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Phase transitions in ZN gauge theory and twisted ZN topological phases

Phase transitions in Z_N gauge theory and twisted Z_N topological phases

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We find a series of non-Abelian topological phases that are separated from the deconfined phase of Z_N gauge theory by a continuous quantum phase transition. These non-Abelian states, which we refer to as the "twisted" Z_N states, are described by a recently studied $U(1) \times U(1) \rtimes Z_2$ Chern-Simons (CS) field theory. The $U(1) \times U(1) \rtimes Z_2$ CS theory provides a way of gauging the global Z_2 electric-magnetic symmetry of the Abelian Z_N phases, yielding the twisted Z_N states. We introduce a parton construction to describe the Abelian Z_N phases in terms of integer quantum Hall states, which then allows us to obtain the non-Abelian states from a theory of Z_2 fractionalization. The non-Abelian twisted Z_N states do not have topologically protected gapless edge modes and, for $N > 2$, break time-reversal symmetry.

I. INTRODUCTION

Landau symmetry breaking theory[1,2] not only classifies a large class of symmetry breaking phases, it also tell us which pairings of phases are connected by continuous phase transitions. Now we know that there are many topologically ordered states[3] that cannot be described by Landau symmetry breaking theory. The next important issue is to understand which pairs of topological phases are connected by continuous phase transitions and what the critical properties of the transitions are.

One class of topological phases are Abelian fractional quantum Hall (FQH) states. Those states can be systematically described by the K matrix and the associated $U(1) \times U(1) \times \cdots$ Chern-Simons (CS) theory.[4,5] For such a class of topological states, we know that two Abelian FQH states described by K and

$$K' = \begin{pmatrix} K & l \\ l^T & p \end{pmatrix} \tag{1}$$

are connected by continuous phase transitions, provided that there is a periodic potential with a proper period.[6,7] This class of phase transitions is induced by anyon condensation described by Ginzburg-Landau Chern-Simons theory.[8] Another class of topological phases are described by gauge theory with a discrete, possibly non-Abelian, gauge group G. For such classes of non-Abelian topological states,[9,10] we also know that a pair of discrete gauge theories described by gauge groups G and G' can be connected by continuous phase transitions if $G' \subseteq G$ or $G \subseteq G'$. This class of phase transitions are induced by boson condensation described by the standard Anderson-Higgs mechanism of "gauge symmetry" breaking.[11,12]

But there exist more general classes of topological phases described by pattern of zeros,[13–16] Z_n vertex algera,[17–19] and/or string-net condensates.[20] The above picture of topological phase transitions is clearly incomplete. One attempt to find new classes of continuous topological phase transitions is introduced in Refs. 21 and 22, which describe a class of continuous phase transitions between Abelian FQH and non-

Abelian FQH[17,23] states, induced by anyon condensation. A special case of such class of continuous topological phase transition is induced by fermion condensation, which was first discussed in Refs. 24–26.

In Ref. 27, we studied $U(1) \times U(1) \rtimes Z_2$ CS theory with integral coupling constants (k,l) [see Eq. (24)]. When $l = 3$, it was found that the topological properties of this theory agree with those of the Z_4 parafermion FQH states[28] at filling fraction $\nu = k/(2k - 3)$, leading us to suggest that this was the long wavelength field theoretic description of these non-Abelian FQH states. This formulation of the effective field theory allowed us to show that there is a continuous phase transition, in the 3D Ising universality class, between the Z_4 parafermion states and the Abelian $(k,k,k - 3)$ states in bilayer quantum Hall systems.[21]

Subsequently, it was found that for more general values of the coupling constants, $k,l \neq 0$, the $U(1) \times U(1) \rtimes Z_2$ CS theory describes a series of non-Abelian FQH states—the orbifold FQH states.[22] The Z_4 parafermion FQH states are then a special case of these more general orbifold FQH states, which are separated from the $(k,k,k - l)$ states by a continuous 3D Ising phase transition.

