6.972: Game Theory February 24, 2005

Lecture 7: Supermodular games

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1 Agenda

In this lecture, we discuss:

- Existence of Nash equilibrium for continuous strategy spaces (Glicksberg's theorem).
- Supermodular Games.

2 Existence of Nash Equilibria for continuous strategy spaces

In last lecture, we proved the existence of a pure strategy NE for continuous strategy spaces (Debreu, Glicksberg, Fan Theorem) under assumptions of:

- nonempty, convex, compact strategy spaces S_i.
- payoff functions $u_i(s)$ continuous in s.
- $u_i(s_i, s_{-i})$ concave in s_i (quasi-concave is sufficient).

Question: What if we relax the quasi concavity assumption? Think about approximating finite strategy spaces by continuous spaces: the corresponding payoff functions will not be quasi concave. We use mixed strategies to obtain convex-valued best responses:

Theorem 1 (Glicksberg) *Consider a strategic form game $< \mathcal{I}, (S_i), (u_i) >$. If the utility functions $u_i(s)$ are continuous in s, then there exists a mixed strategy Nash Equilibrium.*

But continuity of the payoffs is still a strong assumption. For discontinuous payoffs, use Dasgupta and Maskin Theorem.
Examples: Hotelling competition, Section 12.2 of Fudenberg and Tirole.

3 Supermodular Games

Idea: Supermodular games are characterized by strategic complementarities. When a player takes a higher action according to a defined order, the other is better off if he also takes a higher action; we have increasing best responses.
Motivation:

- It ensures existence of a pure strategy NE without requiring quasi concavity of payoffs.
- The equilibrium set has an order structure with extremal elements, for which there exists a simple algorithm.
- It behaves well under various learning rules.

- Relation to global games and equilibrium selection.

Literature:

- Topkis (1979): Lattice programming and monotonicity of optimal solutions.
- Vives (1990).
- Milgram and Roberts (1990).
- Vives (2002, 2001: Oligopoly pricing).

3.1 Monotonicity of Optimal Solutions

Consider $x(t) = \arg\max_{x \in X} f(x,t)$. When is $x(t)$ increasing in t?

Definition 1 (Order Structure) *We consider a finite dimensional Euclidian space* $\mathbb{R}^K$ *with order* $\geq$ *defined as :* $x \geq y \Leftrightarrow x_k \geq y_k \ \forall k = 1, \ldots, K$.

Note, this relation defines a partial order on the space.

Definition 2 (Definition) *Let* T *be a partially ordered set w.r.t order structure* $\geq$ *and* $X \subseteq \mathbb{R}$. *A function* $f : X \times T \longrightarrow \mathbb{R}$ *has* increasing differences *in* (x,t) *if:*

$$\forall x' \geq x, \ \forall t' \geq t$$
$$f(x',t') - f(x,t') \geq f(x',t) - f(x,t)$$

i.e $f(x',t) - f(x,t)$ *is increasing in* t *and symmetrically* $f(x,t') - f(x,t)$ *is increasing in* x.

Lemma 1 *For a function* $f \in C^2$ *(twice continuously differentiable),* f *has increasing differences iff:*

$$t' \geq t \ \Rightarrow \ \frac{\partial f}{\partial x}(x,t') \geq \frac{\partial f}{\partial x}(x,t)$$
$$\Leftrightarrow \ \frac{\partial^2 f}{\partial x \partial t}(x,t) \geq 0 \ \forall x \in X, \ \forall t \in T.$$

3.1.1 Examples

Wireless uplink power control

References

- Saraydar, Mandayan, Goodman (2001): *Efficiency power control via pricing in wireless data networks.*
- Altman (2003): *Supermodular games and power control in wireless networks.*

Characteristics of uplink power control problem in a single cell CDMA wireless data system. The resource to allocate is the power. There is interference between users. In the classical power control for voice, decisions are done centrally by the base station.

In a CDMA wireless data system, all users access the channel using orthogonal codes at the same time, utilizing the entire available frequency spectrum (unlike TDMA, FDMA). The new paradigm is that users are assigned utilities as a function of the power they consume and the SIR (Signal-to-interference Ratio) they attain; each user competes selfishly for power.

