Lecture 15

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1 Agenda

In this lecture, we discuss:

- Nash bargaining solution.
- Relation of axiomatic and strategic models.

2 Rubinstein's Model

As we've seen last class, the Rubinstein bargaining model allows players to offer counter-proposals indefinitely, and it assumes impatient players with discount rates $\delta_1, \delta_2 \in (0,1)$.

The following is an SPE for this game:

- Player 1 proposes x_1^* and accepts offer y if, and only if, $y \geq y_1^*$.
- Player 2 proposes y_2^* and accepts offer x if, and only if, $x \geq x_2^*$.

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \quad (1)$$

$$y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \quad (2)$$

$$x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \quad (3)$$

$$y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2} \quad (4)$$

Clearly, an agreement is reached immediately for any values of δ_1 and δ_2.
Now suppose $\delta_1 = \delta_2$:

- If 1 moves first, the division will be $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$.
- If 2 moves first, the division will be $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$.

The first mover's advantage is clearly related to the impatience of the players (δ):

- If $\delta \to 1$, the FMA disappears and the outcome tends to $(\frac{1}{2}, \frac{1}{2})$.

- If $\delta \to 0$, the FMA dominates and the outcome tends to $(1, 0)$.

More interestingly, let's assume the discount factor is derived from some interest rates r_1 and r_2.

$$\delta_1 = e^{-r_1 \Delta t} \tag{5}$$
$$\delta_2 = e^{-r_2 \Delta t} \tag{6}$$

These equations represent a continuous-time approximation of interest rates. It is equivalent to interest rates for very small periods of time Δt: $e^{-r_i \Delta t} \simeq \frac{1}{1+r_i \Delta t}$.

Taking $\Delta t \to 0$, we get rid of the first mover's advantage.

$$\lim_{\Delta t \to 0} x_1^* = \lim_{\Delta t \to 0} \frac{1-\delta_2}{1-\delta_1\delta_2} = \lim_{\Delta t \to 0} \frac{1-e^{-r_2 \Delta t}}{1-e^{-(r_1+r_2)\Delta t}} = \frac{r_2}{r_1+r_2} \tag{7}$$

3 Alternative Bargaining Model: Nash's Axiomatic Model

Bargaining problems represent situations in which:

- There is a conflict of interest about agreements.
- Individuals have the possibility of concluding a mutually beneficial agreement.
- No agreement may be imposed on any individual without his approval.

Strategic / Noncooperative model: Explicit model of the bargaining process (game form).

Nash's Model: Abstract yourself from the process of bargaining, details, how to effectively bargain. Consider only the set of outcomes / agreements that satisfy "reasonable" properties.

Nash 1953: "One states as axioms several properties that would seem natural for the solution to have and then one discovers that axioms actually determine the solution uniquely."

What are the "reasonable" axioms ?

ex: Suppose 2 players must split one unit of a good.

- If no agreement is reached, then neither receives anything.
- If preferences are identical, then expect each to obtain half.

Two desirable properties are efficiency and symmetry of the outcome.

What if the preferences are not identical ?

Consider a more general scenario:

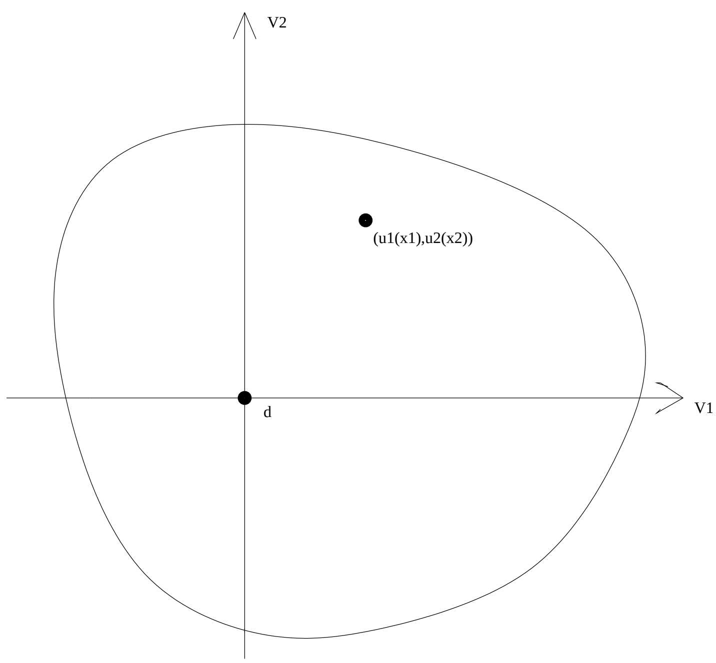
X: set of possible agreements.
D: disagreement outcome.

ex: $X = \{(x_1, x_2) | x_1 + x_2 = 1, x_i \geq 0\}$, $D = (0, 0)$.

