Problem Set #2 Solutions

Course 14.454 – Macro IV

Distributed: November 9, 2004

Due: Tuesday, November 23, 2004 [in class]

1. **Financial Constraints (via Costly State Verification)**

Consider an economy composed of entrepreneurs and outside investors. Both types are risk neutral and can always invest their wealth in outside capital markets and earn an expected gross return $\bar{R}$. Each entrepreneur has wealth w, where w is distributed uniformly between zero and two among the entrepreneurs. Each entrepreneur also has the option to undertake a project that requires an indivisible investment of 1 and has an i.i.d. return of $x \sim U[0, 2\bar{x}]$.

Outside investors have lots of wealth but no access to projects. They are willing to lend money to entrepreneurs if their expected return from lending money is not exceeded by the return to their outside option. However, outside investors cannot verify the returns from the project unless they pay a fixed cost c.

Assume that the contract between the investor and entrepreneur takes the form of a debt contract: the entrepreneur pays a return D to the outside investor whenever he can do so. When he cannot afford to pay, the outside investor pays the verification cost c and takes all the profits. That is:

If $x \geq D$, the investor gets D and the entrepreneur gets $x - D$
If $x < D$, the investor gets $x - c$ (which may be negative) and the entrepreneur gets 0

There is no bargaining in this model. Investors are perfectly competitive, so the entrepreneurs will never offer more than necessary to get the financing $(1-w)$ that they need to do the project.

(a) Assume that the entrepreneur is willing to undertake the project, and analyze the problem from the point of view of the outside investor.

i. First, find the investor's expected gain if she invests in the project. What are the expected verification costs of the investor?

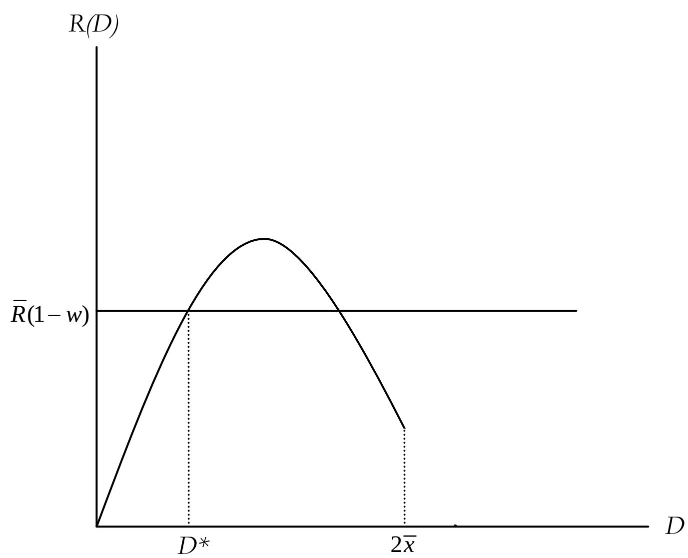
Expected Return, R(D) is given by:

$$R(D) = \int_D^{2\bar{x}} \frac{D}{2\bar{x}} dx + \int_0^D \frac{x-c}{2\bar{x}} dx$$
$$R(D) = \left(D - \frac{D^2}{2\bar{x}} \right) + \left(\frac{D^2}{4\bar{x}} - \frac{cD}{2\bar{x}} \right)$$
$$R(D) = D - \frac{D^2}{4\bar{x}} - \frac{cD}{2\bar{x}}$$

Notice that a higher debt payment D has a number of effects. A higher D increases the return to the investor in good states but also reduces the probability of the entrepreneur being able to repay her debt. Thus, a higher D also increases the expected verification costs, $cD/2\bar{x}$, of the investor.

ii. Graph this expected return as a function of D and show graphically how the equilibrium value D^* will be chosen.

