| **6.972: Game Theory** | March 28, 2005 |
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# Lecture 14

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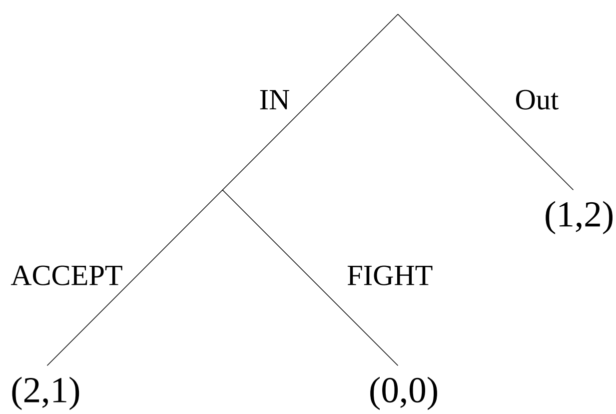
## 1 Agenda

In this lecture, we discuss:

- Ultimatum game
- Rubinstein-Stahl bargaining model
- Finite and infinite horizon

## 2 Extensive Game Model




| | Accord | Fight |
|---|---|---|
| In | 2,1 | 0,0 |
| Out | 1,2 | 1,2 |

- Strategies: $s_i^k : H^K \to A_i(H^K)$
- Payoffs over terminal histories : $u_i(s)$
- NE $\to$ some may be sustained by *incredible* (uncredible) threats

## 3 Subgame Perfect Equilibrium

$s$ is an SPE iff no player $i$ can gain by deviating from a $s$ in a single stage and conforming to $s$ thereafter. We proved this for the finite horizon case in last lecture.

**Idea:** Inductive, if a strategy satisfies the one stage deviation principle then that strategy cannot be improved upon in a finite number of deviations.

$\Rightarrow$ A player may gain by an $\infty$ seq of deviations, therefore we exclude these type of games by some "continuous at $\infty$" condition

**Definition 1** *Consider an extensive form game with $\infty$ horizon denoted by some $G^\infty$. Let $h$ denote an $\infty$-horizon history,*

$$h = (a^0, a^1, a^2 ...) : (\infty \text{ seq of actions})$$
$$\text{Let } h^t = (a^0, ...a^{t-1})\text{: restriction to first } t \text{ periods}$$
$$G^\infty \text{ is cont at } \infty \text{ if } \forall i, u^i \text{ satisfies:}$$
$$\sup_{t\to\infty} |u_i(h) - u_i(\tilde{h})| \to 0$$
$$h, \tilde{h} | h^t = \tilde{h}^t$$

**Remark:** Continuity at $\infty$ is satisfied if overall payoffs are a discounted sum of payoffs

$$u_i = \sum_{t=0}^{\infty} \delta_i^t g_i^t(a^t) \text{ : stage payoffs} \delta | i < 1$$
$$\text{and}$$
$$\max_{t,a^t} |g_i^t(a^t)| < \beta \text{ : uniformly bounded}$$

**Theorem 1** *Consider an infinite-horizon game, $G^\infty$, that is continuous at $\infty$. Then one stage deviation principle holds:*

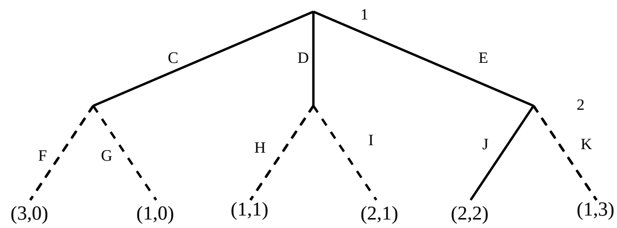
$$s^* \text{ is an SPE iff } \forall i, h^t, t$$
$$u_i(s_i^*, s_{-i}^* | h^t) \leq u_i(s_i, s_{-i}^* | h^t)$$
$$\forall s_i(h^t) \neq s_i^*(h^t)$$
$$s_i | h^t(h^{t+k}) = s_i^* | h^t(h^{t+k})$$
$$\forall h^{t+k} \in G(h^t), \forall k > 0$$

## 4 Backward Induction

Consider subgames of length 1, find optimal actions of players. Start at the last period of the game, and work backward.