Each player has preferences (u_i) over $X \cup \{D\}$.

Let the set of possible payoffs be:

$$U = \{(v_1, v_2) \mid u_1(x_1) = v_1, u_2(x_2) = v_2 \text{ for some } x \in X\} \quad (8)$$
$$d = (u_1(D), u_2(D)) \quad (9)$$



A **bargaining problem** is a pair (U, d) where $U \subset \mathbb{R}^2$ *and* $d \in U$.

- U is convex and compact.
- $\exists\, v \in U \text{ s.t. } v > d \ (v_i > d_i \ \forall i)$

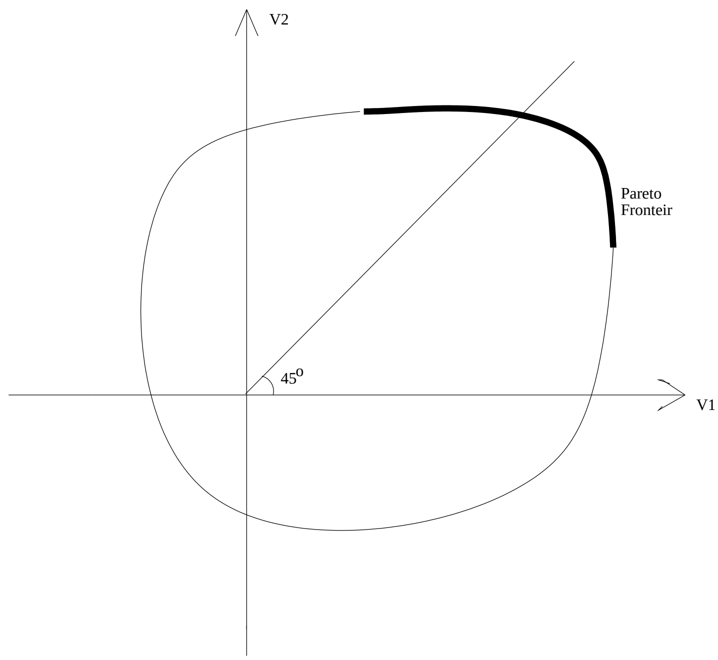
Denote the set of all possible bargaining problems by $\mathcal{B}$. A **bargaining solution** is a function f: $\mathcal{B} \rightarrow U$. We will study bargaining solutions (f) that satisfy a list of reasonable conditions.

Axioms:

1 - Pareto Efficiency

A bargaining solution f(U,d) is Pareto efficient if there does not exist a $(v_1, v_2) \in U$ s.t. $v \geq f(U,d)$ and $v_i > f_i(U,d)$ for some i.

Justification: An inefficient outcome is unlikely, since it leaves space for renegotiation.



2 - Symmetry

Let (U,d) be such that $(v_1, v_2) \in U$ iff $(v_2, v_1) \in U$ and $d_1 = d_2$. Then $f_1(U,d) = f_2(U,d)$.

If the players are indistinguishable, the agreement should not discriminate between them.

3 - Invariance to Equivalent Payoff Representations

Given a bargaining problem (U,d), consider a different bargaining problem (U',d') for some $\alpha > 0, \beta$:

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) \mid (v_1, v_2) \in U\} \quad (10)$$

$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2) \quad (11)$$

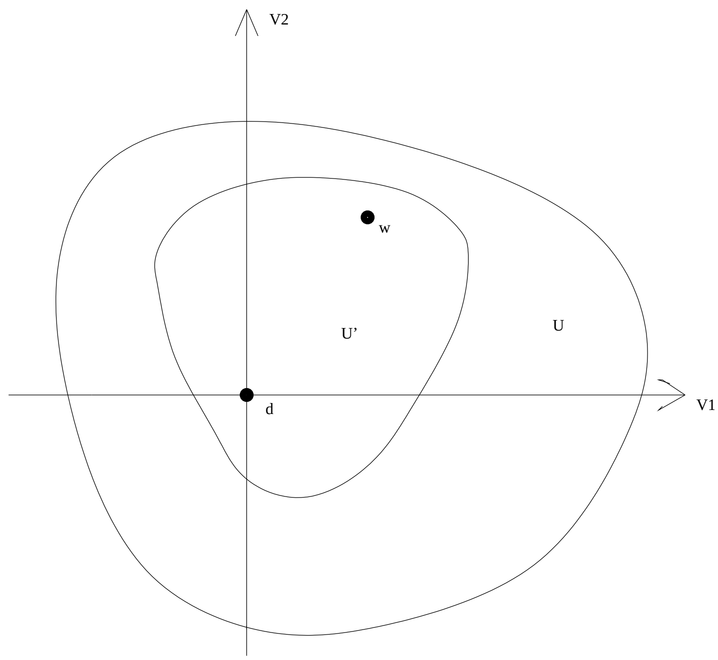
Then, $f_i(U', d') = \alpha_i f_i(U,d) + \beta_i$.