$R(D)$ is a parabola that has x-intercepts at $D = 0$ and $D = 2(2\bar{x} - c)$ as shown below. The equilibrium value D^* will be chosen such that the expected return of the investor just equals her opportunity costs. i.e. $R(D^*) = \bar{R}(1-w)$. As shown in the graph, this can occur at two points, but we know that competition among the investors will ensure that the lowest D^* will be chosen. Why is this true? Well, we know that at any $D > D^*$, another investor will find it optimal to underbid the other investors by lowering the D he requests from the entrepreneur. This underbidding will continue until we hit D^* and investors are just indifferent between lending and not lending.



iii. What is the sign of the derivative $\partial D^*/\partial w$? Interpret

It is easy to see from our graph that the derivative is negative. Thus, the larger amount of wealth posted by the entrepreneur, the smaller amount he will have to pay back the investor via D^*.

iv. Under what circumstances will there be no lending?

There will be no lending in this model, if there doesn't exist a D such that $R(D) = \bar{R}(1-w)$. This will happen if $\bar{R}(1-w) > R(D)$ for all D. Three potential causes could be:

1. The wealth of the borrower, w, is very low.
2. The outside opportunity cost of funds $\bar{R}$ is very high.
3. Verification costs, c, are very high.
4. Or, the expected return, $\bar{x}$, is low.

(b) Taking D^* as given (as seen in part (a, ii), it is a function of the model parameters).

i. Write down the condition under which the entrepreneur is willing to undertake the project. Call this the entrepreneurs [IR] constraint. Don't substitute D^* out of your equation.

Taking D^* as given, the expected return to the entrepreneur is:

$$\int_{D^*}^{2\bar{x}} \frac{x - D^*}{2\bar{x}} dx$$
$$\left(\frac{x^2 - 2D^* x}{4\bar{x}} \right) \Bigg|_{D^*}^{2\bar{x}}$$
$$\bar{x} - D^* + \frac{D^{*2}}{4\bar{x}}$$

The entrepreneur will take the project if this expected return exceeds his opportunity cost of $\bar{R}w$.

$$\bar{x} - D^* + \frac{D^{*2}}{4\bar{x}} \geq \bar{R}w$$

ii. Now use the equilibrium condition for D^* found in part (a,ii) to express the entrepreneur's [IR] constraint in terms of $\bar{x}$, the expected return of the project, $\bar{R}$, the outside return, and the expected cost of verification for the bank found in part (a,i).

The equilibrium condition for D^* found in part (a,ii) is:

$$D^* - \frac{D^{*2}}{4\bar{x}} - \frac{cD^*}{2\bar{x}} = \bar{R}(1-w)$$
$$-D^* + \frac{D^{*2}}{4\bar{x}} = -\bar{R}(1-w) - \frac{cD^*}{2\bar{x}}$$

Plugging this into the entrepreneurs [IR] constraint, we have:

$$\bar{x} - \bar{R} - \frac{cD^*}{2\bar{x}} \geq 0$$
$$\bar{x} \geq \bar{R} + \frac{cD^*}{2\bar{x}}$$

Assuming that the investor is willing to lend, his or her expected return must exceed the outside opportunity cost and the expected cost of monitoring in order for the project to be undertaken.

(c) Using your answer from part (b), show that there are projects implemented in an efficient economy that are not implemented here. Which

entrepreneurs will not be able to start a project in the economy with positive verification costs?

In the efficient economy, the projects will always be undertaken if their expected return exceeds the outside opportunity cost, i.e. $\bar{x} \geq \bar{R}$.

In the economy with verification costs, however, an entrepreneur will only invest if and only if:

$$\bar{x} \geq \bar{R} + \frac{cD^*}{2\bar{x}}$$

Now, the expected return of the project must exceed the outside opportunity cost by the amount the entrepreneur expects he'll have to pay the bank for its expected verification costs, $cD^* / 2\bar{x}$. From earlier, we know that D^* is decreasing in wealth. Therefore, there will now exist a cutoff point where individuals with insufficient wealth will be unable to take the project now. [**Note**: I've implicitly assumed $\bar{x}, c$ are such that investors are willing to lend for any given w. Technically, we also have to check that the lenders actually want to lend for all these w.] If $c = 0$, we return to the result of the efficient economy, and the wealth of the entrepreneurs will not matter for whether a project is undertaken or not.

2. Amplification and Persistence (via Kiyotaki and Moore)

Consider an economy with two types of agents: farmers and gatherers. There is a continuum 1 of each type. There are also two goods: an ordinary nondurable product (fruit) and a durable productive asset (land). The total supply of land is equal to $\bar{K}$.

