem 1, we need to construct a line bundle $(\mathcal{L}, \nabla)_{\ker}$ with connection on $\mathscr{T}_1 \times_{(\mathscr{B}^0)^{(1)}} \mathscr{T}_c$ satisfying

$$\widetilde{\mathrm{curv}}((\mathcal{L}, \nabla)_{\ker}) = c \cdot \widetilde{\mathrm{curv}}(\mathrm{pr}_1^*(\mathcal{L}, \nabla)_{\det}) + c^{-1} \cdot \widetilde{\mathrm{curv}}(\mathrm{pr}_2^*(\mathcal{L}, \nabla)_{\det})\,. \tag{4.5}$$

The sought-for bundle with connection will be given by

$$(\mathcal{L}, \nabla)_{\ker} := a^*(\mathcal{L}'_{\det}, \nabla_{\det}) \otimes \mathrm{pr}_1^*(\mathcal{L}'^{\otimes -1}_{\det}, \nabla^*_{\det}) \otimes \mathrm{pr}_2^*(\mathcal{L}'^{\otimes -1}_{\det}, \nabla^*_{\det}) \tag{4.6}$$

where $(\mathcal{L}'_{\det}, \nabla_{\det})$ is the universal line bundle with connection on Loc, and a is the "addition" map $\mathscr{T}_1 \times_{(\mathscr{B}^0)^{(1)}} \mathscr{T}_c \to \mathscr{T}_{1+c}$ (one can check that the torsor $\mathscr{T}_{1+c}$ is the sum of the torsors $\mathscr{T}_1$ and $\mathscr{T}_c$). Define $\tilde{\theta} = \widetilde{\mathrm{curv}}(\mathcal{L}, \nabla)_{\det}$. Then substituting (4.6) in (4.5) yields

$$\begin{aligned}\mathrm{LHS} - \mathrm{RHS} &= \widetilde{\mathrm{curv}}((\mathcal{L}, \nabla)_{\mathrm{ker}}) - c \cdot \mathrm{pr}_1^* \tilde{\theta} - c^{-1} \cdot \mathrm{pr}_2^* \tilde{\theta} \\ &= a^* \tilde{\theta} - \mathrm{pr}_1^* \tilde{\theta} - \mathrm{pr}_2^* \tilde{\theta} - c \cdot \mathrm{pr}_1^* \tilde{\theta} - c^{-1} \cdot \mathrm{pr}_2^* \tilde{\theta} \\ &= a^* \tilde{\theta} - (1+c) \cdot \mathrm{pr}_1^* \tilde{\theta} - (1 + c^{-1}) \cdot \mathrm{pr}_2^* \tilde{\theta}\,.\end{aligned} \tag{4.7}$$

So we need to show that

$$(1+c)^{-1} a^* \tilde{\theta} = \mathrm{pr}_1^* \tilde{\theta} + c^{-1} \mathrm{pr}_2^* \tilde{\theta}\,. \tag{4.8}$$

To prove formula (4.8), we proceed as follows. Let $\tilde{\alpha} = (1+c)^{-1} a^* \tilde{\theta} - \mathrm{pr}_1^* \tilde{\theta} - c^{-1} \mathrm{pr}_2^* \tilde{\theta}$; we want to prove $\tilde{\alpha} = 0$. Consider the two projections

$$\mathrm{pr}_{1,3}, \mathrm{pr}_{2,3} \colon \mathscr{T}_1 \underset{((\mathscr{B}^0)^{(1)}}{\times} \mathscr{T}_1 \underset{(\mathscr{B}^0)^{(1)}}{\times} \mathscr{T}_c \rightrightarrows \mathscr{T}_1 \underset{(\mathscr{B}^0)^{(1)}}{\times} \mathscr{T}_c.$$

As a first step, we prove that the difference between two pullbacks $\mathrm{pr}_{1,3}^* \tilde{\alpha} - \mathrm{pr}_{2,3}^* \tilde{\alpha} = 0$. Let pr_i' $(i = 1,2,3)$ be the projection from $\mathscr{T}_1 \times_{(\mathscr{B}^0)^{(1)}} \mathscr{T}_1 \times_{(\mathscr{B}^0)^{(1)}} \mathscr{T}_c$ to the i'th factor, and $a_{i,3} = a \circ \mathrm{pr}_{i,3}$ $(i = 1,2)$. We also have a "difference" map $s \colon \mathscr{T}_1 \times_{(\mathscr{B}^0)^{(1)}} \mathscr{T}_1 \to (\mathrm{Higgs}^0)^{(1)}$; denote $s_{1,2} = s \circ \mathrm{pr}_{1,2}$. Now we calculate

$$\begin{aligned}\mathrm{pr}_{1,3}^* \tilde{\alpha} - \mathrm{pr}_{2,3}^* \tilde{\alpha} &= (1+c)^{-1} a_{1,3}^* \tilde{\theta} - \mathrm{pr}_1'^* \tilde{\theta} - c^{-1} \mathrm{pr}_3'^* \tilde{\theta} \\ &\quad - [(1+c)^{-1} a_{2,3}^* \tilde{\theta} - \mathrm{pr}_2'^* \tilde{\theta} - c^{-1} \mathrm{pr}_3'^* \tilde{\theta}] \\ &= (1+c)^{-1} (a_{1,3}^* \tilde{\theta} - a_{2,3}^* \tilde{\theta}) - (\mathrm{pr}_1'^* \tilde{\theta} - \mathrm{pr}_2'^* \tilde{\theta}) \\ &= (1+c)^{-1} \delta\big(s_{1,2}^*((1+c)\theta^{(1)})\big) - \delta(s_{1,2}^* \theta^{(1)}) = 0\,,\end{aligned} \tag{4.9}$$

where in the last line we used formula (4.1).

The formula (4.9) implies that $\tilde{\alpha} = \mathrm{pr}_2^* \tilde{\alpha}'$ for some $\tilde{\alpha}' \in \Gamma(\mathscr{T}_c, \mathfrak{F}_{\mathscr{T}_c})$. Similarly, we can show that $\tilde{\alpha} = \mathrm{pr}_1^* \tilde{\alpha}''$ for $\tilde{\alpha}'' \in \Gamma(\mathscr{T}_1, \mathfrak{F}_{\mathscr{T}_1})$. It follows that $\tilde{\alpha} = (\chi^0 \times_{\mathscr{B}^0} \chi^0)^{(1)*} \tilde{\beta}$ for some $\tilde{\beta} \in \Gamma(\mathscr{B}^0, \mathfrak{F}_{\mathscr{B}^0})$. So we need to show that $\tilde{\beta} = 0$.

4.2 Alternative construction of $\tilde{\theta}$

Let Ω_{Loc} be the canonical symplectic form on the smooth part Loc^{sm} of Loc, so that $\Omega_{\mathrm{Loc}} = F_{\nabla_{\det}}$. We will construct a 1-form θ_0 on Loc^{sm} such that $\Omega_{\mathrm{Loc}} = d\theta_0$ and $\delta(\theta_0) = \tilde{\theta}$ (where δ is defined in 2.5). Since Ω_{Loc} is symplectic, constructing θ_0 satisfying the first condition is equivalent to constructing a vector field ξ_0 which is Liouville, i.e., $L_{\xi_0}\Omega_{\mathrm{Loc}} = \Omega_{\mathrm{Loc}}$ where L denotes the Lie derivative.

First, we need another interpretation of the form Ω_{Loc}. Note that for a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on a smooth variety X, and two $\mathcal{G}$-modules $\mathcal{M}$ and $\mathcal{N}$ with proper support, we have the following version of Serre duality: $\mathrm{RHom}(\mathcal{M},\mathcal{N})^* \cong \mathrm{RHom}(\mathcal{N},\mathcal{M}\otimes\omega_X)[\dim X]$ where ω_X is the sheaf of top degree differential forms on X. In particular, if X is a symplectic surface and $\mathcal{M} = \mathcal{N}$, we have a nondegenerate (in fact, antisymmetric) bilinear form on the space $\mathrm{Ext}^1(\mathcal{M},\mathcal{M})$ which is identified with the tangent space at $\mathcal{M}$ to the moduli space $\mathscr{M}_{X,\mathcal{G}}$ of $\mathcal{G}$-modules on X, so we get a non-degenerate 2-form on this moduli space. One can use properties of the Serre duality to show that this form is closed.

