| **6.972: Game Theory** | April 7, 2005 |
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# Lecture 17: Repeated Games 2

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**Agenda:**

- Repeated games
- Imperfect monitoring
  - price trigger strategies

Recall: $v^*$ set of feasible and strictly individually rational pay off set.

**Theorem 1** *(Nash folk theorem:)* $\forall v \in V^*; \exists\ \underline{\delta} < 1$ *s.t.* $\forall \delta \geq \underline{\delta}$ *there is a NE of the* $G^\infty(\delta)$ *with payoffs* $v$.

*Proof:* Strategy constructed $\Rightarrow$ grim trigger with punishment by minmax.
If $i$ plays the strategy $\Rightarrow$ he gets $v_i$
If $i$ deviates from the strategy in some period t; then denoting $\overline{v_i} = \max_{\mathbf{a}} g_i(\mathbf{a})$, the most that I could get. $(1-\delta)[v_i + \delta v_i + \cdots + \delta^{t-1} v_i + \delta^t \overline{v}_i + \delta^{t+1} \underline{v}_i + \delta^{t+2} \underline{v}_i + \cdots]$
following the suggested strategy will be optimal if,

$$\begin{aligned} v_i &\geq (1-\delta^t) v_i + \delta^t (1-\delta) \overline{v}_i + \delta^{t+1} \underline{v}_i \\ &= v_i - \delta^t [v_i + (1-\delta)\overline{v}_i - \delta \underline{v}_i + (\delta v_i - \delta v_i)] \end{aligned}$$

so if we can show that the expression in the bracket, is non negative then we are done. So we need to show that:

$$\begin{aligned} &\Rightarrow \quad (1-\delta)(v_i - \overline{v}_i) + \delta(v_i - \underline{v}_i) \geq 0 \\ &\Rightarrow \quad \frac{\delta}{(1-\delta)}(v_i - \underline{v}_i) \geq \overline{v}_i - v_i \end{aligned}$$

simply pick: $\underline{\delta} = \max_i \frac{v_i - \overline{v}_i}{2v_i - \overline{v}_i - \underline{v}_i}$. Check that $\underline{\delta} < 1$. (Since $\underline{v}_i < v_i$). □

**Remark:** Nash folk theorem says that essentially anything goes as a NE when players are patient enough.
$\Longrightarrow$ strategies involve unrelenting punishment which may be very costly for the punisher to carry out $\Longrightarrow$ non-credible threat.

**Example:**

| | L $(q)$ | R $(1-q)$ |
|---|---|---|
| U | $6, 6$ | $0, -100$ |
| D | $7, 1$ | $0, -100$ |

The unique NE: $(D, L)$. On the other hand by minmax: $\underline{v}_1 = 0$, $\underline{v}_2 = 1$
$\Longrightarrow$ strategies constructed in the proof require play 2 to play $R$ in every period following a deviation!

**Perfect folk theorems:**

1. Friedman: Any feasible payoff $> e_i$, where $e = [e_i] \quad i = 1, ..., I$, the pay off vector for some static equilibrium, can be obtained as an SPE of $G^\infty(\delta)$

   Proof: grim trigger with punishment by NE in phase 2 ( subgame perfect)

2. What happens if static NE payoffs> minmax payoff?

   - In PD minmax = NE payoff

**Theorem 2** *Perfect folk theorem:(Fudenburg & Maskin) Assume that dim($v^*$) = I (non empty interior) then for any $v \in V^*$; $\exists \underline{\delta} < 1$; s.t. $\forall \delta \geq \underline{\delta}$ there is an SPE of $G^\infty(\delta)$ with payoffs v.*

*Proof Sketch:* Fix $v \in V^*$ . Construct an SPE that achieves it.

Again assume that $\exists$ some action profile $\mathbf{a} = (a_1, ..., a_I)$ s.t. $g_i(\mathbf{a}) = v_i$
Key to the proof: Find strategies that allows us ro "reward" agents $j \neq 1$ in the event i deviates and has to be minmaxed.

Steps:

- choose $\acute{v} \in Int(V^*)$ s.t. $\underline{v}_i < \acute{v}_i < v_i \; \forall i$
- choose $\epsilon > 0$ s.t. the vector

$$\acute{v}(\epsilon) = (\acute{v}_1 + \epsilon, \acute{v}_2 + \epsilon, ..., \acute{v}_i, \acute{v}_{i+1} + \epsilon, ..., \acute{v}_I + \epsilon)$$

- let $\mathbf{a}^i$ be the profile with $g(\mathbf{a}^i) = v^i(\epsilon)$
- choose T (punishment length) s.t. $\forall i$;

$$\max_a g_i(a) + T\underline{v}_i < \min_a g_i(a) + T\acute{v}_i$$

- let $m_i = (m_i^i, m_{-i}^i)$ be the profile that minmaxes $i$

$$g_i(m_i^i, m_{-i}^i) = \underline{v}_i$$

Now consider the following strategy for $i = 1, .., I$

1) play $a_i \quad (g(\mathbf{a}) = v)$ so long as no one deviates. If $j$ alone deviates, go to $2_j$

$2_j$) play $m_j^i$ for T periods then go to $3_j$, If no one deviates. If during $2_j$, k deviates, restart phase $2_k$

$3_j$) play $\mathbf{a}^j \quad (g(\mathbf{a}^j) = v^j(\epsilon))$ so long as no one deviates. If k deviates, go to $2_k$
□

**remarks:**

1. Note that strategies involve punishments & rewards.
2. Check if is an SPE
3. dim $v^* = I \Rightarrow$ ensures that each paper $i$ can be *singled out* for a punishment for a deviation.

- Folk theorem for finitely repeated game:
  - if the stage game has a unique NE, then unique perfect equilibrium is to play the static NashE in every period od the subgame.
  - with multiple stage NE$\Rightarrow$ can use rewards & punishments
- Equilibrium concept do very little to pin down play by patient player?...?

## Repeated games with imperfect public monitoring

Before each player observed the actions of others at the end of each period.
$\Rightarrow$ instead, assume players observe only an imperfect signal of stage game actions.

**Motivational Example** Green & Porter (84) Cournot competition with noisy demand.
Firms set output $q_1^t, ..., q_I^t$ privately. Level of demand stochastic. Each firm's pay off depends on his own output and on the publicly *observed market price.* Firms do not observe each other's output levels.

$$\text{Low market price} \rightarrow \begin{cases} \text{high produced output} \\ \text{low demand} \end{cases}$$

**General Model:**

- $A_1, ..., A_I$ finite action sets: $A = A_1 \times \cdots \times A_I$ Each action profile induces a probability distribution over $y \in Y$
- $y_i$ rv $\in Y$ : finite set.
  $\pi(., \mathbf{a})$ : pmf on $y$ parametrized by the action profile $\mathbf{a}$.
- Players i's realized payoff $r_i(a_i, y)$
- Expected stage payoff.

$$g_i(\mathbf{a}) = \sum_{y \in Y} \pi(y, \mathbf{a}) r_i(a_i, y)$$

- Strategy $\{s_i^t\}$

$$s_i^t : h^t \longrightarrow A_i$$
$$h^t = (y^0, ..., y^{t-1})$$

- Player $i$'s average discounted payoff.

$$(1-\delta) \sum_{t=0}^{\infty} \delta^t g_i^t(\mathbf{a})$$

**example:** Green & Porter's Noisy Cournot Model: Consider the following trigger strategy:

1. Play $q_1, ..., q_I$ (desirable outcome) If $p^t \leq f$, go to phase 2
2. Play $q_1^c, ..., q_I^c$ (static NE) for T periods then return to 1