The idea of this axiom is that utility functions are only representation of preferences over outcomes. A transformation of the utility function that maintains the some ordering over preferences (such as a linear transformation) should not alter the outcome of the bargaining process.

4 - Independence of Irrelevant Alternatives

Let (U,d) and (U'd) be two bargaining problems such that U' ⊆ U.

If f(U,d) ∈ U', then f(U',d) = f(U,d).



4 Nash Bargaining solution

Definition 1 *A pair of payoffs (v_1^*, v_2^*) is a Nash bargaining solution if it solves the following optimization problem:*

$$\max_{v_1,v_2}(v_1 - d_1)(v_2 - d_2)$$
$$s.t \;\; (v_1, v_2) \in U$$

Denote $f^N(U, d)$ the Nash bargaining solution.

- *Existence of a solution*: U is compact and the objective function is continuous, hence the problem has a solution.
- *Uniqueness of the solution*: The objective function is strictly quasi concave.
  Recall that a real function $f : S \to \mathbb{R}$ where S is nonempty convex is quasi concave if

$\forall x, y \in S$, $\forall \lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda)y) \geq \min(f(x), f(y))$$

Another characterization is that the level sets of f are convex $\{x \in S | f(x) \leq C\}$.
f is strictly quasi concave if the above inequality holds strictly $\forall \lambda \in (0,1)$.
Here the level sets of the objective function are hyperbolic.

Property 1 *Nash bargaining solution $f^N(U,d)$ is the only solution satisfying the 4 axioms.*

Proof: The proof has 2 steps: first we prove that Nash bargaining solution that satisfies the 4 axioms; then we prove that if a bargaining solution satisfies the 4 axioms, it is equal to $f^N(U,d)$.
Step 1:

1. Pareto efficiency: the objective function is increasing in v_1 and v_2. Assume it were not Pareto efficient: then there is a v verifying $v \geq f^N(U,d)$ and $v_i > f^N(U,d)$ for some i. Then the objective function evaluated at v is greater than at $f^N(U,d)$ since it is increasing. This contradict the optimality of $f^N(U,d)$.

2. Symmetry: Assume $d_1 = d_2$. Let $v^* = (v_1^*, v_2^*) = f^N(U,d)$ be the Nash bargaining solution. Then we verify that (v_2^*, v_1^*) is also solution. By uniqueness of the solution, it holds that $v_1^* = v_2^*$, i.e $f_1^N(U,d) = f_2^N(U,d)$.

3. Independence of irrelevant alternatives: Let $U' \subseteq U$. $f^N(U',d)$ is solution to the optimization problem with the same objective function as $f^N(U,d)$ and a smaller feasible set. Hence the objective function value at $f^N(U,d)$ is greater than or equal to that at $f^N(U',d)$. If $f^N(U,d) \in U'$, then the objective function values must be equal, i.e $f^N(U,d)$ is optimal for U' and by uniqueness of the solution $f^N(U,d) = f^N(U',d)$.

4. Invariance to equivalence payoff representation: $f(U',d')$ is solution of:

$$\max_{v_1,v_2} (v_1 - \alpha_1 d_1 - \beta_1)(v_2 - \alpha_2 d_2 - \beta_2)$$
$$\text{s.t} \quad (v_1, v_2) \in U'$$

   Perform the change of variables $v_1' = \alpha_1 v_1 + \beta_1$ $v_2' = \alpha_2 v_2 + \beta_2$, then it follows that $f_i^N(U',d') = \alpha_i f_i^N(U,d) + \beta_i, \; i = 1,2$.

Step 2: Let $f(U,d)$ be a bargaining solution satisfying the 4 axioms.
Prove that $f(U,d) = f^N(U,d)$.