Lemma 17. *The form Ω_{Loc} coincides with the one just described, where we put $X = T^*C^{(1)}$, and $\mathcal{G} = \mathcal{G}_{\mathcal{D}}$ is the gerbe corresponding to the Azumaya algebra $\mathcal{D}_C$.*

Proof sketch. It is known that (in any characteristic) the category of $\mathcal{D}$-modules on a smooth variety Z admits a Serre functor $\mathcal{S}_Z$ which, moreover, is canonically isomorphic to the shift by $2\dim Z$. The same is true for $\mathcal{D}'$-modules where $\mathcal{D}'$ is any twisted differential operator algebra. We will need the case $\mathcal{D}' = \mathcal{D}_{C,\omega^{1/2}}$. The lemma will follow from the following two statements:

- The curvature of $(\mathcal{L},\nabla)_{\det}$ coincides with the 2-form constructed from this isomorphism $\mathcal{S}_C \cong [2]$.
- The composite equivalence $\mathcal{D}_{C,\omega^{1/2}} \overset{\otimes\omega^{\otimes(1-p)/2}}{\sim} \mathcal{D}_C\text{-mod} \sim \mathcal{G}_\theta\text{-mod}$ is compatible with the trivializations of Serre functors.

The first statement makes sense in any characteristic and should be well known. As for the second statement, the difference between two trivializations is an invertible function on $T^*C^{(1)}$, therefore a constant. With a little more work, one can show that this constant is equal to 1. $\square$

In the situation above, consider the open substack $\mathscr{M}^0_{X,\mathcal{G}} \subset \mathscr{M}_{X,\mathcal{G}}$ consisting of modules which (locally after splitting $\mathcal{G}$) look like (pushforward of) a line bundle on a smooth curve in X. Denote by $\mathscr{B}^0_X$ the moduli space of proper smooth curves in X, and let $\mathscr{C}^0_X \subset \mathscr{B}^0_X \times X \to \mathscr{B}^0_X$ be the universal family of curves. Then we have a map $\chi\colon \mathscr{M}^0_{X,\mathcal{G}} \to \mathscr{B}^0_X$ given by taking supports, which presents $\mathscr{M}^0_{X,\mathcal{G}}$ as a torsor for the relative Picard stack $\mathrm{Pic}(\mathscr{C}^0_X/\mathscr{B}^0_X)$. One can show that this defines an integrable system (i.e., that the fibers are Lagrangian).

Denote $\mathfrak{D} = \mathrm{Spec}(\mathsf{k}[\varepsilon]/\varepsilon^2)$. A vector field on Loc^{sm} is the same as an automorphism of $\mathrm{Loc}^{sm} \times \mathfrak{D}$ over $\mathfrak{D}$ which is identity on $\mathrm{Loc}^{sm} \subset \mathrm{Loc}^{sm} \times \mathfrak{D}$. Let h be the automorphism of $T^*C^{(1)} \times \mathfrak{D}$ corresponding to the Euler vector field on $T^*C^{(1)}$. Since $H^2(T^*C^{(1)}, \mathcal{O}) = 0$, there exists a (non-unique) equivalence

$$\Phi\colon\ \mathrm{pr}_1^* \mathcal{G}_\mathcal{D} \sim h^* \, \mathrm{pr}_1^* \mathcal{G}_\mathcal{D}$$

which is identity on $T^*C^{(1)} \times \mathrm{pt} \subset T^*C^{(1)} \times \mathfrak{D}$. Now, if we have a $\mathcal{D}$-module $\mathcal{M}$ on C, let $\mathcal{M}'$ be the corresponding $\mathcal{G}_\mathcal{D}$-module, and let $\overline{\mathcal{M}'} = \Phi^{-1} h^*(\mathcal{M}' \boxtimes \mathcal{O}_\mathfrak{D})$ – this is a $\mathrm{pr}_1^* \mathcal{G}_\mathcal{D}$-module on $T^*C^{(1)} \times \mathfrak{D}$, and denote by $\overline{\mathcal{M}}$ the corresponding $\mathcal{D}_C \boxtimes \mathcal{O}_\mathfrak{D}$-module. By construction, $\overline{\mathcal{M}}/\varepsilon\overline{\mathcal{M}} \cong \mathcal{M}$, so $\overline{\mathcal{M}}$ defines a tangent vector to Loc^{sm} at $\mathcal{M}$. This way we get a vector field on Loc^{sm} which is the desired field ξ_0. Denote $\theta_0 = \iota_{\xi_0} \Omega_{\mathrm{Loc}}$.

Proposition 18. *The vector field ξ_0 is Liouville. Equivalently, $d\theta_0 = \Omega_{\mathrm{Loc}}$.*

Proof sketch. The proposition follows from the functoriality of the Serre duality. Namely, if we have a symplectic surface (X, Ω) with a $\mathbb{G}_m$-gerbe $\mathcal{G}$ and an automorphism ϕ of the pair $(X, \mathcal{G})$ such that $\phi^*\Omega = \lambda\Omega$ for some $\lambda \in \mathsf{k}^\times$, then the corresponding automorphism $\tilde{\phi}$ of the moduli space $\mathscr{M}_\mathcal{G}$ of (coherent, properly supported) $\mathcal{G}$-modules will satisfy $\tilde{\phi}^*\Omega_{\mathscr{M}_\mathcal{G}} = \lambda\Omega_{\mathscr{M}_\mathcal{G}}$ where $\Omega_{\mathscr{M}_\mathcal{G}}$ is the symplectic form constructed by the Serre duality. One can also formulate an in-families version of this statement. In particular, if we take $X = T^*C^{(1)}$, $\mathcal{G} = \mathcal{G}_\theta$ and the $\mathfrak{D}$-family of automorphisms given by (h, Φ), then the equality $h^*\Omega_{T^*C^{(1)}} = (1+\varepsilon)\Omega_{T^*C^{(1)}}$ implies that $\tilde{h}^*\Omega_{\mathrm{Loc}} = (1+\varepsilon)\Omega_{\mathrm{Loc}}$ which means (since $\tilde{h} = 1 + \varepsilon\xi_0$ by definition) that $L_{\xi_0}\Omega_{\mathrm{Loc}} = \Omega_{\mathrm{Loc}}$. □

Proposition 19. *The class of ξ_0 modulo Hamiltonian vector fields does not depend on the choice of Φ.*

Proof sketch. Suppose we have two equivalences $\Phi_1, \Phi_2 \colon \mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}} \sim h^* \mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}}$. Then they differ by an auto-equivalence $\Phi_1^{-1} \circ \Phi_2$ of $\mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}}$ which corresponds to an element $\phi \in H^1(T^*C^{(1)}, \mathcal{O})$. From any such element we can construct a function f'_ϕ on $\mathscr{B}^{(1)}$ as follows. If a point $b \in \mathscr{B}^{(1)}$ corresponds to a smooth spectral curve $\tilde{C} \subset T^*C^{(1)}$ (i.e., if $b \in (\mathscr{B}^0)^{(1)}$) then we just put $f'_\phi(b) = \langle \phi|_{\tilde{C}}, \theta|_{\tilde{C}} \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the Serre duality pairing. (It can also be defined for $b \notin (\mathscr{B}^0)^{(1)}$.) One can check that the pullback f_ϕ of f'_ϕ to Loc^{sm} satisfies $H_{f_\phi} = \xi_{0,1} - \xi_{0,2}$ where $\xi_{0,1}$ and $\xi_{0,2}$ are the ξ_0's corresponding to Φ_1 and Φ_2, respectively. □

Recall (see 2.3.2) that for a smooth variety X and a 1-form θ on $X^{(1)}$ we denote by $\mathcal{G}_\theta$ the $\mathbb{G}_m$-gerbe on $X^{(1)}$ corresponding to the Azumaya algebra $\mathcal{D}_\theta$. It is straightforward to generalize this to the case of families $X \to S$ for arbitrary k-scheme S. We will need the case when $S = \mathfrak{D}$. We know that the gerbe $\mathcal{G}_{\mathcal{D}}$ on $T^*C^{(1)}$ is equivalent to $\mathcal{G}_\theta$ where θ is the canonical 1-form on T^*C. One easily checks that $\mathcal{G}_\theta$ depends additively on θ in the sense that $\mathcal{G}_{\theta_1+\theta_2} \sim \mathcal{G}_{\theta_1} \cdot \mathcal{G}_{\theta_2}$. Therefore on $T^*C^{(1)} \times \mathfrak{D}$ we have

$$\mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}}^{-1} \cdot h^* \mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}} \sim \mathcal{G}_{h^*\theta - \theta} = \mathcal{G}_{\varepsilon\theta} = \mathcal{G}_{e^*\theta} \sim e^* \mathrm{pr}_1^* \mathcal{G}_{\mathcal{D}}$$

where θ here is considered as a relative 1-form on $T^*C^{(1)} \times \mathfrak{D}$ over $\mathfrak{D}$, and $e \colon T^*C^{(1)} \times \mathfrak{D} \to T^*C^{(1)} \times \mathfrak{D}$ is given by fiberwise multiplication by ε. Now, e factors through the 1st infinitesimal neighborhood Z_1 of the zero section in $T^*C^{(1)}$. Therefore the above equivalence Φ can be constructed from any trivialization Ψ of $\mathcal{G}_{\mathcal{D}}$ on Z_1 which coincides with the canonical trivialization on the zero section. We assume from now on that Φ is obtained in this way.