$\Rightarrow$ The strategy constructed is an SPE

**Question:** What happens if in some subgame more than one action is optimal? Consider all optimal actions and trace back implications of each in all of the longer subgames.
Player 2's optimal strategies (all strategy choices not involving J):




| | |
|---|---|
| FHK | FIK |
| GHK | GIK |

Now consider player 1's optimal strategies for every combination of optimal actions for player 2:

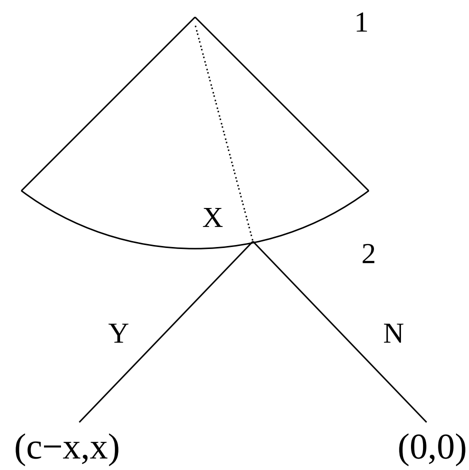
| 2's opt. strt. | | 1's BR | 6 SPE's |
|---|---|---|---|
| | | | (C, FHK) |
| FHK | → | C | (C, FIK) |
| FIK | → | C | (C, GHK) |
| GHK | → | C,D,E | (D, GHK) |
| GIK | → | D | (E, GHK) |
| | | | (D, GIK) |

Bargaining over the division of a pie is naturally modeled as an extensive game

## 5 Ultimatum Game:

The ultimatum game is a simple game that is the basis of a richer model. Two people use the following procedure to split \$$c$:

- 1 offers 2 some amount $x \leq c$
- if 2 accepts the outcome is: $(c - x, x)$
- if 2 rejects the outcome is: $(0, 0)$
- Note: each person cares about the amount of money he receives – $x$ can be any number, not necessarily integral




**Question:** What is an SPE for this game?

Let's use an extensive game model for the negotiation process: It is a finite horizon game, so we can use backward induction. There is a different possible subgame for each value of $x$, so we just need to find the optimal action of player 2 for each such subgame:

- if $x > 0 \rightarrow$ Yes
- $x = 0 \rightarrow$ indifferent between Yes and No

How many different optimal strategies does player 2 have?

1. Yes $\forall x >= 0$
2. Yes if $x > 0$ – No if $x = 0$

Trace back the implications of each of player 2's optimal strategies, i.e., consider player 1's optimal strategy for each of these strategies:

- For (1): player 1's optimal offer is $x = 0$
- For (2): player 1's optimal offer is:
  $x = 0 \rightarrow 0$
  $x > 0 \rightarrow c - x \quad \max_{x>0}(c - x) \Rightarrow$ no optimal solution
  $\Rightarrow$ no offer of person 1 is optimal!
- Unique SPE:
  - 1 offers 0
  - person 2 accepts all offers
  - Outcome: $(0, y) \Rightarrow$ 1 gets all the pie

**Remarks**

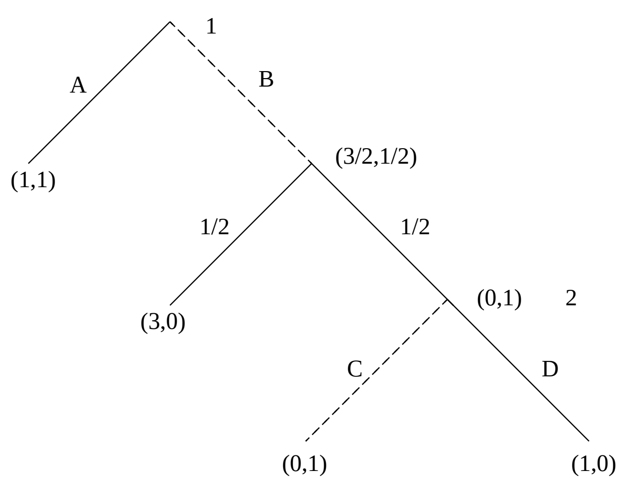
1. One-sided outcome → one-sided structure of the game should allow 2 to make a counter offer after rejection then it is more like bargaining
2. This SPE is not supported by experimental evidence (culture, equity-conscious players)
   - Roth et. al. (91) AER
   - Hoffman et. al. (98)
   - behavior exhibits some sort of concern for fair outcomes or reciprocity

**Questions:**

1. What if amount of money available is in multiples of a cent? Then there are 2 SPE's instead of 1:
   - Player 1 offers 0, and player 2 says Yes to all offers
   - Player 1 offers 1 cent, and player 2 says Yes to all offers except 0
2. Show that for every $\bar{x} \in [0, c]$, there are NE in which 1 offers $\bar{x}$. Find 2's optimal strategy.

## 6 Chance moves:

Extensive game with perfect information and chance moves. Player 1 makes an offer, chance determines the type of person 2, and finally 2 accepts or rejects 1's offer.