- Voice traffic as a function of the SIR is a step function with support $[\gamma_{min};\infty)$.
- The data utility function $u(\gamma)$ is a function of the SIR γ.
- p_i denotes the power (e.g battery life on a portable device).
- Matched filter receiver:

$$\gamma_i = \frac{h_i p_i}{\sum_{j\neq i} h_j p_j + \sigma^2}$$

  where h_i is the channel gain from mobile to base, and σ^2 is the noise variance.

- $u_i(p_i, p_{-i}) = f(\gamma_i) - cp_i$ with $f(.)$ increasing not necessarily concave.

We need to check whether the function u_i is supermodular, the partial derivatives are:

$$\frac{\partial u_i}{\partial p_i}(p_i, p_{-i}) = f'(\gamma_i)\frac{\gamma_i}{p_i} - c$$

$$\forall j \neq i \quad \frac{\partial^2 u_i}{\partial p_i \partial p_j}(p_i, p_{-i}) = \frac{\gamma_i^2 h_j}{p_i^2 h_i}(-\gamma_i f''(\gamma_i) + f'(\gamma_i))$$

The elasticity is less than 1, i.e $\frac{\gamma f''(\gamma)}{f'(\gamma)} < 1$, or $-\gamma_i f''(\gamma_i) + f'(\gamma_i) > 0$. Hence the cross partial derivatives are positive, and u_i is supermodular.
One examples of such a function is $f(\gamma) = \gamma^{1+\alpha}$ with $\alpha < 1$; the elasticity is $\alpha < 1$.

Oligopoly Models Example: Cournot competition in a duopoly:
2 firms choose the quantity they produce $q_i \in [0;\infty)$.
The inverse demand function is: $p(q_i, q_j) \in C^2$; it is a function of $Q = q_i + q_j$.
The payoff functions of the firms are: $u_i(q_i, q_j) = q_i p(q_i + q_j) - c(q_i)$.
The marginal revenue is: $\frac{\partial u_i}{\partial q_i}(q_i, q_j) = p(q_i, q_j) + q_i \frac{p}{q_i} = p + q_i p'$: it is decreasing in q_j.
Hence $\frac{\partial^2 u_i}{\partial q_i \partial q_j} = p' + q_i p'' < 0$.
Let us consider the transformed game: $s_1 = q_1$, $s_2 = -q_2$: it is easy to see that this game is supermodular.

Monotonicity of optimal solutions (continued). Consider $x(t) = \arg\max_{x\in X} f(x,t)$. When is $x(t)$ increasing in t?

Theorem 2 (Topkis) *Assume that $f : X \times T \longrightarrow \mathbb{R}$ where $X \subseteq \mathbb{R}$ is compact and T is a partially ordered set.*
Also assume $f(.,t)$ to be upper semi-continuous in x $\forall t \in T$.
Define $x(t) = \arg\max_{x\in X} f(x,t)$. Then we have:

1. *$\forall t \in T$, $x(t) \neq \emptyset$ and has a greatest and least element denoted respectively $\bar{x}(t)$ and $\underline{x}(t)$.*
2. *$\bar{x}(t)$ and $\underline{x}(t)$ are increasing in t.*

Proof:

1. Compactness (i.e closeness and boundedness in finite dimension) and upper semi-continuity of f imply the existence of a maximum. Hence $x(t)$ is nonempty $\forall t$:
   Let $\{x^k\}$ be a sequence of points in $x(t)$: since X is compact, there is a limit point $\bar{x}$:
   $f(x^k, t) \geq f(x, t)\ \forall x \in X$.
   By u.s.c of f, $f(\bar{x}, t) \geq \limsup_{k\to\infty} f(x^k, t) \geq f(x, t)\ \forall x \in X$.
   Hence $\bar{x}$ belongs to $x(t)$.
   $x(t)$ is therefore closed, and it is bounded since contained in X, i.e it is compact. It therefore has a greatest and least element.