Note that, given Ψ, one can construct a family of equivalences

$$\Phi_c\colon \operatorname{pr}_1^* \mathcal{G}_{c\theta} \sim h^* \operatorname{pr}_1^* \mathcal{G}_{c\theta} \tag{4.10}$$

parametrized by $c \in \mathsf{k}$ (we just have to replace e by fiberwise dilation by $c\varepsilon$). One can check that Φ_c is additive in c, i.e., compatible with the equivalence $\mathcal{G}_{(c+c')\theta} \sim \mathcal{G}_{c\theta} \cdot \mathcal{G}_{c'\theta}$ and, if $c \in \mathsf{k}^\times$, Φ_c is obtained from Φ by conjugation with dilation by c on $T^*C^{(1)}$.

Proposition 20. *If the equivalence Φ is from the class just described then, in the notation of* (4.8) *(for any $c \in \mathsf{k} \setminus \mathbb{F}_p$), we have*

$$(1+c)^{-1} a^* \theta_0 = \operatorname{pr}_1^* \theta_0 + c^{-1} \operatorname{pr}_2^* \theta_0 \,. \tag{4.11}$$

Proof sketch. First we'll show that

$$(1+c)^{-1} a^* \Omega_{\mathrm{Loc}} = \operatorname{pr}_1^* \Omega_{\mathrm{Loc}} + c^{-1} \operatorname{pr}_2^* \Omega_{\mathrm{Loc}} \,. \tag{4.12}$$

This will follow from a more general statement:

Lemma 21. *Let X be a symplectic surface, $\mathcal{G}, \mathcal{G}', \mathcal{G}''$ three $\mathbb{G}_m$-gerbes on it, and suppose we are given an equivalence $\mathcal{G} \sim \mathcal{G}' \cdot \mathcal{G}''$. Consider the corresponding $\operatorname{Pic}(\mathscr{C}_X^0/\mathscr{B}_X^0)$-torsors $\mathscr{T} = \mathscr{M}_{X,\mathcal{G}}^0$, $\mathscr{T}' = \mathscr{M}_{X,\mathcal{G}'}^0$, $\mathscr{T}'' = \mathscr{M}_{X,\mathcal{G}''}^0$ endowed with symplectic structures. Clearly, $\mathscr{T}$ is identified with the sum of torsors $\mathscr{T}'$ and $\mathscr{T}''$, so we can define the graph of addition $\Gamma \subset \mathscr{T} \times_{\mathscr{B}_X^0} \mathscr{T}' \times_{\mathscr{B}_X^0} \mathscr{T}''$. Then Γ is a Lagrangian subvariety in $(\mathscr{T} \times \mathscr{T}' \times \mathscr{T}'', \Omega_{\mathscr{T}} - \Omega_{\mathscr{T}'} - \Omega_{\mathscr{T}''})$.*

Proof sketch. Suppose we want to prove that Γ is Lagrangian in a formal neighborhood of some point $\gamma \in \Gamma$, and let $\tilde{C} \subset X$ be the corresponding spectral curve. Clearly one can replace X by the formal neighborhood $\tilde{C}^\wedge$ of $\tilde{C}$ in X. Since $H^2(\tilde{C}^\wedge, \mathcal{O}) = 0$, we can trivialize $\mathcal{G}$ on $\tilde{C}^\wedge$. So we can assume that $\mathcal{G}$ is trivial. Then Γ is the graph of addition on $\operatorname{Pic}(\mathscr{C}_X^0, \mathscr{B}_X^0)$. It is a well-known fact that this graph is Lagrangian (at least for the usual Hitchin fibration, i.e., for $X = T^*C$). $\square$

Note that for $X = T^*C^{(1)}$, $\mathcal{G} = \mathcal{G}_{c\theta}$ the torsor $\mathscr{M}^0_{X,\mathcal{G}}$ is identified with $\mathscr{T}_c$ from the previous subsection. Moreover, under this identification, we have $c^{-1}\Omega_{\mathrm{Loc}} = \Omega_{\mathscr{M}^0_{X,\mathcal{G}}}$. Therefore, formula (4.12) follows from Lemma 21 applied to $\mathcal{G} = \mathcal{G}_{(1+c)\theta}$, $\mathcal{G}' = \mathcal{G}_\theta$, $\mathcal{G}'' = \mathcal{G}_{c\theta}$.

Denote by $\xi_{0,c}$ the vector field on $\mathscr{T}_c$ obtained from ξ_0 under the identification $\mathscr{T}_c \cong \mathrm{Loc}^0$. Then, in order to prove Proposition 4.11, we need to prove that the vector field $\eta_c = \mathrm{pr}_1^* \xi_{0,1+c} + \mathrm{pr}_2^* \xi_{0,1} + \mathrm{pr}_3^* \xi_{0,c}$ on $\mathscr{T}_{1+c} \times \mathscr{T}_1 \times \mathscr{T}_c$ preserves Γ (because Γ is Lagrangian, and this vector field corresponds to the 1-form $(1+c)^{-1}\,\mathrm{pr}_1^*\,\theta_0 - \mathrm{pr}_2^*\,\theta_0 - c^{-1}\,\mathrm{pr}_3^*\,\theta_0$). To see this, note that $\xi_{0,c}$ can be obtained the same way as ξ_0 with Φ replaced by Φ_c from (4.10). Now, η_c comes from an infinitesimal automorphism of the quadruple $(X, \mathcal{G}_\theta, \mathcal{G}_{c\theta}, \mathcal{G}_{(1+c)\theta})$ given by $(h, \Phi_1, \Phi_c, \Phi_{1+c})$. Additivity of Φ_c in c implies that this automorphism is compatible with the equivalence $\mathcal{G}_\theta \cdot \mathcal{G}_{c\theta} \sim \mathcal{G}_{(1+c)\theta}$. Therefore η_c preserves Γ, which is what we want. $\square$

Lemma 22. *The extended curvature $\tilde{\theta}$ of the determinant line bundle with connection $(\mathcal{L}, \nabla)_{\det}$ on Loc^{sm} is equal to $\delta(\theta_0)$.*

Since δ is k-linear and compatible with pullbacks, Lemma 22 implies formula (4.8) and therefore Theorem 1.

We'll prove Lemma 22 in subsection 4.2. Here we prove the following partial result:

Proposition 23. *We have $\tilde{\theta} - \delta(\theta_0) = \mathsf{P}(\chi'^{(1)*}\beta_0')$ where χ' is the map $\mathrm{Loc}^{sm} \to \mathscr{B}^{(1)}$, δ and P are defined in 2.5, and β_0' is some form on $\mathscr{B}^{(2)}$.*

Proof sketch. Denote $\tilde{\alpha}_0 = \tilde{\theta} - \delta(\theta_0)$. We have already mentioned above that $\mathsf{Q}(\delta(\theta_0)) = d\theta_0 = \Omega_{\mathrm{Loc}} = \mathsf{Q}(\tilde{\theta})$. So $\mathsf{Q}(\tilde{\alpha}_0) = 0$, which means that $\tilde{\alpha}_0 = \mathsf{P}(\alpha_0)$ for some $\alpha_0 \in \Gamma(\mathscr{B}^{(1)}, \Omega^1_{\mathscr{B}^{(1)}})$. We want to prove that $\alpha_0 = \chi'^{(1)*}\beta_0'$ for some β_0'. We'll show that $\alpha_0|_{\mathrm{Loc}^0}$ is a pullback of some 1-form β_0 on $\mathscr{B}^{0(1)}$. Using the properties of χ', we can then deduce that β_0 extends to the whole $\mathscr{B}^{(1)}$ since $\chi'^{(1)*}\beta_0$ extends to Loc^{sm}.

