6.972: Game Theory	February 15, 2005

Lecture 5: The Existence of the Nash Equilibrium

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1 Introduction

Recall a *correlated equilibrium* (CE) is a pdf $s = \{s_1..s_I\} \in S$ such that $\forall i \in \mathcal{I}, t_i \in S_i$,

$$\sum_{S_{-i}} P(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{S_{-i}} P(t_i, s_{-i})u_i(t_i, s_{-i}).$$

Remark:

1. A mixed strategy NE is a CE (i.e. $NE \subseteq CE$)
2. The Set of CE is convex (specified by a finite number of linear inequalities).
3. Contains convex hull of Mixed Strategy Nash Equilibria
4. $NE \subseteq CE \subseteq R^\infty \subseteq D^\infty$ where R^∞ is the set of rationalizable strategies, and D^∞ is the set of strategies which survive dominance.

Traffic Intersection Problem

	Stop	Go
Stop	$4,4$	$1,5$
Go	$5,1$	$0,0$

Previously Considered:

- 2 pure NEs $(1,5)$, $(5,1)$
- 1 mixed NE , each player choosing Stop or Go with $\frac{1}{2}$ probability
- 1 CE (there may be more), a traffic light with equal probability being either (Red, Green) or (Green, Red)

Exercise: Consider biased coins.

Claim: The following pdf representing a pdf over the profiles results in a CE.

	Stop	Go
Stop	1/3	1/3
Go	1/3	0

Definition 1 (Alternative Definition of a CE) *A* Correlated Equilibrium (CE) *is a pdf* $s = \{s_1..s_I\} \in S$ *such that* $\forall i \in \mathcal{I}, s_i \in S_i$,

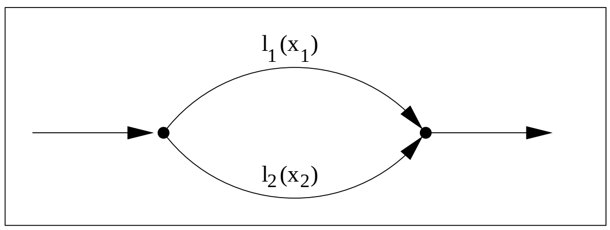
$$P(s_i) > 0 \Rightarrow \forall t_i \in S_i, \sum_{s_{-i} \in S_{-i}} P(s_{-i}|s_i) u_i(s_{-i}|s_i) \geq \sum_{s_{-i} \in S_{-i}} P(s_{-i}|s_i) u_i(s_{-i}|t_i).$$

For Player 1:

- $s_i = Stop:$
  $E(\text{Payoff} \mid s_i = Stop) = 4 \times \frac{1}{2} + 1 \times \frac{1}{2} \geq 5\frac{1}{2} + 0\frac{1}{2} = E(\text{Payoff} \mid s_i = Go)$
- $s_i = Go:$
  $E(\text{Payoff} \mid s_i = Go) = 5 \times \frac{1}{1} \geq 0 \times \frac{1}{1} = E(\text{Payoff} \mid s_i = Stop)$

There is no incentive for Player 1 to deviate unilaterally. By symmetry this also holds for Player 2. Therefore this is a CE.

Pricing-Congestion Problem



- The above represents a number of small consumers (whose usage sums to at most 1) who have the option of using either link.
- $l_i(x_i)$ is the latency of link i based on the total flow x_i over the link i.
- Two providers p_1, p_2 setting self-named prices per unit bandwidth over their associated link.
- The effective cost for a consumer using link i is $p_i + l_i(x_i)$.
- If $p_i + l_i(x_i) > 1$, consumers will choose to not to participate.

Question: If $l_1(x_1) = 0$ and $l_2(x_2) = \frac{3x_2}{2}$, how is flow allocated over the two links?