However, it is also known that $U(1) \times U(1)$ CS theory need not describe only quantum Hall states. For a particular set of coupling constants, it can also describe time-reversal invariant topological phases. The Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} a_I \partial a_J, \tag{2}$$

with $K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$, describes the long-wavelength properties of the deconfined phase of Z_N gauge theory. Such a phase has N^2 topologically distinct Abelian quasiparticles. The quasiparticles can be labeled by their electric and magnetic charge, (e,m), for $e,m = 0, \ldots, N - 1$, and have spin em/N.

The above observation suggests that the $U(1) \times U(1) \rtimes Z_2$ CS theory, with $k = 0$, may also describe non-Abelian topological phases, although ones that may exist in frustrated spin models and that do not require the presence of a strong

external magnetic field. This raises many questions regarding whether such phases can be obtained from microscopic models with local interactions, how to develop a more complete theory for these possible states, how to understand the conditions under which they may occur, and how to understand their full topological order.

In this paper, we study the topological properties of these non-Abelian states and, by developing a slave-particle description of them, provide further evidence that they are physical in that they can be realized in a microscopic model with local interactions. This leads to a series of non-Abelian states, without protected gapless edge modes, that are separated from the Abelian Z_N states by a continuous 3D Ising phase transition. For $N > 2$, these non-Abelian states can only be accessed when time-reversal symmetry is broken.

The specific system that we use to establish our results is one with two flavors of strongly interacting bosons. However, the main purpose of the present work is to show that the topological phases and phase transitions that we discuss here exist, in principle, and to understand their topological properties; they may also appear in other models with different microscopic degrees of freedom.

We begin in Sec. II by developing a parton construction for the Abelian Z_N states, which allows us to describe these conventional phases in terms of integer quantum Hall states. The use of this formulation is that it allows us to access the $U(1) \times U(1) \rtimes Z_2$ phases through a theory of Z_2 fractionalization. In Sec. III, we then use this parton construction, in conjunction with a recently developed slave Ising formulation, to describe the twisted Z_N phases and argue that their low-energy theory should be the $U(1) \times U(1) \rtimes Z_2$ CS theory with suitably chosen coupling constants. In Sec. IV, we review the results of the $U(1) \times U(1) \rtimes Z_2$ CS theory. In Sec. V, we give a prescription, using conformal field theory, to derive the full topological order of these non-Abelian states; wherever comparison is possible, we find agreement with highly non-trivial results from the $U(1) \times U(1) \rtimes Z_2$ CS theory.

II. PARTON CONSTRUCTION FOR Z_N TOPOLOGICAL ORDER

In this section, we show how to construct a state with Z_N topological order by projecting from $\nu = 1$ IQH states.[29] We begin with two flavors of bosons, $b_\uparrow$ and $b_\downarrow$, and decompose them in terms of $3N$ partons as follows:

$$b_\uparrow = \prod_{i=1}^{N} \psi_i \prod_{j=2N+1}^{3N} \psi_j, \quad b_\downarrow = \prod_{i=N+1}^{2N} \psi_i \prod_{j=2N+1}^{3N} \psi_j, \quad (3)$$

where we have suppressed the space indices. Note that the partons $\psi_{2N+1}, \ldots, \psi_{3N}$ are shared between $b_\uparrow$ and $b_\downarrow$. We can rewrite the original theory of these two flavors of bosons in terms of a theory of these partons coupled to a gauge field. The gauge field projects the expanded parton Hilbert space onto the physical Hilbert space, which is generated by the physical operators $b_\uparrow$ and $b_\downarrow$.

Next, we assume a mean-field ansatz where $\psi_1, \ldots, \psi_{2N}$ form a $\nu = 1$ IQH state, while $\psi_{2N+1}, \ldots, \psi_{3N}$ form a $\nu = -1$ IQH state. The maximal gauge group that respects this mean-

field ansatz is $SU(N) \times SU(N) \times SU(N) \times U(1)$, which we will write as $SU(N)^3 \times U(1)$.