Now let Γ be the graph of addition in $\mathrm{Higgs}^{0(1)} \times_{\mathscr{B}^{(1)}} \mathrm{Loc}^0 \times_{\mathscr{B}^{(1)}} \mathrm{Loc}^0$. The argument of the proof of Proposition 4.11 applied to $c = 0$ shows that on Γ we have $\mathrm{pr}_1^*\,\theta^{(1)} + \mathrm{pr}_2^*\,\theta_0 = \mathrm{pr}_3^*\,\theta_0$ where θ is the canonical 1-form on $\mathrm{Higgs} = T^*\,\mathrm{Bun}$. (We use that the vector field $\xi_{0,0}$ on Higgs coincides with the Euler vector field, and therefore $\theta_{0,0} = \theta$.) Applying δ yields $\delta(\mathrm{pr}_1^* \theta^{(1)}) + \delta(\mathrm{pr}_2^* \theta_0) = \delta(\mathrm{pr}_3^* \theta_0)$. On the other hand, we know from (4.1) that $\delta(\mathrm{pr}_1^* \theta^{(1)}) + \mathrm{pr}_2^* \tilde{\theta} = \mathrm{pr}_3^* \tilde{\theta}$, so subtracting the previous equation, we get $\mathrm{pr}_2^{(1)*} \tilde{\alpha}_0 = \mathrm{pr}_3^{(1)*} \tilde{\alpha}_0$, so $\tilde{\alpha}_0|_{\mathrm{Loc}^0}$ is a pullback of some $\tilde{\beta}_0' \in \Gamma(\mathscr{B}^{0^{(1)}}, \mathcal{F}_{\mathscr{B}^{0^{(1)}}})$. Since $\tilde{\alpha}_0 \in \mathrm{Im}(\mathsf{P})$, we must have $\tilde{\beta}_0' \in \mathrm{Im}(\mathsf{P})$, so $\tilde{\beta}_0' = \mathsf{P}(\beta_0)$ for some β_0. □

4.3 Proof of Lemma 22

Denote by $\mathrm{Bun}^{[d]}$ the connected component of Bun consisting of vector bundles of degree d and $\mathrm{Loc}^{[d]}$ its preimage in Loc. Note that $\mathrm{Loc}^{[d]}$ is nonempty only for d divisible by p.

We can deduce Lemma 22 from the following statement:

Lemma 24. *For generic curve C the fibers of the maps $q\colon \mathrm{Loc} \to \mathrm{Bun}$ and $\chi\colon \mathrm{Loc} \to \mathscr{B}^{(1)}$ are transversal generically on $\mathrm{Loc}^{[0]}$.*

Namely, we will show the following:

Proposition 25. *The 1-form $\beta_0 = \chi^{(1)*}\beta_0'$ (where β_0' is defined in Proposition 23) vanishes on $q^{-1}(b)$ if $b \in \mathrm{Bun}$ is such that q is smooth over b.*

Clearly, together with Lemma 24, this implies the desired equality $\beta_0' = 0$.

Consider the stack $\widetilde{\mathrm{Loc}}$ over $\mathbb{A}^1$ whose fiber over $\lambda \in \mathbb{A}^1(\mathsf{k}) = \mathsf{k}$ is the stack Loc_λ of rank N bundles on C with λ-connection. The stack $\widetilde{\mathrm{Loc}}$ has a canonical $\mathbb{G}_m$-action lifting the dilation action on $\mathbb{A}^1$. Consequently, we have an isomorphism $\widetilde{\mathrm{Loc}} \times_{\mathbb{A}^1} \mathbb{G}_m \cong \mathrm{Loc} \times \mathbb{G}_m$. Let t be the coordinate function on $\mathbb{A}^1$. Denote by Loc^{sm} the smooth part of Loc and by $\widetilde{\mathrm{Loc}}^{sm}$ the maximal open substack in $\widetilde{\mathrm{Loc}}$ smooth over $\mathbb{A}^1$. The stack $\widetilde{\mathrm{Loc}}$ defines a filtration on functions, differential forms, etc. on Loc (and on open substacks thereof): a form η on Loc belongs to the k'th filtered piece iff the pullback of η to $\mathrm{Loc} \times \mathbb{G}_m \hookrightarrow \widetilde{\mathrm{Loc}}$ has pole of order not greater than k along $\mathrm{Higgs} = \widetilde{\mathrm{Loc}} \times_{\mathbb{A}^1} \{0\}$. This filtration is compatible with the de Rham differential. Similarly, we get a filtration on $\Gamma(\mathcal{F}_{\mathrm{Loc}^{sm}}) = \Gamma(\Omega^1_{\mathrm{Loc}^{sm}}/d\mathcal{O}_{\mathrm{Loc}^{sm}})$. All these filtrations will be denoted by $F^\bullet$.

For example, there is a relative symplectic form on $\widetilde{\mathrm{Loc}}^{sm}/\mathbb{A}^1$ of weight 1 with respect to the $\mathbb{G}_m$-action. Its restriction to the fibers over $0, 1 \in \mathbb{A}^1$ are the standard symplectic forms on Higgs^{sm} and Loc^{sm}. This means that $\Omega_{\mathrm{Loc}} \in F^1\Omega^2(\mathrm{Loc}^{sm})$. It is also straightforward to check that

$$\tilde{\theta} \in F^p\mathcal{F}(\mathrm{Loc}^{sm}). \tag{4.13}$$

Lemma 26. *We have $\theta_0 \in F^{p+1}\Omega^1(\mathrm{Loc}^{sm})$. Equivalently, $\xi_0 \in F^p\mathcal{T}(\mathrm{Loc}^{sm})$.*

Proof. We need to prove that $t^p\xi_0$ extends to $\widetilde{\mathrm{Loc}}^{sm}$. Recall that the value of ξ_0 at a point corresponding to a bundle with connection (or $\mathcal{O}$-coherent $\mathcal{D}$-module) $\mathcal{M} = (\mathcal{E}, \nabla)$ is given by the infinitesimal deformation $\overline{\mathcal{M}} = (\overline{\mathcal{E}}, \overline{\nabla})$ of $\mathcal{M}$ constructed in 4.2.

We will think of $\overline{\mathcal{M}}$ as an extension of $\mathcal{M}$ by itself. Recall that the construction of $\overline{\mathcal{M}}$ in 4.2 uses the splitting of the gerbe $\mathcal{G}$ on the 1st infinitesimal neighborhood of zero section in $T^*C^{(1)}$. This splitting can be thought of as an extension of $\mathcal{D}$-modules $0 \to \mathcal{T}_C^{(C)} = \mathcal{T}_C^{\otimes p} \to \mathcal{M}_0 \to \mathcal{O}_C \to 0$. Denote by $v \in \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{O}_C, \mathcal{T}_C^{(C)})$ the class of this extension. We will also need the p-curvature of $\mathcal{M}$ thought of as a map of $\mathcal{D}$-modules $\mathrm{curv}_p(\mathcal{M})\colon \mathcal{M} \otimes \mathcal{T}_C^{(C)} \to \mathcal{M}$. By unwinding the definition of $\overline{\mathcal{M}}$, it is not hard to check the following:

Claim 27. *The class of $\overline{\mathcal{M}}$ in $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is given by*

$$\mathrm{class}(\overline{\mathcal{M}}) = \mathrm{curv}_p(\mathcal{M}) \cdot (\mathrm{id}_{\mathcal{M}} \otimes v)$$

where $\cdot$ denotes the composition $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M} \otimes \mathcal{T}_C^{(C)}, \mathcal{M}) \otimes \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{M} \otimes \mathcal{T}_C^{(C)}) \to \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$. More precisely, the exact sequence $0 \to \mathcal{M} \to \overline{\mathcal{M}} \to \mathcal{M} \to 0$ is canonically isomorphic to the pullback of $\mathcal{M} \otimes (0 \to \mathcal{T}_C^{(C)} \to \mathcal{M}_0 \to \mathcal{O}_C \to 0)$ by $\mathrm{curv}_p(\mathcal{M})$.

In order to construct the vector field $\tilde{\xi}_0$ on $\widetilde{\mathrm{Loc}}^{sm}$ extending $t^p\xi_0$, recall the notion of p-curvature of a λ-connection from 2.2. It allows to extend the above construction of $\overline{\mathcal{M}}$ to λ-connections to get the desired vector field $\tilde{\xi}_0$. (We just need to multiply the connection on $\mathcal{M}_0$ by λ and use tensor product of λ-connections.) $\square$

Proof of Proposition 25. From Lemma 26 and formula (4.13) we see that $\mathsf{P}(\beta_0) = \tilde{\theta} - \delta(\theta_0) \in F^{p+1}\mathcal{F}(\mathrm{Loc}^{sm})$. So we must have

$$\beta_0 \in F^1\Omega^1((\mathrm{Loc}^{sm})^{(1)}) \tag{4.14}$$

(otherwise $\mathsf{P}(\beta_0) \boxtimes 1_{\mathcal{O}(\mathbb{G}_m)}$ extended to $\widetilde{\mathrm{Loc}}^{sm}$ would have a pole of order $\geq 2p > p+1$ along Higgs). Let $\mathscr{Y}$ be the open part of Higgs given by $\mathscr{Y} := \mathrm{Higgs}^{sm[p\mathbb{Z}]} = \widetilde{\mathrm{Loc}}^{sm} \times_{\mathbb{A}^1} \{0\}$. Then we get a 1-form β_1 on $\mathscr{Y}^{(1)}$ obtained by extending the (relative) 1-form $t\beta_0$ from $(\mathrm{Loc}^{sm})^{(1)} \times \mathbb{G}_m$ to $(\widetilde{\mathrm{Loc}}^{sm})^{(1)}$ and then restricting to $\mathscr{Y}^{(1)}$. The form β_1 has weight 1 with respect to the $\mathbb{G}_m$-action on $\mathscr{Y}^{(1)} \subset \mathrm{Higgs}^{(1)}$. For $b \in \mathrm{Bun}$ as in the statement of Proposition 25, (4.14) implies that $\beta_0|_{\mathrm{Loc}_b}$ is a translation-invariant form on Loc_b, and the restriction of β_1 to the fiber Higgs_b of Higgs at b is the corresponding translation-invariant form on Higgs_b (recall that Loc_b is an affine space over the vector space Higgs_b).