The farmer has constant returns to scale technology: he uses k_t units of time t land to produce ak_t units of time $t+1$ fruit. The farmer is also is also subject to the flow of funds constraint, which implies his investment expenditure is financed by his output and net borrowing:

$$q_t\left(k_t - k_{t-1}\right) + Rb_{t-1} = ak_{t-1} + b_t \tag{1}$$

where R is one plus the real interest rate, q_{t+1} is the land price in terms of fruit at time $t+1$, and b_t is the value of debt undertaken at time t .

For simplicity, you should assume that each farmer is *always* eager to expand (due to their great enjoyment of farming), but faces the following credit constraint:

$$Rb_t \leq q_{t+1}k_t \tag{2}$$

(a) Combine equations (1) **and** (2) **to prove the following condition:**

$$k_t = \frac{1}{q_t - \frac{q_{t+1}}{R}}\left[\left(a + q_t\right)k_{t-1} - Rb_{t-1}\right] \tag{3}$$

This is straightforward. The credit constraint will bind exactly by our assumption that the farmers always want to borrow more. So plugging into (1) using

$$b_t = \frac{q_{t+1}k_t}{R}$$

We then have:

$$q_t\left(k_t - k_{t-1}\right) + Rb_{t-1} = ak_{t-1} + \frac{q_{t+1}k_t}{R}$$

With a little rearranging, we get equation (3).

i) Why can we interpret $\mu = q_t - q_{t+1}/R$ as the amount of down payment necessary per unit of capital purchased?

Farmers always want to buy as much capital as possible by our assumptions. Hence they will borrow the maximum amount they can to do so. q_t is the price per unit of capital they must pay, and q_{t+1}/R is the maximum amount they can borrow per unit of capital purchased (because it is the most they can promise to pay back tomorrow). Thus, $\mu = q_t - q_{t+1}/R$ is the amount left over per unit of capital purchased that the farmer must pay for directly. i.e. it is his/her down payment.

ii) How do we interpret the expression inside the bracket?

The expression inside the bracket is the farmer's net worth. $\left(a + q_t\right)k_{t-1}$ is the amount of fruit the farmer receives at time t. He receives a return a for each unit of land he has, k_{t-1}, and he receives a price q_t for selling the land at time t. (Implicitly, the farmers all sell their existing land in each period of time before purchasing new land.) He must then pay back the Rb_{t-1} he borrowed last period.

iii) Why is $\partial k_t / \partial a$ positive?

This derivative is positive because an increase in a increases the farmer's net worth, and he or she uses this increase to purchase more land.

(b) Consider equation (3)**, suppose q_t and q_{t+1} increase by 1%. Explain how this changes the necessary down payment and net worth of the farmer and how each change impacts farmers' land demand. Which effect is stronger when $ak_{t-1} < Rb_{t-1}$?**

This clearly increases the farmer's down payment by 1% also.

$$(1.01)q_t - (1.01)q_{t+1}/R = 1.01\mu$$

The net worth of the farmer increases since he now receives a higher price for the land he has.

The increase in the necessary down payment causes the farmer to purchase less land. The is easy to see in equation (3). However, the increase in net worth induces the farmer to purchase more land. The two effects work in opposite directions.

Plugging into equation (3), we have:

$$k_t^* = \frac{1}{(1.01)q_t - \frac{(1.01)q_{t+1}}{R}}\left[\left(a + (1.01)q_t\right)k_{t-1} - Rb_{t-1}\right]$$

$$k_t^* = \frac{q_t k_{t-1}}{q_t - \frac{q_{t+1}}{R}} + \frac{1}{(1.01)\left(q_t - \frac{q_{t+1}}{R}\right)}\left[ak_{t-1} - Rb_{t-1}\right] > k_t$$

when $ak_{t-1} < Rb_{t-1}$, the new amount of land, k_t^*, is greater. i.e. the increased demand because of higher net worth exceeds the decrease in demand because of the higher necessary down payment.