In order to motivate the above construction, note that bilayer ($NN0$) FQH states can be obtained through the parton construction by decomposing the electron operator in each layer as

$$\Psi_{e\uparrow} = \psi_1 \cdots \psi_N, \quad \Psi_{e\downarrow} = \psi_{N+1} \cdots \psi_{2N}, \quad (4)$$

and assuming a mean-field ansatz where the ψ_i each form $\nu = 1$ IQH states. It can be shown that the low-energy field theory for such a state is $U(1) \times U(1)$ CS theory with K matrix, $K = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$. In order to describe more general bilayer FQH states such as $(N + m, N + m, m)$, we simply multiply each electron operator by an additional set of operators:

$$\begin{aligned} \Psi_{e\uparrow} &= \psi_1 \cdots \psi_N \times \psi_{2N+1} \cdots \psi_{2N+m}, \\ \Psi_{e\downarrow} &= \psi_{N+1} \cdots \psi_{2N} \times \psi_{2N+1} \cdots \psi_{2N+m}, \end{aligned} \quad (5)$$

and we assume again that all of the partons ψ_i form a $\nu = 1$ IQH state. At the level of the wave functions, this has the effect of multiplying the ($NN0$) wave function by a Jastrow factor to give the $(N + m, N + m, m)$ wave function:

$$\Phi_{(N+m,N+m,m)} = \Phi_{(NN0)}\Phi_{(mmm)}. \quad (6)$$

Since the Z_N gauge theory is described by a K matrix,

$$K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} = \begin{pmatrix} N & N \\ N & N \end{pmatrix} - \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \quad (7)$$

a natural guess is the decomposition in Eq. (3), where $\psi_{2N+1}, \ldots, \psi_{3N}$ are assumed to form $\nu = -1$ IQH states.

The low-energy theory for such a state will involve the partons interacting with a gauge field from the gauge group $SU(N)^3 \times U(1)$. It is not at all clear that such a complicated field theory, with many non-Abelian gauge groups, has simply the Z_N topological order.

In Appendix B, we compute the ground-state degeneracy of the $SU(N)^3 \times U(1)$ theory on a torus. We find that it is given by

$$\text{torus degeneracy} = N^2, \quad (8)$$

which agrees with that of the Z_N topological order.

Unfortunately, besides the torus ground-state degeneracy, it is extremely difficult to compute any other topological properties of a theory with such a complicated non-Abelian gauge group. In order to proceed, we choose a mean-field ansatz for the partons that breaks the gauge group down to the center of $SU(N)^3 \times U(1)$, which is $U(1)^{3N-2}$. One way to do this, for example, is to assume various condensates such that in the low-energy field theory, the partons all have different masses, while still forming the IQH states described above. Since the gauge group is now Abelian, it is possible to compute all topological properties of the resulting states. In the following section, we show that such a gauge theory coupled to the partons describes the topological properties of the Z_N phase and directly yields the $U(1) \times U(1)$ mutual CS theory as its low-energy effective field theory.

A. Mutual $U(1) \times U(1)$ CS theory from parton construction

The effective field theory is described by the Lagrangian:

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + \psi^\dagger \frac{M^{-1}}{2}(\partial - iAQ)^2 \psi$$
$$+ \text{Tr}\left(j^\mu a_\mu^I p^I\right) + \cdots, \qquad (9)$$

where $\psi^T = (\psi_1, \ldots, \psi_{3N})$, $M_{ab} = m_a \delta_{ab}$, and m_a is the mass of the ath parton, A_i describes a magnetic field seen by the partons, $j^\mu_{ab} = \psi_a \partial^\mu \psi_b$ describes the current of the partons, a_μ is the gauge field, and the $3N - 2$ generators of the gauge group $U(1)^{N-1} \times U(1)^{N-1} \times U(1)^{N-1} \times U(1)$ are given by the matrices p^I:

$$p^I_{ij} = \delta_{ij}(\delta_{i,I} - \delta_{i,I+1}), \quad I = 1, \ldots, N-1,$$
$$p^I_{ij} = \delta_{ij}(\delta_{i,I+1} - \delta_{i,I+2}), \quad I = N, \ldots, 2N-2,$$
$$p^I_{ij} = \delta_{ij}(\delta_{i,I+2} - \delta_{i,I+3}), \quad I = 2N-1, \ldots, 3N-3,$$
$$p^I_{ij} = \delta_{ij}(\delta_{i,1} + \delta_{i,N+1} - \delta_{i,2N+1}), \quad I = 3N-2. \qquad (10)$$