Denoting by Eu the differential of the $\mathbb{G}_m$-action on $\mathscr{Y}$, we get a function $F = \iota_{\mathrm{Eu}}\beta_1$ on $\mathscr{Y}^{(1)}$ of $\mathbb{G}_m$-weight 1. The restriction of F to Higgs_b is a linear function whose differential is $\beta_1|_{\mathrm{Higgs}_b}$. Since the projection $\mathscr{Y} \to \mathscr{B}$ is proper over $\mathscr{B}^0$, the restriction of F to each component of Higgs must be a pullback of a function F' on $\mathscr{B}$. This function must also have degree 1 with respect to the standard $\mathbb{G}_m$-action on $\mathscr{B}$. We want to show that $F' = 0$ and hence $F = 0$. This will imply that $\beta_1|_{\mathrm{Higgs}_b} = 0$ and therefore $\beta_0|_{\mathrm{Loc}_b} = 0$.

Since we know that β_0 descends to $\mathscr{B}$, the function F' does not depend on the choice of connected component of Higgs. Now consider the Serre duality involution σ on $\mathrm{Loc}_{\omega^{1/2}}$. Via the identification $\mathrm{Loc}_{\omega^{1/2}} \xrightarrow{\sim} \mathrm{Loc}$ given by $\mathcal{M} \mapsto \mathcal{M} \otimes \omega^{\otimes(p-1)/2}$ it corresponds to an involution on Loc given by $\mathcal{M} \mapsto \mathcal{M}^{\vee} \otimes \omega^{\otimes p}$ which we will also denote by σ. It is easy to see that the determinant line bundle with connection $(\mathcal{L}, \nabla)_{\det}$ is invariant under σ, hence so is its curvature Ω_{Loc} and extended curvature $\tilde{\theta}$. The vector field ξ_0 can also be shown to be invariant under σ. So the 1-form θ_0 is

σ-invariant as well. Recalling that $\mathsf{P}(\beta_0) = \tilde{\theta} - \delta(\theta_0)$, we see that β_0 must also be σ-invariant. Thus for the function F we get that it is invariant under an analogous involution on Higgs. But then for F' it means that it should be invariant under the action of $-1 \in \mathbb{G}_m$ on $\mathscr{B}$, whereas in the preceding paragraph we saw it is *anti*-invariant under the same element. The desired equality $F' = 0$ follows. $\square$

Appendix A

Twisted cotangent bundle to the moduli stack of coherent sheaves

The main goal of this appendix is to show that the twisted cotangent bundle to the stack of coherent sheaves on a smooth projective curve is identified with the stack of half-form twisted coherent $\mathcal{D}$-modules on that curve. This is a characteristic-independent statement, except that we have to assume that the characteristic is not 2. In fact, we prove it for arbitrary base scheme S defined over $\mathbb{Z}[1/2]$.

We will understand all derived categories in the higher-categorical (i.e., $(\infty, 1)$ or DG) sense, so that it makes sense to talk about homotopies between (1-)morphisms. (In fact, the $(2,1)$-categorical level would suffice: all our morphism spaces will be 1-groupoids.)

A.1 Twisted cotangent bundles to stacks

Let S be a (Noetherian?) scheme and $\mathscr{X} \xrightarrow{\pi} S$ a smooth Artin stack over S. For an R-point x of $\mathscr{X}$ where R is a commutative ring, consider the groupoid $\mathfrak{T}_{\mathscr{X}/S,x}$ of all dotted arrows in the diagram

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{x} & \mathscr{X} \\ {\scriptstyle \iota}\downarrow & \overset{\tilde{x}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathcal{D}_R & \xrightarrow[\pi \circ x \circ p]{} & S \end{array} \tag{A.1}$$

where $\mathcal{D}_R = \operatorname{Spec}(R[\varepsilon]/\varepsilon^2)$ and $\iota\colon \operatorname{Spec} R \to \mathcal{D}_R$, $p\colon \mathcal{D}_R \to \operatorname{Spec} R$ are the natural morphisms. This is an R-linear Picard groupoid, so it corresponds to a 2-step complex of R-modules $\mathcal{T}^\bullet_{\mathscr{X}/S,x}$ living in degrees 0 and -1. This complex is perfect, compatible with derived base change and satisfies descent, and therefore defines a perfect object $\mathcal{T}^\bullet_{\mathscr{X}/S} \in D^b(\mathscr{X})$. It is called the *tangent complex* of $\mathscr{X}$ over S. The *cotangent stack* is then defined as

$$T^*(\mathscr{X}/S) := \operatorname{Spec}_{\mathscr{X}}(\operatorname{Sym}_{\mathcal{O}_{\mathscr{X}}} \mathcal{H}^0(\mathcal{T}^\bullet_{\mathscr{X}/S}))$$

where $\mathcal{H}^0$ is taken with respect to the natural t-structure on $D^b(\mathscr{X})$.

Now let $\mathcal{L}$ be a line bundle on $\mathscr{X}$ and $\mathscr{P} \to \mathscr{X}$ the corresponding principle $\mathbb{G}_m$-bundle. Then $\mathcal{T}^\bullet_{\mathscr{P}/S}$ is a $\mathbb{G}_m$-equivariant complex and therefore descends to a complex $\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}}$ on $\mathscr{X}$ which fits into an exact triangle

$$\mathcal{O}_{\mathscr{X}} \xrightarrow{i} \tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}} \to \mathcal{T}^\bullet_{\mathscr{X}/S} \xrightarrow{\delta} \mathcal{O}_{\mathscr{X}}[1] \tag{A.2}$$

We will sometimes refer to $\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}}$ as the *extended tangent complex.* Now define the *twisted cotangent stack* $\tilde{T}^*_{\mathcal{L}}(\mathscr{X}/S)$ as

$$\tilde{T}^*_{\mathcal{L}}(\mathscr{X}/S) := \operatorname{Spec}_{\mathscr{X}}(\operatorname{Sym}_{\mathcal{O}_{\mathscr{X}}} \mathcal{H}^0(\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}})) \underset{\mathbb{A}^1_S}{\times} \{1\}_S \tag{A.3}$$

where the morphism from the first factor to $\mathbb{A}^1_S$ is induced by i.

Remark 5. There are several alternative interpretations of the stack $\tilde{T}^*_{\mathcal{L}}(\mathscr{X}/S)$:

1. It is the spectrum of the quotient of $\operatorname{Sym} \mathcal{H}^0(\mathcal{T}^\bullet_{\mathscr{X}/S})$ by the ideal generated by $1 - i(1_{\mathcal{O}_{\mathscr{X}}})$.

2. Its R-points lying over $x\colon \operatorname{Spec} R \to \mathscr{X}$ are given by "splittings" of the pullback

under x of triangle (A.2).

3. These splittings are the same as null-homotopies of δ_x.

There is a closed substack $\mathscr{Z} \subset \mathscr{X}$ consisting of points $x \in \mathscr{X}$ where i acts non-trivially on local cohomology. Then the map $\tilde{T}^*_{\mathcal{L}}(\mathscr{X}/S) \to \mathscr{X}$ factors through $\mathscr{Z}$ and over $\mathscr{Z}$ it looks like a torsor for $T^*(\mathscr{X}/S)$.

Another way to define the complex $\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}}$ is as follows. For an R-point x of $\mathscr{X}$ let $B\mathbb{G}_a(R)$ denote the classifying groupoid for the group $\mathbb{G}_a(R) = (R,+)$. We will define a functor $\delta'\colon \mathfrak{T}_{\mathscr{X}/S,x} \to B\mathbb{G}_a(R)$. Namely, for any $\tilde{x}\colon \mathfrak{D}_R \to \mathscr{X}$ as in (A.1), we set $\delta'_x(\tilde{x})$ to be the torsor of isomorphisms $\tilde{x}^*\mathcal{L} \xrightarrow{\sim} p^*x^*\mathcal{L}$ whose restriction to $\operatorname{Spec} R \subset \mathfrak{D}_R$ is $\mathrm{id}_{x^*\mathcal{L}}$. The action of $\mathbb{G}_a(R)$ on $\delta'_x(\tilde{x})$ is given by the composition $\mathbb{G}_a(R) \xrightarrow{\sim} 1 + \varepsilon R \subset R[\varepsilon]/\varepsilon^2 = \mathcal{O}(\mathfrak{D}_R) \to \operatorname{End}(\tilde{x}^*\mathcal{L})$. The map δ'_x corresponds to a map $\delta_x\colon \mathcal{T}^\bullet_{\mathscr{X}/S,x} \to R[1]$. The maps δ_x for all R and x glue to a map $\delta\colon \mathcal{T}^\bullet_{\mathscr{X}/S} \to \mathcal{O}_{\mathscr{X}}[1]$. Then $\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}}$ can be reconstructed as "the" cone of δ.[1] Below we write $\delta^{\mathcal{L}}$ instead of δ to show explicitly the dependence of δ on $\mathcal{L}$.