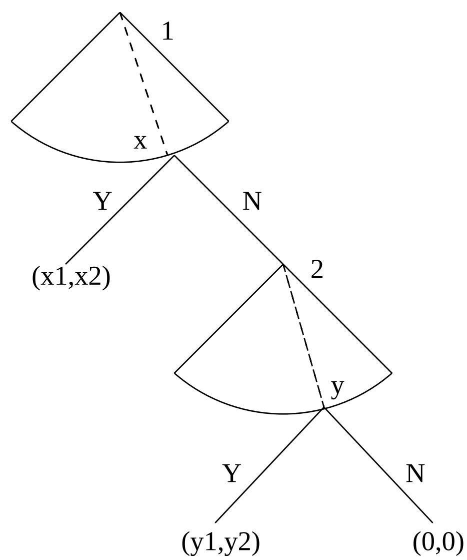


**Unique SPE**

- $1 \to B$
- $2 \to C$
- work with expected payoffs

# 7 Bargaining as an extensive game

In the ultimatum game, player 2 is powerless. His only alternative to accepting is to reject which results in him getting no pie. We'd like to extend the model to give player 2 more power:




- $c = 1$
- $x = (x_1, x_2)$ with $x_1 + x_2 = 1$
- 2 rejects all offers $x_2 < 1$
- $y = (y_1, y_2)$
- $y_1 + y_2 = 1$
- The second part of the game is just an ultimatum game in which 2 moves first
  $\Rightarrow$ Unique SPE
  - 2 offers nothing to 1
  - 1 accepts all offers

$\Rightarrow$ In every SPE, 2 obtains all the pie

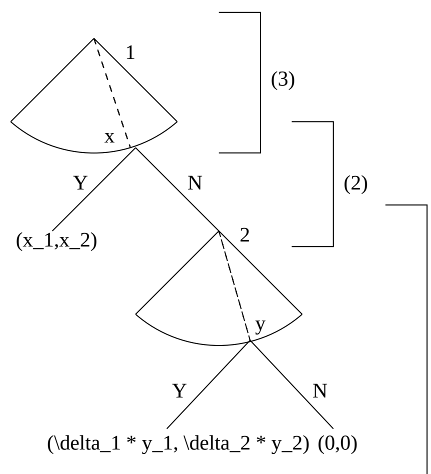
**Last Mover's Advantage:** Similar result with alternating offers. In every SPE, the player who makes the offer in the last period obtains all the pie.

In our model so far, players indifferent about timing of an agreement. In real life however, bargaining take time and time is valuable. Players preferences should reflect the fact that they have bias toward early agreement.

## 7.1 Example: Finite horizon game with alternating offers

Players alternate proposals, future discounted using the constant discount factor $\delta_i$ at each period.
$\Rightarrow 0 < \delta_i < 1$

### 7.1.1 Two Periods:




**Find the SPE by backward induction:**

1. Subgame unique SPE in which player 2 offers $(0, 1)$
   - 1 accepts all proposals
   - outcome: $(0, \delta_2)$
2. $N \rightarrow (0, \delta_2)$
   $Y \rightarrow (x_1, x_2)$
   2 strategies
   (a) Y if $x_2 >= \delta_2$
   N if $x_2 < \delta_2$
   (b) Y if $x_2 > \delta_2$
   N if $x_2 <= \delta_2$
3. Players 1's optimal strategy: $(1 - \delta_2, \delta_2)$

**Unique SPE:**

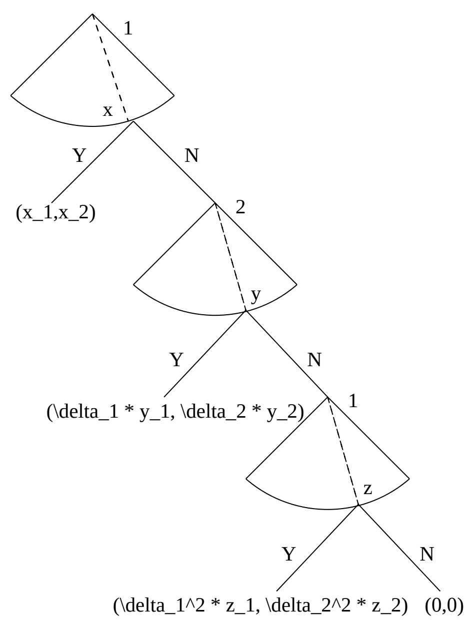
- Player 1's initial proposal $(1 - \delta_2, \delta_2)$
- Player 2 accepts all proposals where $x_2 >= \delta_2$ and rejects everything where $x_2 < \delta_2$
- Player 2 proposed $(0, 1)$ after any history in which he rejects a proposal of player 1
- Player 1 accepts all proposals of player 2 (after a hist in which 2 rejects 1's opening proposal)

**Outcome:**

- Player 1 proposes $(1 - \delta_2, \delta_2)$
- Player 2 accepts
- Resulting payoff : $(1 - \delta_2, \delta_2)$

### 7.1.2 Three Periods

Still finite, use backward induction.