Since the partons are in $\nu = 1$ IQH states, their action is each given by a $U(1)_1$ CS theory; because of the gauge constraint, they will be coupled to the gauge field as well:

$$\mathcal{L} = \mathcal{L}_{\text{parton}} + \mathcal{L}_{\text{constraint}},$$
$$\mathcal{L}_{\text{parton}} = \frac{1}{4\pi} \sum_{i=1}^{2N} b^i \partial b^i - \frac{1}{4\pi} \sum_{i=2N+1}^{3N} b^i \partial b^i, \qquad (11)$$
$$\mathcal{L}_{\text{constraint}} = \sum_{i=1}^{3N} j^i_\mu p^I_{ii} a^I_\mu,$$

where b^i is a $U(1)$ gauge field describing the current density of the ith parton:

$$j^i_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu b^i_\lambda. \qquad (12)$$

From the definition of the p^I, we see that integrating out the a gauge fields enforces the constraints:

$$j^1 = \cdots = j^N, \quad j^{N+1} = \cdots = j^{2N},$$
$$j^{2N+1} = \cdots = j^{3N}, \quad j^{2N+1} = j^{N+1} + j^1. \qquad (13)$$

Therefore the effective action becomes

$$\mathcal{L} = -\frac{N}{4\pi}(b^1 \partial b^{N+1} + b^{N+1} \partial b^1), \qquad (14)$$

which is exactly the action for the mutual $U(1) \times U(1)$ CS theory description of Z_N topological order. Actually, this analysis is essentially the same analysis that we intuited by analyzing wave functions in Eqs. (5)–(7).

When the masses of the partons are all equal, we see that the theory has the enhanced $SU(N) \times SU(N) \times SU(N) \times U(1)$ gauge symmetry. Since the number of states on the torus does not change when this gauge symmetry is broken to its Abelian subgroup by assuming different mean-field masses for the partons, we conjecture that it also describes the topological properties of the Z_N phases. This is a surprising result, for it provides an example in which gauge symmetry breaking does not actually change the topological properties of a state. This is related to the fact that sometimes a non-Abelian CS theory is equivalent to an Abelian CS theory. For example, non-Abelian

$SU(k)$ level 1 CS theory is equivalent to Abelian $U(1)$ level k CS theory.[30,31]

III. SLAVE ISING DESCRIPTION

The parton description presented in the previous section yields the mutual $U(1) \times U(1)$ CS theory at long wavelengths in a way that is amenable to a certain Z_2 "twisting." To do this, we follow the slave-Ising construction presented in Ref. 22 in the context of the orbifold non-Abelian FQH states. We start with two boson operators defined on a lattice, $b_{i\sigma}$, and we consider the positive and negative combinations:

$$b_{i\pm} = \frac{1}{\sqrt{2}}(b_{i\uparrow} \pm b_{i\downarrow}). \qquad (15)$$

We introduce two new fields at each lattice site i: an Ising field $s_i^z = \pm 1$ and a bosonic field d_{i-}, and we rewrite b_{i-} as

$$b_{i+} \equiv d_{i+}, \quad b_{i-} = s_i^z d_{i-}. \qquad (16)$$

This introduces a local Z_2 gauge symmetry, associated with the transformation

$$s_i^z \to -s_i^z, \quad d_{i-} \to -d_{i-}. \qquad (17)$$

The electron operators are neutral under this Z_2 gauge symmetry, and therefore the physical Hilbert space at each site is the gauge-invariant set of states at each site:

$$(|\uparrow\rangle + |\downarrow\rangle) \otimes |n_{d_-} = 0\rangle, \quad (|\uparrow\rangle - |\downarrow\rangle) \otimes |n_{d_-} = 1\rangle, \qquad (18)$$

where $|\uparrow\rangle$ $(|\downarrow\rangle)$ is the state with $s^z = +1(-1)$, respectively. In other words, the physical states at each site are those that satisfy

$$\left(s_i^x + 1\right)/2 + n_{d_{i-}} = 1. \qquad (19)$$

If we imagine that the bosons $d_{i\pm}$ form some gapped state, then we would generally expect two distinct phases:[32] the deconfined/Z_2 unbroken phase, where

$$\langle s_i^z \rangle = 0, \qquad (20)$$

and the confined/Higgs phase, where upon fixing a gauge, we have

$$\langle s_i^z \rangle \neq 0. \qquad (21)$$

We seek a mean-field theory where the deconfined phase has the properties described by the $U(1) \times U(1) \rtimes Z_2$ CS theory, and the confined/Higgs phase corresponds to the Z_N topological phases. To do this, observe that in the Higgs phase, we have

$$b_{i\pm} = d_{i\pm}, \qquad (22)$$

since we may set $s_i^z = 1$ in this phase. Now for this to describe the Z_N phases, we use the parton construction of Sec. II:

$$d_{i\pm} = \frac{1}{\sqrt{2}}(d_{i1} \pm d_{i2}), \quad d_{i1} = \psi_{1i} \cdots \psi_{Ni}\psi_{2N+1,i} \cdots \psi_{3N,i},$$
$$d_{i2} = \psi_{N+1,i} \cdots \psi_{2N,i}\psi_{2N+1,i} \cdots \psi_{3N,i}, \qquad (23)$$

and we assume that $\psi_1, \ldots, \psi_{2N}$ form a $\nu = 1$ IQH state while $\psi_{2N+1}, \ldots, \psi_{3N}$ form a $\nu = -1$ IQH state.

Clearly, the low-energy field theory of the confined phase is the mutual $U(1) \times U(1)$ CS theory, describing the Abelian

Z_N topological order. In the deconfined phase, we see that the parton sector is still described by a $U(1) \times U(1)$ CS theory, but that there is also an additional Z_2 gauge symmetry associated with exchanging the two $U(1)$ gauge fields. This is precisely the content of the $U(1) \times U(1) \rtimes Z_2$ CS theory,[27] which we therefore expect to describe the topological properties of this Z_2 deconfined phase.

Since the transition between these two phases is induced by the condensation of the Ising spin s_i^z, which is coupled to a Z_2 gauge field, we see that as the gap to the s^z excitations is reduced, the low-energy field theory is simply a real scalar field coupled to a Z_2 gauge field. Such a theory was analyzed in Ref. 11, where it was found that the transition is continuous and in the 3D Ising universality class. Therefore the Abelian Z_N and its Z_2 fractionalized neighbor, the "twisted" Z_N states, are separated by a continuous quantum phase transition.

A useful property of this slave Ising formulation is that standard methods of constructing projected trial wave functions will, when applied to the Z_2 deconfined phase, yield possible trial wave functions for these non-Abelian twisted Z_N states.[22,33]

IV. $U(1) \times U(1) \rtimes Z_2$ CS THEORY AND TOPOLOGICAL QUANTUM NUMBERS OF TWISTED Z_N STATES

The $U(1) \times U(1) \rtimes Z_2$ CS theory was studied in detail in Ref. 27. In this section, we review the results for the choice of coupling constants that is relevant here.

The $U(1) \times U(1) \rtimes Z_2$ CS theory is described by the Lagrangian

$$\mathcal{L} = \frac{k}{4\pi}(a\partial a + \tilde{a}\partial\tilde{a}) + \frac{k-l}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a), \quad (24)$$

where a and $\tilde{a}$ are two $U(1)$ gauge fields. Formally, this is the same Lagrangian as that of the $U(1) \times U(1)$ CS theories, although here we also have an additional Z_2 gauge symmetry associated with interchanging the two $U(1)$ gauge fields at each space-time point. This allows, e.g., for the possibility of Z_2 vortices—configurations in which the two $U(1)$ gauge fields transform into each other around the vortex—and twisted sectors on manifolds of nontrivial topology.