We will need the following lemma whose straightforward proof is omitted.

Lemma 28. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of smooth Artin stacks and $\mathcal{L}$ a line bundle on $\mathscr{Y}$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{T}^\bullet_{\mathscr{X}/S} & \xrightarrow{\delta^{f^*\mathcal{L}}} & \mathcal{O}_{\mathscr{X}}[1] \\ {\scriptstyle df}\downarrow & & \| \\ f^*\mathcal{T}^\bullet_{\mathscr{Y}/S} & \xrightarrow{f^*\delta^{\mathcal{L}}} & f^*\mathcal{O}_{\mathscr{Y}}[1] \end{array}$$

A.2 The stack $\operatorname{Coh}(C)$ and its twisted cotangent bundle

Now let $C \to S$ be a smooth proper family of algebraic curves. We work with schemes over S throughout, so we will often drop "S" from the notation writing $\mathcal{T}_{\mathscr{X}}$, $T^*\mathscr{X}$,

[1] In order to reconstruct $\tilde{\mathcal{T}}^\bullet_{\mathscr{X}/S,\mathcal{L}}$ canonically, one needs to understand the derived categories in the $(\infty,1)$-categorical (or DG) sense.

$\times$, etc. instead of $\mathcal{T}_{\mathscr{X}/S}$, $T^*(\mathscr{X}/S)$, $\times_S$, etc. If R is a ring, and $s\colon \operatorname{Spec} R \to S$ is an R-point of S, we will denote by C_s or C_R the base change of C by s, that is, $C_s = C_R := C \times_{S,s} \operatorname{Spec} R$.

We consider the stack $\operatorname{Coh}(C)$ of coherent sheaves on fibers of C. Its groupoid of R-points is given by

$$\operatorname{Coh}(C)(R) = \left\{ (s, \mathcal{F}) \;\middle|\; \begin{array}{c} s \in S(R); \\ \mathcal{F} \text{ is an } S'\text{-flat coherent sheaf on } C_s \end{array} \right\}.$$

Below we abbreviate $\operatorname{Coh} = \operatorname{Coh}(C)$. Consider the *determinant bundle* $\mathcal{L}_{\det}$ on Coh. Its fiber at an R-point x of Coh corresponding to a coherent sheaf $\mathcal{F}_x$ on C_R is given by

$$(\mathcal{L}_{\det})_x = \det_R \mathrm{R}\Gamma(C_R, \mathcal{F}_x).$$

From now on we assume that

$$2 \text{ is invertible on } S, \quad \text{i.e.,} \quad 2 \cdot 1_{\mathcal{O}(S)} \in \mathcal{O}(S)^\times. \tag{$*$}$$

We will be interested in the twisted cotangent stack corresponding to $\mathcal{L}_{\det}$. Namely we will prove the following.

Theorem 29. *There is a canonical isomorphism of stacks over* Coh*:*

$$\tilde{T}^*_{\mathcal{L}_{\det}} \operatorname{Coh} \cong \operatorname{Conn}^{coh}_{1/2} \tag{A.4}$$

where $\operatorname{Conn}^{coh}_{1/2}$ *is the moduli stack of* $\mathcal{O}_C$*-coherent modules over the algebra* $\mathcal{D}_{C,\omega^{1/2}}$ *of differential operators in* $\omega_C^{\otimes 1/2}$.

Denote $\tilde{T}^\bullet_{\operatorname{Coh},\mathcal{L}_{\det}}$ by $\tilde{T}^\bullet_{\operatorname{Coh}}$. Now it follows from the formula (A.3) that the datum of an R-point of $\tilde{T}^*_{\mathcal{L}_{\det}}\operatorname{Coh}$ lying over $x \in \operatorname{Coh}(R)$ is equivalent to the datum of a nilhomotopy of the map

$$\delta_x\colon T^\bullet_{\operatorname{Coh},x} \to R.$$

We can therefore reduce the study of $\tilde{T}^*_{\mathcal{L}_{\det}}\operatorname{Coh}$ to the study of δ_x.

It is known[2] that the tangent complex $\mathcal{T}^\bullet_{\mathrm{Coh},x}$ to Coh at a point $x \in \mathrm{Coh}(R)$ is canonically isomorphic to $(\mathrm{REnd}\,\mathcal{F}_x)[1] \in D^b(R\text{-mod})$. So we have to study the map $\delta_x[-1]\colon \mathrm{REnd}\,\mathcal{F}_x \to R$. Using the Serre duality, we get a map

$$\alpha_x\colon \mathcal{F}_x \to \mathcal{F}_x \otimes \omega_{C_R/R}[1].$$

The map α_x corresponds to an extension

$$0 \to \mathcal{F}_x \otimes \omega_{C_R/R} \to \Phi_s(\mathcal{F}_x) \to \mathcal{F}_x \to 0. \tag{A.5}$$

It is clear that, for any R and a map $s\colon \operatorname{Spec} R \to S$, the assignment $\mathcal{F}_x \mapsto \Phi_s(\mathcal{F}_x)$ defines a functor from the *groupoid* of R-flat coherent sheaves on C_R to itself compatible with base-change of R. In other words, we have a morphism of stacks $\Phi\colon\ \mathrm{Coh} \to \mathrm{Coh}$.

We will show that Φ_s extends to *non-invertible* morphisms of sheaves and, moreover, has the following description:

Proposition 30. *Let $\Delta_C^{\langle 2\rangle}$ be the 2nd infinitesimal neighborhood of the diagonal $\Delta_C \subset C \times C$ and $p, q\colon \Delta_C^{\langle 2\rangle} \rightrightarrows C$ the restriction of the two projections $C \times C \rightrightarrows C$. Denote also by p_R, q_R their base-change by $s\colon \operatorname{Spec} R \to S$. We have a canonical isomorphism*

$$\Phi_s(\mathcal{F}) \cong q_{R*}\big(p_R^*\mathcal{F} \otimes (s \times \mathrm{id}_{\Delta_C^{\langle 2\rangle}})^*(p^*\omega_C \otimes q^*\omega_C^{\otimes -1})^{\otimes 1/2}\big) \tag{A.6}$$

where $^{\otimes 1/2}$ denotes the canonical square root of the line bundle on $\Delta_C^{\langle 2\rangle}$ which is trivial on Δ_C. (It exists and is essentially unique due to the assumption $(*)$.)

Denote by $\tilde{\Phi}_s(\mathcal{F})$ the right-hand side of (A.6).

Proposition 30 implies Theorem 29. According to Remark 5, point 3, for any $x\colon \operatorname{Spec} R \to \mathrm{Coh}$ the elements of $\mathrm{Hom}_{\mathrm{Coh}}(\operatorname{Spec} R, \tilde{T}^*_{\mathcal{L}_{\det}}\mathrm{Coh})$ correspond bijectively to null-homotopies of the map δ_x. By the Serre duality isomorphism, these are the same as null-homotopies of α_x and therefore the same as splittings of (A.5). Now,

[2]At least for the open part of Coh parametrizing vector bundles; but the same proof works for all of Coh.

using (A.6), it is not hard to see that these splittings correspond to $(s \times \mathrm{id}_C)^* \mathcal{D}_{C,\omega^{1/2}}$-module structures on $\mathcal{F}_x$. $\square$

Lemma 31. *For any* $s\colon \operatorname{Spec} R \to S$ *the map of groupoids* $\Phi_s\colon \mathrm{Coh}(R) \to \mathrm{Coh}(R)$ *extends to a self-functor of the category of* R*-flat coherent sheaves on* $C \times_s \operatorname{Spec} R$. *In other words,* Φ_s *extends to non-invertible morphisms.*

Proof. Let $\mathrm{pr}_{1,2}\colon \mathrm{Coh} \times \mathrm{Coh} \rightrightarrows \mathrm{Coh}$ be the projections and $\Sigma\colon \mathrm{Coh} \times \mathrm{Coh} \to \mathrm{Coh}$ the map classifying direct sum of coherent sheaves. By the properties of determinant, we have a canonical isomorphism

$$\mathrm{pr}_1^* \mathcal{L}_{\det} \otimes \mathrm{pr}_2^* \mathcal{L}_{\det} \cong \Sigma^* \mathcal{L}_{\det}. \tag{A.7}$$