**Unique SPE:**

- 2 offers $(\delta_1, 1-\delta_1)$
- 1 accepts
- results in $(\delta_1^2, \delta_2(1-\delta_1))$

**2 says:**

- Y if $x_2 > \delta_2(1-\delta_1) \rightarrow (x_1, x_2)$
- N if $x_2 < \delta_2(1-\delta_1) \rightarrow (\delta_1^2, \delta_2(1-\delta_1))$
- indifferent otherwise

**Player 1's optimal strategy:**

- $(1-\delta_2(1-\delta_1), \delta_2(1-\delta_1)) = (\delta_1^2, \delta_2(1-\delta_1))$
- 2 accepts if:
  - $1-\delta_1^2 > \delta_2(1-\delta_1)$
  - $1+\delta_1 > \delta_2$

### 7.1.3 Stahl's Bargaining Model: Finite Horizon

| # *Periods* | player 1 gets |
|---|---|
| 2 periods | $1-\delta_2$ |
| 3 periods | $1-\delta_2+\delta_1\delta_2$ |
| 5 periods | $1-\delta_2+\delta_1\delta_2(1-\delta_2)+\delta_1\delta_2$ |
| $2k$ periods | $1-\delta_2\left[\frac{1-(\delta_1\delta_2)^k}{1-(\delta_1\delta_2)}\right]$ |
| $2k+1$ periods | $1-\delta_2\left[\frac{1-(\delta_1\delta_2)^k}{1-(\delta_1\delta_2)}\right]+(\delta_1\delta_2)^k$ |
| $\lim_{k\to\infty}$ | $\frac{1-\delta_2}{1-\delta_1\delta_2}$ |

# 8 Rubinstein's infinite horizon model:

Instead of two players alternating offers for a period of time, there is no deadline, they can alternate offers forever.

**Terminal Histories:**

$$
\begin{array}{ll}
(x^1, N, x^2, N.....x^t, N...) & \rightarrow \quad \text{every} \quad \text{offer} \quad \text{rejected} \\
(x^1, N, x^2, N.....x^t, Y) &
\end{array}
$$

**SPE:**

- don't have finite horizon → so we can't directly apply backward induction
- guess the equation and check using one stage deviation principle

**Strategy in this game:**

1. Offer in period 1
2. Response to hist $(x\prime, N, x^2)$
3. counteroffer for history $(x\prime, N, x^2, N)$
4. The subgame looks the same in all periods. The absolute payoffs are different but the preferences are the same, because all options are discounted by the same factor.

Guess a static policy in which each player always make the same proposal and always accepts the same set of proposals.

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$$
$$x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$
$$y_1^* = \frac{1-\delta_1}{1-\delta_1\delta_2}$$
$$y_2^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

**SPE**

- player 1 proposes $x^*$ and accepts $y$ iff $y_1 >= y_1^* \Rightarrow s_1^*$
- player 2 proposes $y^*$ and accepts x iff $x_2 >= x_2^* \Rightarrow s_2^*$

**Check SPE:**

Game has 2 types of subgames:

1. One in which first move is an offer:

   - Suppose offer made by player 1 (symmetric for player 2)
   - Fix 2's strategy at $s_2^*$
   - if player 1 adopts $s_1^* \Rightarrow$ 2 accepts, gets $x_1^*$
   - if 1 offers $> x_2^*$, 2 accepts $\Rightarrow$ lower payoff than $x_1^*$
   - if 1 offers $< x_2^*$, 2 rejects, offers $y^*$
     - player 1 accepts, payoff: $\delta_1 y_1^*$
     - $\delta_1 y_1^* < x_1^*$

2. One in which first move is a response to an offer:

   - Suppose player 1 is responding
   - Fix 2's strategy at $s_2^*$
   - Denote by $(y_1, y_2)$ the offer to which player 1 is responding
   - $s_1^*$ accepts iff $y_1 > y_1^*$
   - if $y_1 < y_1^*$ then $s_1^*$ will reject it and player 1 will get $\delta_1 x_1^* = y_1^*$ and he cannot increase his payoff by deviating

Hence $s^*$ is an SPE (in fact the *unique SPE*)

**Suppose** $\delta_1 = \delta_2$

The outcomes are (all of which end with immediate agreement):

| | |
|---|---|
| 1 moves first | $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ |
| 2 moves first | $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ |
| $\delta \to 1$ | $(\frac{1}{2}, \frac{1}{2})$ : symmetry<br>- first mover advantage disappears |
| $\delta \to 0$ | $(1, 0)$<br>- first mover advantage dominates |