Lecture 21

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1 Agenda

- Optimal mechanisms
- Efficient mechanisms $\rightarrow$ VCG

2 Optimal Mechanism

Mechanism that maximizes expected revenue. Recall: (Q, M): Direct mechanism

$$Q : X \rightarrow \Delta + \{0\}$$

$$M : X \rightarrow R^N$$

$$\boxed{m_i(x_i) = m_i(0) + q_i(x_i)x_i - \textstyle\int_0^{x_i} q_i(t_i)dt_i}$$

- IC: $\Rightarrow q_i(\cdot)$:nondecreasing
- IR: $\Longleftrightarrow m_i(0) \leq 0$

How do we design the mechanism that optimizes (maximizes) the expected revenue to the seller?

$$E[R] = \sum_{i \in N} E[m_i(X_i)]$$

$$E[m_i(X_i)] = \int_0^{w_i} m_i(x_i) f_i(x_i) dx_i$$

$$= m_i(0) + \int_0^{w_i} q_i(x_i) x_i f_i(x_i) dx_i - \int_0^{w_i} \int_0^{x_i} q_i(t_i) dt_i f_i(x_i) dx_i$$

changing order of integration, the second part becomes

$$\int_0^{w_i} \int_{t_i}^{w_i} f_i(x_i) dx_i q_i(t_i) dt_i i$$

inside $f_i(x_i)dx_i$ becomes

$$1 - F_i(t_i)$$

where m_i is ex ante expected payment of buyer i.

$$= m_i(0) + \int_0^{w_i} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) q_i(x_i) f_i(x_i) dx_i$$

$$= m_i(0) + \int_X (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) Q_i(X) f(X) dX$$

Optimize MD problem

$$\max E[R] \qquad s.t. \qquad IC + IR$$

Define: Virtual valuation of a buyer with value x_i

$$U_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$$

Seller should choose Q and M to maximize:

$$\sum_{i \in N} m_i(0) + \int_X (\sum_{i \in N} \Psi_i(x_i) Q_i(x)) f(x) dx$$

The following is an optimal mechanism

- Allocation rule:

$$Q_i(x) > 0 \Longleftrightarrow \Psi_i(x_i) = \max_j \Psi_j(x_j) \geq 0$$

  Remark: Assume that $\Psi_i(x_i)$ is increasing in x_i. Ψ_i: Virtual valuation. Design problem is regular.

- Payment rule:

$$M_i(x) = Q_i(x) x_i - \int_0^x Q_i(z_i, x_{-i}) dz_i$$

  Let's see if it satifies IC and IR.

- $M_i(0, x_{-i}) = 0 \quad \forall x_{-i} \Rightarrow m_i(0) = 0 \quad \surd$
- $z_i < x_i \Rightarrow U_i(z_i) < U_i(x_i) \Rightarrow Q_i(z_i, x_{-i} \leq Q_i(x_i, x_{-i}) \quad \forall x_{-i} \Rightarrow q_i(z_i) \leq q_i(x_i) \quad \surd$

Let's now look at the payment rule:

2.1 Intuition

Define

$$y_i(x_{-i}) = inf\{z_i | \Psi_i(z_i) \geq 0, \Psi_i(z_i) \geq \Psi_j(x_j) \quad \forall j \neq i\}$$

$\therefore$ smallest value for i that wins against x_{-i}

Can write

$$Q_i(z_i, x_{-i}) = \begin{cases} 1 & if \quad z_i > y_i(x_{-i}) \\ 0 & if \quad z_i < y_i(x_{-i}) \end{cases}$$

$$\Rightarrow \int_0^{x_i} Q_i(z_i, x_{-i} = \begin{cases} x_i - y_i(x_{-i}) & if \quad x_i > y_i(x_{-i}) \\ 0 & if \quad x_i < y_i(x_{-i}) \end{cases}$$

$$\Rightarrow M_i(x) = \begin{cases} y_i(x_{-i}) & if \quad Q_i(x) = 1 \\ 0 & if \quad Q_i(x) = 0 \end{cases}$$

2.2 Symmetric case

Suppose that distributions of values identical across buyers,

$$\forall i \; f_i = f \Rightarrow \Psi_i = \Psi$$

Note that

$$y_i(x_{-i}) = \max\{\Psi^{-1}(0), \max_{j \neq i} x_j\}$$

$$\Psi_i(z_i) \geq \Psi_j(x_j) \quad \forall j \neq i$$

$$\Rightarrow z_i \geq x_j \quad \forall j \neq i$$

also $\Psi(z_i) \geq 0$.

Proposition: Assume that the design problem is regular and symmetric. Then a second price auction (Vickrey) with reservation price $r^* = \Psi^{-1}(0)$ is an optimal mechanism.

2.3 Intuition on Virtual Valuations

Consider a seller and a a single buyer. Suppose that the seller makes a take it or leave it offer at a price p.

accept if $x \geq p$

$$\max_{p \geq 0} p(1 - F(p))$$

$1 - F(p)$ is the prob that buyer will accept.

First order conditions: $1 - F(p) - pf(p) = 0 \Rightarrow p^* = \frac{1-F(p^*)}{f(p^*)}$

Recall $\Psi(p) = p - \frac{1-F(p)}{f(p)} = 0 \Rightarrow p^* = \Psi^{-1}(0)$

3 Mechanism Design

Social choice point of view

- I agents with possible types $(\theta_1, \ldots, \theta_I)$

$$\theta_i \in \Theta_i, \Theta = \prod_{i=1}^{I} \Theta_i$$

- These agents must make a collective choice from some set Y (of possible alternatives). Y is allocations and transfers.