Applying Lemma 28 to $\mathcal{L}_{\det}$ and Σ yields a commutative diagram on the left:

$$\begin{array}{ccc} \mathcal{T}^\bullet_{\mathrm{Coh} \times \mathrm{Coh}} & \xrightarrow{\delta^{\Sigma^* \mathcal{L}_{\det}}} & \mathcal{O}_{\mathrm{Coh} \times \mathrm{Coh}}[1] \\ {\scriptstyle d\Sigma}\downarrow & & \Vert \\ \Sigma^* \mathcal{T}^\bullet_{\mathrm{Coh}} & \xrightarrow{\Sigma^* \delta^{\mathcal{L}_{\det}}} & \mathcal{O}_{\mathrm{Coh} \times \mathrm{Coh}}[1] \end{array} \qquad \boxed{\begin{array}{ccc} \mathrm{REnd}(\mathcal{F}_x) \oplus \mathrm{REnd}(\mathcal{F}_y) & \xrightarrow{\delta^{\Sigma^* \mathcal{L}_{\det}}_{(x,y)}} & R[1] \\ \downarrow & & \Vert \\ \mathrm{REnd}(\mathcal{F}_x \oplus \mathcal{F}_y) & \xrightarrow{\Sigma^* \delta^{\mathcal{L}_{\det}}_{\Sigma(x,y)}} & R[1] \end{array}}$$

Pulling it back under some $(x, y)\colon \operatorname{Spec} R \to \mathrm{Coh} \times \mathrm{Coh}$, we obtain the diagram on the right (boxed). From (A.7) we see that the top arrow of this diagram is equal (canonically homotopic) to $\delta_x^{\mathcal{L}_{\det}} p_1 + \delta_y^{\mathcal{L}_{\det}} p_2$ where p_1, p_2 are projections to the summands. Note also that the left vertical arrow is given by direct sum of (derived) endomorphisms.

Now if we apply the Serre duality to the boxed diagram, we get

$$\alpha_{\Sigma(x,y)} = \alpha_x \oplus \alpha_y\colon\ \mathcal{F}_x \oplus \mathcal{F}_y \longrightarrow (\mathcal{F}_x \oplus \mathcal{F}_y) \otimes \omega_{C_R/R}[1].$$

(Note that the 'equals' sign here again really means 'canonically homotopic.') In other words, we have an isomorphism

$$\Phi_s(\mathcal{F}_x \oplus \mathcal{F}_y) \cong \Phi_s(\mathcal{F}_x) \oplus \Phi_s(\mathcal{F}_y).$$

We already know that Φ_s is functorial with respect to isomorphisms, so $\Phi_s(\mathcal{F}_x \oplus \mathcal{F}_y)$ is acted on by the automorphism group of $\mathcal{F}_x \oplus \mathcal{F}_y$. Moreover, since Φ is compatible with base change, we have an action of the *group-scheme* over R given by $R' \mapsto \mathrm{Aut}((\mathcal{F}_x \oplus \mathcal{F}_y) \otimes_R R')$. In particular, consider the automorphism $a_f = \begin{pmatrix} \mathrm{id}_{\mathcal{F}_x} & 0 \\ f & \mathrm{id}_{\mathcal{F}_y} \end{pmatrix}$ for some $f\colon \mathcal{F}_x \to \mathcal{F}_y$. It can be shown that $\Phi(a_f)$ has the form $\begin{pmatrix} \mathrm{id}_{\Phi(\mathcal{F}_x)} & 0 \\ \tilde{f} & \mathrm{id}_{\Phi(\mathcal{F}_y)} \end{pmatrix}$. Now we let $\Phi(f) = \tilde{f}$. Using triple direct sums, one can prove that $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$ for any $g\colon \mathcal{F}_y \to \mathcal{F}_z$. □

Lemma 32. *Let $\mathcal{L}$ be a line bundle on* $\operatorname{Spec} R$ *and p an R-point of C. Let $\gamma_p = (p, \mathrm{id}_{\operatorname{Spec} R})\colon \operatorname{Spec} R \to C_R = C \times \operatorname{Spec} R$ be the embedding of the graph of p. Then Proposition 30 holds for $\mathcal{F} = \gamma_{p*}\mathcal{L}$.*

Proof. Let $x\colon \operatorname{Spec} R \to \mathrm{Coh}$ be the point corresponding to $\mathcal{F}$ and $\Gamma_p = \gamma_p(\operatorname{Spec} R) = \operatorname{supp} \mathcal{F} \subset C_R$. First we will show that the two extensions of $\mathcal{F} \otimes \omega$ by $\mathcal{F}$ in the two sides of (A.6) have the same class in $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega)$. Note that for the "relative skyscraper" as in the lemma, this R-module is identified with R. We claim that the classes of both extensions are equal to 1 under this identification. For the RHS of (A.6) we have pushforward of a line bundle on the 2nd infinitesimal neighborhood of Γ_p whose restriction to Γ_p is $\mathcal{L}$. Now the statement can be easily seen from the construction of the identification.

For the LHS we are interested in the image of α_x under the projection

$$\mathrm{RHom}(\mathcal{F}, \mathcal{F} \otimes \omega[1]) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega).$$

By Serre duality, this map is dual to $(\operatorname{End} \mathcal{F})[1] \cong H^{-1}(\mathcal{T}^\bullet_{\mathrm{Coh},x})[1] \to (\operatorname{REnd} \mathcal{F})[1] \cong \mathcal{T}^\bullet_{\mathrm{Coh},x}$. Therefore we have to study the restriction of δ_x to the (-1)st cohomology of $\mathcal{T}^\bullet_{\mathrm{Coh},x}$. This restriction is responsible for the action of infinitesimal automorphisms of x on $(\mathcal{L}_{\det})_x$. The group scheme of automorphisms of x is identified with $\mathbb{G}_{mR}$, and it acts on $(\mathcal{L}_{\det})_x$ via the tautological character. This means that δ_x restricted to $H^{-1}(\mathcal{T}^\bullet_{\mathrm{Coh},x})[1] \cong (\operatorname{End} \mathcal{F})[1] \cong R[1]$ is the identity. Now the statement about the class in Ext^1 follows from the fact that the Serre duality pairing of the canonical generators of $\operatorname{End} \mathcal{F}$ and $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega)$ is equal to 1.

Thus we showed that the two extensions in (A.6) are non-canonically isomorphic. The set of isomorphisms is a torsor for the group R which is compatible with base change. So we get a $\mathbb{G}_a$-torsor $\mathcal{A}$ on the open piece $\mathrm{Coh}^{[1]} \cong C \times B\mathbb{G}_m$ of Coh classifying length 1 sheaves, and we need to construct a canonical trivialization of $\mathcal{A}$.

Consider the involution σ on Coh given by Serre duality: $\mathcal{F} \mapsto \mathcal{F}^\vee \otimes \omega$. We have $\sigma^*\mathcal{L}_{\det} \cong \mathcal{L}_{\det}$. One can check that the isomorphism $\mathrm{RHom}(\mathcal{F}_x, \mathcal{F}_x \otimes \omega) \cong (\mathcal{T}^\bullet_{\mathrm{Coh},x})^\vee \xrightarrow{\sim} (\mathcal{T}^\bullet_{\mathrm{Coh},\sigma(x)})^\vee \cong \mathrm{RHom}(\mathcal{F}_{\sigma(x)}, \mathcal{F}_{\sigma(x)} \otimes \omega)$ is given by taking dual morphism. Hence the morphism δ_x is dual to $\delta_{\sigma(x)}$, and therefore $\Phi_s(\mathcal{F}_x) \cong (\Phi_s(\mathcal{F}_{\sigma(x)}))^\vee$.

Restricting to $\mathrm{Coh}^{[1]}$, for the torsor $\mathcal{A}$ we get $(\sigma|_{\mathrm{Coh}^{[1]}})^*\mathcal{A} \cong -\mathcal{A}$. On the other hand, any $\mathbb{G}_a$-torsor on $\mathrm{Coh}^{[1]} \cong C \times B\mathbb{G}_m$ descends to a torsor $\tilde{\mathcal{A}}$ on C. We get $\tilde{\mathcal{A}} \cong -\tilde{\mathcal{A}}$ and thus $\tilde{\mathcal{A}}$ is trivial due to $(*)$, so we are done.[3] □

Lemma 33. *Suppose x is an R-point of* Coh *corresponding to a line bundle $\mathcal{L}$ on C_R. Then Proposition 30 holds for x.*

Proof. Consider the S-scheme $S' := C_R$ and let $\Delta\colon C_R \to C_{S'} = C \times_S C \times_S \operatorname{Spec} R$ be the diagonal embedding. Let $\mathcal{L}_{S'}$ be the line bundle on $C_{S'}$ obtained from $\mathcal{L}$ by base-change along $S' \to \operatorname{Spec} R$. Now consider the map of coherent sheaves on $C_{S'}$:

$$f\colon \mathcal{L}_{S'} \to \Delta_*\mathcal{L} = \Delta_*\Delta^*\mathcal{L}_{S'}.$$