- Let $z = f^N(U,d)$, and $U' = \{\alpha' v + \beta | v \in U; \alpha' z + \beta = (1/2, 1/2)'; \alpha' d + \beta = (0,0)'\}$. Since $f(U,d)$ and $f^N(U,d)$ both satisfy axiom 3, then $f(U,d) = f^N(U,d)$ is equivalent to $f(U',0) = f^N(U',0) = (1/2,1/2)$. Hence it is sufficient to prove that $f(U',0) = (1/2,1/2)$.

- Let us show that there is no $v \in U'$ such that $v_1 + v_2 > 1$:
  Assume that there is a $v \in U'$ such that $v_1+v_2 > 1$. Let $t = (1-\lambda)(1/2, 1/2)+\lambda(v_1, v_2)$ for some $\lambda \in (0, 1)$. As U' is convex, $t \in U'$. We can choose λ sufficiently small so that $t_1t_2 > 1/4 = f^N(U', 0)$: this contradicts the optimality of $f^N(U', 0)$.

- Since U' is bounded, we can find a rectangle U'' symmetric w.r.t the line $v_1 = v_2$, such that $U' \subseteq U''$ and $(1/2, 1/2)$ is on the boundary of U''.

- By axioms 1 and 2, $f(U'', 0) = (1/2, 1/2)$.

- By Axiom 4, since $U' \subseteq U''$, we have $f(U', 0) = (1/2, 1/2)$.

□

Example: Dividing a dollar
$X = \{(_1, x_2)|x_1 \geq 0,\ x_1 + x_2 = 1\}$, $D = (0, 0)$.
$U = \{(v_1, v_2)|(v_1, v_2) = (u_1(x_1), u_2(x_2)),\ (x_1, x_2) \in X\}$, $d = (u_1(0), u_2(0))$. U is convex and compact. (U, d) is a bargaining problem.

- Case 1: $u_1 = u_2 = u$: symmetric bargaining problem. Hence $f^N(U, d) = (1/2, 1/2)$: the dollar is shared equally. The Nash bargaining problem is solution to the following problem:

  $$\max_{0 \leq z \leq 1} v_1(z)v_2(1-z) = u(z)u(1-z)$$

  We denote its solution z_u: it verifies the first order optimality conditions: $u'(z)u(1-z) = u(z)u'(1-z)$ i.e $\frac{u'(z_u)}{u(z_u)} = \frac{u'(1-z_u)}{u(1-z_u)}$.

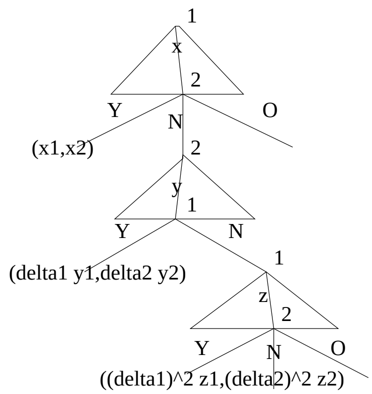
- Case 2: player 2 is more risk averse: $v_1 = u$, $v_2 = h \circ u$, where $h : \mathbb{R} \to \mathbb{R}$ increasing concave function with $h(0) = 0$. The Nash bargaining problem is solution to the following problem:

  $$\max_{0 \leq z \leq 1} v_1(z)v_2(1-z) = u(z)h(u(1-z))$$

  We denote its solution z_v: it verifies the first order optimality conditions: $u'(z)h(u(1-z)) = u(z)h'(u(1-z))u'(1-z)$ i.e $\frac{u'(z_v)}{u(z_v)} = \frac{h'(u(1-z_v))u'(1-z_v)}{u(1-z_v)}$.
  Since h is concave increasing and $h(0) = 0$, we have for $t \geq 0$: $h'(t) \leq \frac{h(t)}{t}$. Hence $\frac{u'(z_v)}{u(z_v)} \leq \frac{u'(1-z_v)}{u(1-z_v)}$. Hence $z_u \leq z_v$.
  Conclusion: player 2 is more risk averse: player 1's share increases.

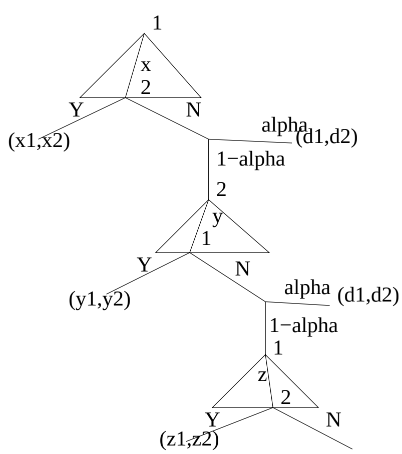
Question: Can we modify Rubinstein's model so that it reaches a Nash bargaining solution?