In order to describe the twisted Z_N topological phases, we choose $k = 0$ and $l = N$. In Ref. 27, we found that such a theory has $N(N+7)/2$ topologically distinct quasiparticles. The ground-state degeneracy on genus g surfaces is

$$S_g(N) = (N^g/2)[N^g + 1 + (2^{2g} - 1)(N^{g-1} + 1)]. \quad (25)$$

From $S_g(N)$, we can obtain the quantum dimensions of all the quasiparticles. The total quantum dimension is

$$D^2 = 4N^2. \quad (26)$$

There are three classes of quasiparticles: $2N$ quasiparticles with quantum dimension $d = 1$, $2N$ quasiparticles with quantum dimension $d = \sqrt{N}$, and $N(N-1)/2$ quasiparticles with quantum dimension $d = 2$.

The fundamental non-Abelian excitations in the $U(1) \times U(1) \rtimes Z_2$ CS theory are Z_2 vortices. In Ref. 27, we studied the number of degenerate ground states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere. The result

for the number of such states is

$$\alpha_n = \begin{cases} (N^{n-1} + 2^{n-1})/2 & \text{for } N \text{ even,} \\ (N^{n-1} + 1)/2 & \text{for } N \text{ odd.} \end{cases} \quad (27)$$

This shows that the quantum dimension of the Z_2 vortices is $d = \sqrt{N}$. We can also compute the number of states that are odd under the Z_2 gauge transformation. The number of these Z_2 noninvariant states turns out to be an important quantity, because it yields important information about the fusion rules of the quasiparticles. The number of Z_2 noninvariant states yields the number of ways for n pairs of Z_2 vortices to fuse to an Abelian quasiparticle that carries Z_2 gauge charge. The ground-state degeneracy of Z_2 noninvariant states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere was computed to be

$$\beta_n = \begin{cases} (N^{n-1} - 2^{n-1})/2 & \text{for } N \text{ even,} \\ (N^{n-1} - 1)/2 & \text{for } N \text{ odd.} \end{cases} \quad (28)$$

Thus if γ labels a Z_2 vortex, these calculations reveal the following fusion rules for γ and its conjugate $\bar{\gamma}$:

$$(\gamma \times \bar{\gamma})^n = \alpha_n \mathbb{I} + \beta_n j + \cdots, \quad (29)$$

where j is a topologically nontrivial excitation that carries the Z_2 gauge charge. The $\{\cdots\}$ represent additional quasiparticles that may appear in the fusion.

Note that the above is true also for $U(1) \times U(1) \rtimes Z_2$ CS theory with coupling constants $(k,l) = (N,0)$, which applies to bilayer FQH states. This indicates a close relation between the FQH phases with $(k,l) = (N,0)$ and the nonquantum Hall ones with $(k,l) = (0,N)$

The above gives us much information about the topological order of the twisted Z_N states, but we have not been able to compute the full topological order of these states directly form the $U(1) \times U(1) \rtimes Z_2$ CS theory. However, since we know that the twisted Z_N states contain a Z_2 charged boson—labeled s_i^z in the previous section and j here—whose condensation yields the Abelian Z_N states, we can deduce even more topological properties of the quasiparticles.

In our case, the two phases are separated by the condensation of a topologically nontrivial bosonic quasiparticle, j, that fuses with itself to a local topologically trivial excitation. Based on general considerations,[12] we expect the following regarding the topological quantum numbers of such phases. Upon condensation of j, quasiparticles that differed from each other by fusion with j become topologically equivalent. Quasiparticles that were nonlocal with respect to j before condensation become confined after condensation and do not appear in the low-energy spectrum. Finally, quasiparticles that fused with their conjugate to the identity and j will, after condensation, split into two topologically distinct quasiparticles. The spins of the quasiparticles remain unchanged through this process, which allows us to obtain information about the spins of some of the quasiparticles in the twisted Z_N states from knowledge of the spins of the quasiparticles in the Abelian Z_N states.