2. Let $t' \geq t$. Let $x \in x(t)$ and $x' = \bar{x}(t')$; then we have:
   $f(x, t) - f(\min(x, x'), t) \geq 0$
   $f(\max(x, x'), t') - f(x', t') \geq 0$
   By increasing differences of f, we get:
   $f(\max(x, x'), t') - f(x', t') \geq 0$
   Thus $\max(x, x')$ maximizes $f(., t')$, i.e $\max(x, x')$ belongs to $x(t')$. As x' is the greatest element of the set $x(t')$, we conclude that $\max(x, x') \leq x'$, thus $x \leq x'$.
   $\therefore$ if f has increasing differences, the set of maxima $x(t)$ is increasing in the sense that both $\bar{x}(t)$ and $\underline{\mathrm{x}}(t)$ are increasing.

□

3.2 Supermodular Games

Definition 3 (Supermodular Game) *$< \mathcal{I}, (S_i), (u_i) >$ is a supermodular game if for all i:*

1. *S_i is a compact subset of $\mathbb{R}$ (S_i is a sublattice of $\mathbb{R}^m$);*

2. *u_i is upper semi continuous in s_i, continuous in s_{-i};*

3. *u_i has increasing differences in (s_i, s_{-i}) (u_i supermodular in (s_i, s_{-i}))*

Definition 4 (Upper semi-continuity) *A function f is said to be upper semi-continuous at x $\Longleftrightarrow$:*
$\forall\{x^k\}$ converging to x, we have: $f(x) \geq \limsup_{k\to\infty} f(x^k)$

Applying Topkis's theorem, the best responses are increasing in actions of others.

Corollary 1 *Assume $< \mathcal{I}, (S_i), (u_i) >$ is a supermodular game.*
Let $B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$. Then:
$B_i(s_{-i})$ has a greatest and least element, denoted $\bar{B}_i(s_{-i})$ and $\underline{B}_i(s_{-i})$;
$\bar{B}_i(s_{-i})$ and $\underline{B}_i(s_{-i})$ are increasing in s_{-i}.

Theorem 3 *Let $< \mathcal{I}, (S_i), (u_i) >$ be a supermodular game.*
Then the set of strategies that survive Iterated Strict Dominance (ISD) (iterated elimination of strictly dominated strategies) has greatest and least elements $\bar{s}$ and $\underline{s}$, which are pure strategy Nash Equilibria.

Corollary 2

1. *Pure strategy NE exist;*

2. *The largest and smallest strategies are compatible with ISD, rationalizability, CE, NE are the same;*

3. If a supermodular game has a unique NE, it is dominance solvable (ISD).

Proof: Idea: iterate best response mapping.

- $S^0 = S$, denote $s^0 = (s_1^0, \ldots, s_I^0)$ the largest element of S.
  Let $s_i^1 = \bar{B}_i(s_{-i}^0)$ and $S_i^1 = \{s_i \in S_i^0 : s_i \leq s_i^1\}$.
  Claim: Any $s_i > s_i^1$ i.e any $s_i \notin S_i^1$ is dominated by s_i^1.
  By increasing differences, $\forall s_{-i}$:
  $u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) \leq u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) < 0$, the last inequality stemming from the fact that s_i is not a best response to s_{-i}^0.
  $s_i^1 \leq s_i^0 \;\Rightarrow\; s^1 \leq s^0$
  iterate k: $s_i^k = \bar{B}_i(s_{-i}^{k-1})$; $S_i^k = \{s_i \in S_i^{k-1} : s_i \leq s_i^k\}$.
  Assume $s^k \leq s^{k-1}$. Then :
  $s_i^{k+1} = \bar{B}_i(s_{-i}^k) \leq \bar{B}_i(s_{-i}^{k-1}) = s_i^k$
  $\therefore$ $\{s^k\}$ is a decreasing sequence, bounded below, hence it has a limit denoted $\bar{s}$.
  Only the strategies $s_i \leq \bar{s}_i$ are undominated.

- Similarly, start with $s^0 = (s_1^0, \ldots, s_I^0)$ the smallest element in S and identify s^0 with $\underline{s}$.

- Show that $\bar{s}$ and $\underline{s}$ are NE:
  $\forall i\ \forall s_i$: $u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$.
  By u.s.c of u_i in s_i and continuity of u_i in s_{-i}, taking the limit as $k \to \infty$ yields:
  $u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i})$.

□