Now consider the gatherers' who use a decreasing returns to scale technology, such that $k_t^{'}$ units of time t land to produce $G(k_t^{'})$ units of time $t+1$ fruit. They do not face any borrowing constraint and will maximizes the expected discounted consumption of fruit with discount factor $1/R < 1$. Land market equilibrium implies $k_t + k_t^{'} = \bar{K}$.

(c) Use the land market equilibrium condition and the FOC of the gatherer's maximization problem to prove the following market clearing condition:

$$q_t - \frac{1}{R}q_{t+1} = \frac{1}{R}G'\left(\bar{K} - k_t\right)$$

The gatherer's maximization problem is:

$$\max_{k_t^{'}} \frac{1}{R}\left[q_{t+1}k_t^{'} + G\left(k_t^{'}\right)\right] - q_t k_t^{'}$$

The gatherer expects to receive a return of $G\left(k_t^{'}\right)$ from his land next period along with the revenues of selling it, $q_{t+1}k_t^{'}$ next period. He discounts this future return by 1/R, and subtracts off the cost of the land, $q_t k_t^{'}$. Taking the FOC, we have:

$$\frac{1}{R}q_{t+1} + \frac{1}{R}G'\left(k_t^{'}\right) - q_t = 0$$

Rearranging and plugging in for $k_t^{'} = \bar{K} - k_t$, we have our solution.

i) What is the sign of $\partial G'/\partial k_t$? Explain

Since $G' > 0$, it is easy to see that an increase in the farmer's use of land (higher k_t), reduces the use of land by gatherers and increases the marginal product of their land. i.e. $\partial G'/\partial k_t > 0$

ii) Given this condition and holding future prices constant, how will today's land prices respond to an increase in k_t? (No math)

q_t will rise because of the increased demand from farmers. This comes directly out of this condition.

(d) Now let's put all the pieces together and analyze the impact of a one-time, temporary, upward shock to the productivity of farmers, a, at time t. Just give 1-2 sentence explanations for each part below.

i) Describe the direct impact on land demanded by farmers at time t.

The immediate impact is an increase in the farmer's net worth. This allows farmers to increase their demand for land.

ii) How does the demand change affect the price of land and cause amplification?

The increased demand by farmers causes the price to rise using our market clearing condition found in part (c). This can cause amplification because this increases farmer's net worth by raising the value of his existing stock of land. This additional collateral allows him to borrow and purchase more land at time t. (Note: I'm assuming that the positive impact from the increase in net worth exceeds the negative impact from the higher down payment that is now necessary. In equilibrium this will actually be true.)

iii) Why does the shock persist and affect farmer's net worth and demand for land tomorrow (after the shock is gone)?

When farmers have more land today, they will produce and sell more land tomorrow. This implies persistence because their net worth tomorrow is also higher than it otherwise would have been, and they will be able to borrow more and buy more land than if the shock had never happened.

iv) Why do these future impacts further amplify the shock today?

Well, if demand for land is higher tomorrow, our market clearing condition ensures that the price of land tomorrow is also higher. This relaxes the farmer's borrowing constraint today further amplifying the initial impact of the shock!

3. Banks and Bank Runs (via Diamond and Dybvig)

Assume there is a continuum 1 of individuals that are each endowed with one unit of currency. There are three time periods, $t = 0,1,2$. At $t = 0$, individuals have two options with regards to how they can invest their money. They can either stuff it in their mattress, where it gets a return equal to 1, or they can invest it in a long-term project that yields a return $R = 4$ in period two. For example, in individual that invests an amount I will receive $4I$ in period two, and have $1 - I$ stuffed under the mattress. However, individuals always have the option of withdrawing their money from the long-term project early in period one at a penalty. If they withdraw early, they only receive a return $L = 1/4$ in period 1, rather than the return $R = 4$ in period 2.

At time $t=1$, a fraction $\pi = 1/2$ of the individuals receive a liquidity shock. These individuals are "impatient" and only value consumption in period one. The fraction $1-\pi$ individuals that do not receive a liquidity shock are "patient" and only value consumption in period two. At time $t=0$, each individual has an equal chance of being hit by the liquidity shock. Assume that individuals do not discount the future, so that their ex-ante expected utility is given by, $U = \pi u(c_1) + (1-\pi)u(c_2)$, where c_1 and c_2 is the consumption period 1 and 2 respectively, and $u(c) = -1/c$.