- Utility: $u_i(y, \theta_i)$ where $y \in Y$ and $y = (x, t)$

Definition: A social choice function $f : \Theta_1 \times \cdots \times \Theta_I \to Y$

Problem: θ_i's are not publicly observable, so for the social choice $f(\theta_1, \ldots, \theta_I)$ to be chosen, each agent must be relied on to disclose their type correctly.

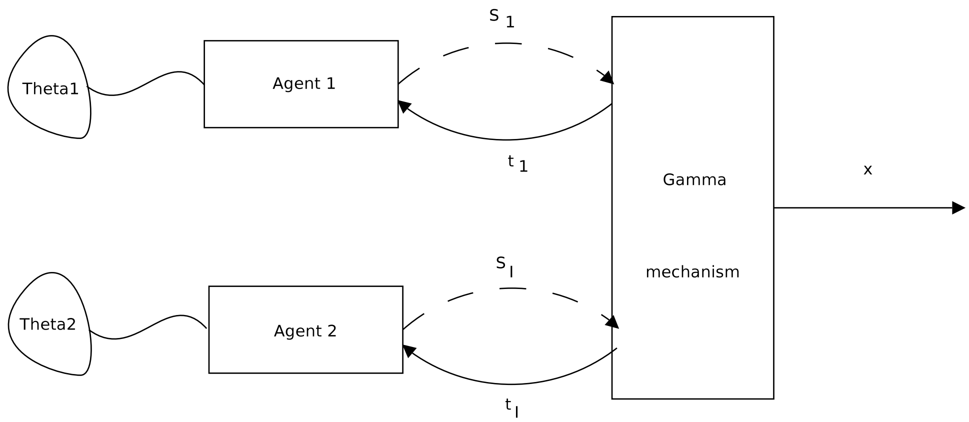

Figure 1: Mechanism

3.1 Mechanism

Definition: A mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ is a collection of strategy sets $(S_1, \ldots, S_I)$ and an outcome function $g : S_1 \times \cdots \times S_I \to Y$.

$\Rightarrow$ Defines a Bayesian game.

- Payoffs: $u_i(g(S_1, \ldots, S_I), \theta_i)$
- Strategy: $s_i : \Theta_i \to S_i$

Consider an equilibrium

3.2 Dominant Strategy Equilibrium of the Mechanism

Dominant strategy equilibrium of the mechanism is $S^*(\cdot)$ if $\forall i, \theta_i$,

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i) \qquad \forall s_i' \; and \; \forall s_{-i}$$

Definition: The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ implements the social choice function $f(\cdot)$ in dominant strategies if exists a dominant strategy equilibrium of Γ, such that $\forall \theta$, $g(S^*(\theta)) = f(\theta)$.

3.3 Direct mechanism

$$S_i = \Theta_i$$

Definition: The social choice function (scf) $f(\cdot)$ is truthfully implementable in dominant strategies (incentive compatible) if the direct mechanism$\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$ has a dominant strategy equilibrium $s^*(\cdot)$ such that $s_i^*(\theta_i) = \theta_i$. Truthfully implemntable in dominant strategies $\to$ Strategy Proof mechanism.

3.4 Quasilinear utilities

$$u_i(y, \theta_i) = v_i(x, \theta_i) + t_i \quad for \quad i = 1, \ldots, I$$

Social choice function: $f(\cdot) = (x(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ interested in efficient social choice functions.

1. Efficient allocations: $\forall \theta, x^*(\theta)$ must satisfy

$$\sum_{i=1}^{I} v_i(x^*(\theta), \theta_i) \geq \sum_{i=1}^{I} v_i(x, \theta_i) \quad \forall x \in X$$

$\therefore x^*(\theta)$ maximizes the social utility.

3.5 Truthful Implementation of Efficient Allocations

Proposition: (Groves) The social choice function (scf) $f(\cdot) = (x^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is truthfully implementable in dominant strategy if

$$\forall i = 1, \ldots, I, t_i(\theta) = [\sum_{j \neq i} v_j(x^*(\theta), \theta_j)] + h_i(\theta_{-i})$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

Proof in the homework.

3.5.1 Special Case

Clarke Mechanism

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(x^*_{-i}(\theta_{-i}), \theta_j)$$

where

$$x^*_{-i}(\theta_{-i})$$

satisfies

$$\sum_{j \neq i} N_j(x^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(x, \theta_j) \qquad \forall x \in X, \forall \theta_{-i}$$

$x^*_{-i}(\cdot)$: efficient allocation with $I - 1$ agents

3.5.2 Transfer in Clarke Mechanism

$$t_i(\theta) = [\sum_{j \neq i} v_j(x^*(\theta), \theta_j)] - [\sum_{j \neq i} v_j(x^*_{-i}(\theta_{-i}), \theta_j)]$$

first sum is I and second sum is II.

3.5.3 Remark

1. $t_i(\theta) = 0$ if his announcement does not change the allocation decision. $t_i(\theta) < 0$ if it does; i.e. if he is "pivotal" to effective allocation. i needs to pay the amount of resource he uses or the damage he imposes on the system and others.

2. Consider allocation of a single indivisible good. *Clarke mechanism*: social choice function implemented by second price auction which is Vickrey auction.

   - $x^*(\theta)$: allocate to the highest valuation buyer.
   - "pivotal" when he has the highest valuation
   - when pivotal, tax $\rightarrow I = 0$ and $II = Second\ Highest\ Valuation$.

   $\Rightarrow$ Hence the name "VCG".