According to Lemma 31, we have a map

$$\Phi_{S'}(f)\colon\ \mathrm{pr}_1^*\,\Phi_s(\mathcal{L}) \to \Phi_{S'}(\Delta_*\mathcal{L})$$

(we used that Φ is compatible with base-change), and therefore, by adjunction, a map

$$g\colon \Phi_s(\mathcal{L}) \to \mathrm{pr}_{1*}\,\Phi_{S'}(\Delta_*\mathcal{L}).$$

By Lemma 32, we have $\Phi_{S'}(\Delta_*\mathcal{L}) \cong \tilde{\Phi}_{S'}(\Delta_*\mathcal{L})$. Also, from the definition of $\tilde{\Phi}$ one can see that $\mathrm{pr}_{1*}\,\tilde{\Phi}_{S'}(\Delta_*\mathcal{L}) \cong \tilde{\Phi}_s(\mathcal{L})$. So g is a map $\Phi_s(\mathcal{L}) \to \tilde{\Phi}_s(\mathcal{L})$. By construction,

[3] Actually, we will also need the statement that the isomorphism (A.6) in the case in question is compatible with morphisms of $\mathcal{L}$'s that are not necessarily isomorphisms.

g is a map between extensions of $\mathcal{L}$ by $\mathcal{L}\otimes\omega$, so it gives the desired isomorphism. $\square$

Proof of Proposition 30 (sketch). Locally on $\operatorname{Spec} R$, we can find a resolution of a coherent sheaf $\mathcal{F}$ on C_R by direct sums of line bundles. Any two such resolutions are related by a sequence of homotopy equivalences. Thus, due to the functoriality of Φ (Lemma 31), we can reduce Proposition 30 to Lemma 33. $\square$

A.3 The determinant bundle with connection on $\mathrm{Conn}_{1/2}^{coh}$

For a smooth stack $\mathscr{X}$ and a line bundle $\mathcal{L}$ on $\mathscr{X}$, the pullback of $\mathcal{L}$ to $\tilde{T}^*_{\mathcal{L}}\mathscr{X}$ carries a canonical connection. Applying this observation to $\mathscr{X} = \mathrm{Coh}(C)$ and $\mathcal{L} = \mathcal{L}_{\det}$, and taking Theorem 29 into account, we get a connection $\nabla_{\det}$ on the pullback $\mathcal{L}'_{\det}$ of $\mathcal{L}_{\det}$ to $\mathrm{Conn}_{1/2}^{coh}$. In this subsection we state some properties of $\mathcal{L}_{\det}$ and $\nabla_{\det}$.

Suppose we have an S-scheme S', and a short exact sequence of S'-families of coherent $\mathcal{D}_{\omega^{1/2}}$-modules on C, i.e., of S'-flat coherent sheaves on $C_{S'}$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{A.8}$$

with compatible (relative over S') $\omega^{1/2}$-connections. Let $x, x', x'' \in \mathrm{Coh}(C)(S')$ be the corresponding points of Coh. Then we have the following relation between the pullbacks of the determinant bundle:

$$(\mathcal{L}_{\det})_x \cong (\mathcal{L}_{\det})_{x'} \otimes (\mathcal{L}_{\det})_{x''}. \tag{A.9}$$

Now from the connection $\nabla_{\det}$ we get connections (over S) on both sides of the above isomorphism.

Lemma 34. *The isomorphism* (A.9) *is compatible with connections.*

Proof. We first reduce the statement to the case where the exact sequence (A.8) splits. Namely, note that for a general exact sequence we can construct a family of coherent sheaves with $\omega^{1/2}$-connections on C parametrized by $S' \times \mathbb{A}^1/\mathbb{G}_m$ whose restriction to $C_{S'} \xrightarrow{\sim} C_{S'} \times (\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m \hookrightarrow C_{S'} \times \mathbb{A}^1/\mathbb{G}_m$ is isomorphic to $\mathcal{F}$ and whose restriction to $C_{S'} \times \{0\} \hookrightarrow C_{S'} \times \mathbb{A}^1 \to C_{S'} \times \mathbb{A}^1/\mathbb{G}_m$ is isomorphic to $\mathcal{F}' \oplus \mathcal{F}''$. Denote by $\tilde{x}$ the corresponding map $S' \times (\mathbb{A}^1/\mathbb{G}_m) \to \mathrm{Conn}^{coh}_{1/2}$. Pulling back $(\mathcal{L}'_{\det}, \nabla_{\det})$ under $\tilde{x}$ we get a line bundle with connection on $S' \times (\mathbb{A}^1/\mathbb{G}_m)$ over $\mathbb{A}^1/\mathbb{G}_m$. We want to compare this connection with the one obtained as direct sum of pullbacks by x' and x''. The difference is an $\mathbb{A}^1/\mathbb{G}_m$-family of one-forms on S'/S. But any such family is necessarily constant. So it is enough to prove that it is 0 on $\{0\}/\mathbb{G}_m$ where the exact sequence splits.

For split exact sequences, the statement is equivalent to compatibility of the isomorphism (A.4) with symmetric monoidal structure on both sides given by direct sum. □

A.3.1 The case of characteristic p

Now suppose that S is a scheme in characteristic p, i.e., for a prime p we have $p\mathcal{O}_S = 0$. Then there is a component $\mathscr{I}$ of $\mathrm{Conn}^{coh}_{1/2}$ classifying irreducible $\mathcal{D}$-modules. We identify $\mathrm{Conn}^{coh}_{1/2}$ with Conn^{coh} by $\cdot \otimes \omega^{\otimes(p-1)/2}$. Recall the Azumaya algebra $\tilde{\mathcal{D}}_{C/S}$ on $T^*(C^{(S)}/S)$ whose pushforward to $C^{(S)}$ is isomorphic to $\mathrm{Fr}_{C/S*}\,\mathcal{D}_{C/S}$. For an S-scheme S', the S'-points of $\mathscr{I}$ correspond to pairs $(y, \mathcal{E})$ where y is an S'-point of $T^*C^{(S)}$, and $\mathcal{E}$ is a splitting of $y^*\tilde{\mathcal{D}}_{C/S}$. In other words, $\mathscr{I}$ is isomorphic to (the "total space" of) the $\mathbb{G}_m$-gerbe $\mathcal{G}_\theta$ on $T^*C^{(1)}$ where θ is the canonical 1-form on T^*C. We want to describe the restriction of $(\mathcal{L}'_{\det}, \nabla_{\det})$ to $\mathscr{I}$.

Because of the equivalence $\tilde{\mathcal{D}}_C$-mod $\sim \mathcal{D}_{T^*C,\theta}$-mod, the gerbe $\mathcal{G}_\theta$ also classifies the irreducible modules over $\mathcal{D}_{T^*C,\theta}$. Therefore we can consider the universal object: this is a coherent sheaf on $\mathscr{I} \times T^*C$ with connection in the T^*C-direction and with support given by $\mathscr{I} \times_{T^*C^{(1)}} T^*C$. Now if we apply the relative Frobenius twist over $\mathscr{I}$ to this sheaf, the resulting sheaf on $\mathscr{I} \times T^*C^{(1)}$ will have connection along both factors. So we get a $\mathcal{D}$-module on $\mathscr{I} \times T^*C^{(1)}$ supported on the Frobenius neighborhood of the "diagonal" or, more precisely, of the graph of $v\colon \mathscr{I} \to T^*C^{(1)}$. The restriction of this $\mathcal{D}$-module to the graph itself is a line bundle on $\mathscr{I}$ with connection which we will denote by $(\mathcal{L}_{\mathrm{univ}}, \nabla_{\mathrm{univ}})$.

Proposition 35. *There is a canonical isomorphism of line bundles with connections on* $\mathscr{I}$*:*

$$(\mathcal{L}'_{\mathrm{det}}, \nabla_{\mathrm{det}}) \cong (\mathcal{L}_{\mathrm{univ}}, \nabla_{\mathrm{univ}} - v^*\theta) \tag{A.10}$$

Bibliography

[A] D. Arinkin, Appendix to [DP]

[BD] A. Beilinson, V. Drinfeld, *Quantization of Hitchin integrable system and Hecke eigen-sheaves*, preprint, available at `http://www.math.uchicago.edu/~mitya/langlands.html`

[BB] R. Bezrukavnikov, A. Braverman, *Geometric Langlands for $\mathcal{D}$-modules in prime characteristic: the $GL(n)$ case*, `arXiv:math/0602255`

[BMR] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semi-simple Lie algebra in prime characteristic*, `arXiv:math.RT/0205144`

[DP] R. Donagi, T. Pantev, *Torus fibrations, gerbes, and duality*, preprint, `arXiv:math.AG/0306213`

[Gr] M. Gröchenig, *Moduli of flat connections in positive characteristic*, preprint, `arXiv:1201.0741`