(a) Assume there are no markets available to individuals, so that individuals must simply invest on their own. Given that the individual has invested an amount I at time $t=0$, what will be the optimal levels of consumption, c_1, c_2, if:

i) the individual receives a liquidity shock (i.e. is impatient)

$$c_1^1 = 1 - I + LI = 1 - (3/4)I$$
$$c_2^1 = 0$$

ii) the individual does not receive a liquidity shock (i.e. is patient)

$$c_1^2 = 0$$
$$c_2^2 = 1 - I + RI = 1 + 3I$$

(b) What is the optimal level of investment, I^*? Given I^*, what is the ex-ante expected utility of an individual? Explain in 1-2 sentences why both patient and impatient individuals regret their initial investment decision ex-post in period 1 after their type is realized.

The individual's maximization problem is given by:

$$\max_I \ \pi u\left(c_1^1\right) + (1-\pi)u\left(c_2^2\right)$$
$$\max_I \ -\left(\frac{1}{2}\right)\left(1-(3/4)I\right)^{-1} - \left(\frac{1}{2}\right)\left(1+3I\right)^{-1}$$

The FOC is thus,

$$-\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(1-(3/4)I\right)^{-2} + \left(\frac{1}{2}\right)3\left(1+3I\right)^{-2} = 0$$
$$\left(1+3I\right)^{-2} = \left(\frac{1}{4}\right)\left(1-(3/4)I\right)^{-2}$$
$$1+3I = 2-(3/2)I$$
$$I^* = 2/9$$

Thus, the ex-ante expected utility is given by:

$$U = -\left(\frac{1}{2}\right)\left(1-(1/6)\right)^{-1} - \left(\frac{1}{2}\right)\left(1+2/3\right)^{-1}$$
$$U = -\left(\frac{1}{2}\right)\frac{6}{5} - \left(\frac{1}{2}\right)\frac{3}{5}$$
$$U = -\frac{9}{10}$$

Both types of individuals are ex-post unhappy. The patient individuals will have wished they saved everything. This would give a payoff of 4 in period 2 and a utility of $-1/4$ which is better than his current utility of $-3/5$. The impatient will have wished they saved nothing, allowing them to consume 1 in period 1 and get a utility of -1 which is better than his current utility of $-6/5$.

(c) Now suppose an ex-post financial market exists where individuals can trade bonds at time $t=1$. Each bond costs p units of goods at time $t=1$, and the bond pays 1 unit of goods at time $t=2$. Assume all individuals invest an initial amount $I=1/2$.

i) What is the aggregate demand and supply of bonds at $t=1$?

$$\text{Aggregate demand} = \begin{cases} (1-\pi)\left(\dfrac{1-I}{p}\right) & \text{if } p<1 \\ \left[(1-\pi)(1-I),0\right] & \text{if } p=1 \\ 0 & \text{if } p>1 \end{cases}$$

$$\text{Aggregate supply} = \begin{cases} 0 & \text{if } p<L \\ \left[0,\pi RI\right] & \text{if } p=L \\ \pi RI & \text{if } p>L \end{cases}$$

ii) What is the equilibrium price p?

From part (i), we see that demand = supply when $p=1/4$.

iii) How much do "impatient" individuals consume in each period?

$$c_1^1 = 1-I+pRI = 1$$
$$c_2^1 = 0$$

iv) How much does a "patient' individual consume in each period?

$$c_1^2 = 0$$
$$c_2^2 = \frac{1-I}{p} + RI = R$$

(d) When ex-post financial markets exist, what is the ex-ante expected utility of individuals? Compare this with part (b). Are individuals better off? And, do individuals now have any regrets about their initial investment decision?

The expected ex-ante utility is given by:

$$\pi u\left(c_1^1\right)+(1-\pi)u\left(c_2^2\right)$$
$$-\frac{1}{2}(1)-\frac{1}{2}\left(\frac{1}{4}\right)=-\frac{5}{8}$$

Comparing this to part (b), it is clear individuals are better off now. Moreover, we see that individuals do not have any ex-post regrets.