5 Outside Options Model



When responding to an offer, player 2 may pursue an outside option.
Claim:
If $d_2 \leq x_2^*$ then the strategy pair of Rubinstein's bargaining model is the unique SPE.
If $d_2 > x_2^*$ then the game has a unique SPE in which:
Player 1 proposes $(1 - d_2, d_2)$ and accepts a proposition y iff $y_1 \geq \delta_1(1 - d_2)$.
Player 2 proposes $(\delta_1(1 - d_2), 1 - \delta_1(1 - d_2))$ and accepts a proposition x iff $x_2 \geq d_2$.
Intuition: Player 2's outside option has value only if it is worth more than her equilibrium payoff in the game without the option, otherwise, the outside option has no effect.
Compare with the Nash bargaining solution x_N:
Let $u_1(x) = x$ and $u_2(x) = x$. The Nash bargaining solution is $x_N = \arg\max_{x \geq 0}(x - d_1)(1 - x - d_2) = 1/2 + 1/2(d_1 - d_2)$.
Special Case of disagreement outcome: $(0, d_2)$
If $d_2 < x_2^*$: no effect.
If $d_2 > x_2^*$: then $x_1^* = 1 - d_2$.

6 bargaining with risk of breakdown



There is a probability α of breaking down. Assume $\delta \to 1$: the possibility of a breakdown puts pressure to reach an agreement.
The game has a unique SPE:
1 proposes $\hat{x}$ and accepts an offer y iff $y_1 \geq \hat{y}_1$.
2 proposes $\hat{y}$ and accepts an offer x iff $x_1 \geq \hat{x}_1$.

$$\hat{x}_1 = \frac{1 - d_2 + (1-\alpha)d_1}{2-\alpha}$$
$$\hat{y}_1 = \frac{(1-\alpha)(1-d_2) + d_1}{2-\alpha}$$

Let $\alpha \to 0$: then $\hat{x}_1 \to 1/2 + 1/2(d_1 - d_2)$: this is the Nash bargaining solution: For risk averse players, the game's outcome is the NBS.

7 Annex 1: Solution of outside options model:

Suppose there is an equilibrium in which an offer is accepted:
Player 1 accepts iff $\delta_1 y_1 \geq \delta_1^2 z_1$ i.e $y_1 \geq \delta_1 z_1$.
Player 2's strategy is: $(\delta_1 z_1, 1 - \delta_1 z_1)$.
Player 2 accepts iff $x_2 \geq \max(d_2, \delta_2(1 - \delta_1 z_1))$.
1's optimal offer is : $x_2 = \max(d_2, \delta_2(1 - \delta_1 z_1))$.
Assume $d_2 \leq \delta_2(1 - \delta_1 z_1)$: then $z_1 = 1 - \delta_2(1 - \delta_1 z_1)$.
Hence $z_1 = \frac{1-\delta_2}{1-\delta_1\delta_2} = x_1^*$.
Then replacing z_1 by its value: $d_2 \leq \delta_2(1 - \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2}) = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} = x_2^*$.
If $d_2 \leq x_2^*$ then 1 offers x^* and accepts $y_1 \geq y_1^*$; 2 offers y^* and accepts $x_2 \geq x_2^*$.

If $d_2 > x_2^*$, then 1 offers $(1 - d_2, d_2)$ and accepts $y_1 \geq \delta_1(1 - d_2)$; 2 offers $(\delta_1(1 - d_2), 1 - \delta_1(1 - d_2))$ and accepts $x_2 \geq d_2$.

8 Annex 2: Solution of risk of breakdown model:

1 accepts iff $y_1 \geq (1-\alpha)z_1 + \alpha d_1$
2's optimal strategy is $((1-\alpha)z_1 + \alpha d_1, 1 - ((1-\alpha)z_1 + \alpha d_1))$.
2 accepts iff $x_2 \geq \alpha d_2 + (1-\alpha)(1 - ((1-\alpha)z_1 + \alpha d_1))$
1 offers $z_1 = 1 - \alpha d_2 - (1-\alpha)(1 - ((1-\alpha)z_1 + \alpha d_1)$.
Solving for z_1: let $z_1 = \hat{x}_1$:

$$\hat{x}_1 = \frac{1 - d_2 + (1-\alpha)d_1}{2-\alpha}$$
$$\hat{y}_1 = \frac{(1-\alpha)(1-d_2) + d_1}{2-\alpha}$$