By Wardrop's Principle, $x = [x_i]$ is an equilibrium if:

$$x_i > 0 \Rightarrow p_i + l_i(x_i) = \min_j (p_j + l_j(x_j))$$

$$p_i + l_i(x_i) \leq 1$$

$$\sum_i x_i \leq 1.$$

The second inequality above essentially puts a price cap on the providers so their prices do not go to infinity. Alternatively, we can view it as consumers having a reservation utility 1 for not sending any flow over the links, so they would send flow only if their effective cost is less than 1. From above, given p_1 and p_2, we can determine the usage of each link.

$$(x_1, x_2) = \begin{cases} (1 - \frac{2}{3}(p_1 - p_2), \frac{2}{3}(p_1 - p_2)), & p_2 \leq p_1 < 1 \\ (x_1 \leq 1 - \frac{2}{3}(p_1 - p_2), \frac{2}{3}(p_1 - p_2)), & p_2 \leq p_1 = 1 \\ (x_1 \leq 1, 0), & p_1 < p_2,\ p_1 \leq 1 \\ (0,0), & \text{otherwise} \end{cases}$$

The payoffs for p_1 and p_2:

$$p_1\!: \ u_1(p_1, p_2) = p_1 * X_1(p_1, p_2)$$
$$p_2\!: \ u_2(p_1, p_2) = p_2 * X_2(p_1, p_2)$$

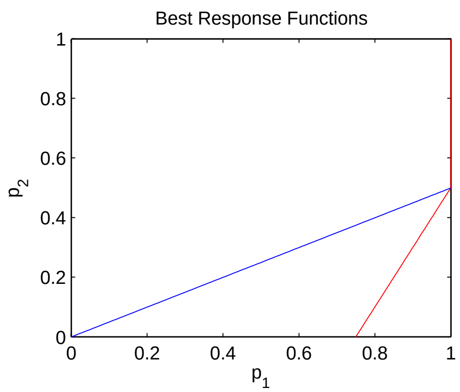
Strategic Form: 2 players with $S_i = [0, 1]$.

We attempt to find pure strategy NE by finding the intersection of the best response functions

$$B_1(p_2) = arg \max_{p_1, x_1, x_2} s.t. \begin{cases} p_1 = p_2 + \frac{3x_2}{2} \\ p_1 \leq 1 \\ x_1 + x_2 \leq 1 \end{cases}$$

therefore:

$$B_1(p_2) = \min(1, \frac{3}{4} + \frac{p_2}{2}) \ \& \ B_2(p_1) = \frac{p_1}{2}.$$



In the figure above, $B_1(p_2)$ is in red and $B_2(p_1)$ is blue. We see these functions intersect at $(p_1, p_2) = (1, \frac{1}{2})$, which is the only pure strategy equilibrium.

Question: If $l_1(x_1) = 0$ and $l_2(x_2) = \begin{cases} 0 & x_2 \leq \frac{1}{2} \\ \infty & x_2 > \frac{1}{2} \end{cases}$, how is flow allocated over the two links?

Claim: Any possible pure strategy NE has a unilateral strategy change for a player.

- $p_1 = p_2 = 0 \rightarrow x_1 \geq \frac{1}{2}, x_2 \leq \frac{1}{2}$. p_1 increases to get positive profit.
- $p_1 = p_2 > 0, x_1 = 1$. p_2 reduces price by ϵ to get $\frac{1}{2}$ flow
- $p_1 = p_2 > 0, x_1 < 1$. p_1 reduces price by ϵ to get full flow
- $p_1 < p_2$. p_1 increases to $p_2 - \epsilon$
- $p_1 > p_2$. p_2 increases to $p_1 - \epsilon$

Therefore no pure strategy NE exists.

Existence of NE

Consider a finite Strategic Game.

Matching Pennies Game: has no Pure Strategy NE, but does have a mixed strategy NE with equal probability for each profile.

Recall: σ^* is a NE if $\forall \sigma_i \in \Sigma_i$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$.

Further, $\sigma_i^* \in B_{-i}^*(\sigma_{-i}^*)$ where $B_{-i}^*(\sigma_{-i}^*)$ is the best response of player i, given that the other players' strategies are σ_{-i}^*.

We define:

$$B(\sigma^*) = \begin{cases} B_1(\sigma_{-1}^*) \\ \vdots \\ B_I(\sigma_{-I}^*) \end{cases} \ni \begin{bmatrix} \sigma_1^* \\ \vdots \\ \sigma_I^* \end{bmatrix} = \sigma^*$$

More precisely this defines a correspondence $B : \Sigma \rightarrow \Sigma$ with $B(\sigma) = [B_i(\sigma)]_{i \in \mathcal{I}}$

Question: Does there exist σ^*, such that $\sigma^* \in B(\sigma^*)$?

More on this next lecture.