(e) Now suppose that when types are revealed in period 1, this information is publicly observable. Suppose there exists a social planner that individual's entrust all of their endowment to at time 0. The social planner will pay impatient individuals c_1* in period 1 and patient individuals c_2* in period 2.

i) Solving the social planner's problem, what is c_1* and c_2*?

The social planner's problem is given by the following:

$$\max_{c_1, c_2} \ \pi u\left(c_1\right) + (1-\pi)u\left(c_2\right) \quad \text{s.t. } \pi c_1 + (1-\pi)\left(\frac{1}{R}\right)c_2 = 1$$

Using a Langrangian (with multiplier λ) the FOCs are as follows:

$$c_1^{-2} = \lambda$$
$$Rc_2^{-2} = \lambda$$

Thus, we have,

$$c_2 = R^{\frac{1}{2}}c_1$$
$$c_2 = 2c_1$$

Plugging this back into the budget constraint, we solve for the solution:

$$c_1^* = 4/3$$
$$c_2^* = 8/3$$

ii) How much does the social planner invest? (i.e. what is I?)

The social planner simply saves $(1-\pi)\left(\frac{1}{R}\right)c_2 = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{8}{3}\right) = 1/3$

iii) What is an individual's ex-ante expected utility now?

$$U = \pi u\left(c_1\right) + (1-\pi)u\left(c_2\right)$$
$$U = \left(\frac{1}{2}\right)\left(-\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(-\frac{3}{8}\right) = -\frac{9}{16}$$

iv) Why is the social planner able to improve the individual's ex-ante utility relative to that found in part (d)?

The social planner improves welfare because it is able to provide insurance to the individuals. It is always the case that individuals without a liquidity shock (i.e. patient) are better off than those that realize a liquidity shock at time 1. The social planner just provides insurance against the shock.

(f) Now suppose an agent's type is private information, and the social planner can only offer a contract contingent only an individual's announcement of his or her type at time 1. (i.e. she cannot condition the contract on other agents' announcements). Furthermore, at time 1, she meets each agent once with the meeting order randomly determined. If individual's report honestly, can the social planner offer the same contract as in part (e)? Is it optimal for an individual to report honestly when everyone else does? Explain in 1-2 sentences how this planner can be interpreted as a bank.

If individual's report honestly, than the social planner can offer the same contract. It is easy to see that individuals will always want to be honest. An impatient person would never report as patient since that would yield $-\infty$ utility. A patient person would never report as impatient since this would entail a drop in consumption from 8/3 to 4/3.

The interpretation of the planner as a bank with a "sequential service" constraint is straightforward. The bank takes deposits at time 1, and it promises to payout a small return at time 1 for those that need to withdraw their funds early. Those that withdraw at time 2 get a larger return. It does this by keeping some funds on hand for 'impatient' individuals and by investing the rest.

(g) Suppose all agents fear a bank run, and each agent reports to the bank at time 1 as being impatient. How many individuals will get paid by the bank before it runs out of money in period 1? Given this, explain why this bank run can be an equilibrium… i.e. why is it optimal for a "patient" individual to run on the bank when he/she expects a bank run?

If everyone reports, the bank must convert the $I = 1/3$ it invested in the illiquid long-term project at time 1. Because of this, its return on these funds will only be 1/12. Combining this with the $1 - I = 2/3$ that the bank kept in liquid assets, it has a total of 3/4 assets to pay out in period 1. Dividing this by the promised first period payoff of 4/3, we see that only the first 9/16 of individuals making claims in period 1 get paid before the bank runs out of cash to make payments.

It is now clear why a bank run is an equilibrium. When a patient individual expects a bank run, his expected utility of waiting to withdraw in period 2 is negative infinity (zero consumption) because the bank will be bankrupt by then. Therefore, he or she is better off running on the bank also where he or she has a probability 9/16 of getting a payout of 4/3 before the bank collapses.

(h) Suppose the bank implements a policy of only paying the first π individuals that show up at time 1, and the rest will get paid at time 2. (i.e. it suspends convertibility). Will this eliminate the bank run as an equilibrium?

Yes, this will eliminate the bank run equilibrium. A patient person will never have an incentive to run because he knows the bank will never collapse and hence the bank-run equilibrium is no longer sustainable.