In the case of the twisted Z_N states, we have the following. The $2N$ Abelian quasiparticles, which contain the quasiparticle j, become N Abelian quasiparticles after condensation. The Z_2 vortices are clearly nonlocal with respect to the Z_2 charges, so they become confined. Finally, the $N(N-1)/2$

TABLE I. Some topological quantum numbers for quasiparticle excitations based on considerations of Sec. IV. A_l, for $l = 0, \ldots, 2N - 1$, labels the $2N$ Abelian quasiparticles. B_l, for $l = 0, \ldots, 2N - 1$, labels the Z_2 vortices. C_{mn}, for $m, n = 0, \ldots, N - 1$ and $m < n$, labels the $N(N-1)/2$ quasiparticles with quantum dimension 2. Note that the quasiparticles (e, m) in the Abelian Z_N states have spin em/N. Also note that the spin is meaningful only modulo 1.

	Spin	Quantum dimension
A_l	l^2/N	1
B_l	$\ldots$	$\sqrt{N}$
C_{mn}	mn/N	2

quasiparticles with quantum dimension 2 each split into two distinct quasiparticles. This yields the N^2 quasiparticles of the Abelian Z_N states. The natural interpretation is that the $N(N-1)/2$ quasiparticles correspond to the Z_2 invariant combinations of quasiparticles in the Abelian states: $(e,m) + (m,e)$ for $e \neq m$, while the $2N$ Abelian quasiparticles of the twisted Z_N states consist of the N diagonal quasiparticles (l,l), and their N counterparts that differ by fusion with j. Therefore we can infer the spins of these two classes of quasiparticles. The results are listed in Table I.

We still have not been able to compute the spins of the Z_2 vortices or the complete fusion rules of the quasiparticles. In the following section, we will present a prescription that enables us to calculate all of the topological properties of these twisted Z_N states.

V. CONFORMAL FIELD THEORY CONSTRUCTION AT $c - \bar{c} = 0$

The use of CFT techniques to compute topological quantum numbers for FQH states has been very powerful.[17–19] Physically, this is possible because the edge theory is described by CFT, and there is a correspondence between the spectrum of states in CFT and the topological properties of quasiparticles in the bulk of FQH states.[34] The prescription in those cases is to identify an appropriate set of CFTs, choose an appropriate electron operator, and then the quasiparticles are those operators that can be constructed that are mutually local with respect to the electron operator. Two quasiparticles that are related by electron operators are topologically equivalent. The topological spin of the quasiparticles then is believed to follow from the scaling dimension of the quasiparticle operator in the CFT, while the fusion rules of the quasiparticles are equivalent to the fusion rules, with respect to the electron chiral algebra, of the quasiparticle operators in the CFT.

In the case of the twisted Z_N states, we do not expect to have topologically protected edge modes. However, under certain symmetry, gapless edge modes described by CFT can exist.[35] So the CFT prescription can still be used to yield possible full sets of topological quantum numbers. Physically, we can think of this as the CFT that describes gapless edge excitations for these states, although it is unstable to opening up a gap. In this section, we will give a prescription to compute the topological properties from CFT. While we cannot prove that the topological quantum numbers are precisely those of the

$U(1) \times U(1) \rtimes Z_2$ CS theory, they are consistent with all of the highly nontrivial results of the previous section. Additionally, based on the relation of these twisted Z_N states to their FQH counterparts, the orbifold FQH states,[22] we have even more reason to believe that the prescription given here is the correct one. We expect it possible to prove that the topological quantum numbers found using this prescription are in fact the unique consistent set that is also consistent with results that can be deduced from the $U(1) \times U(1) \rtimes Z_2$ CS theory.

The construction is analogous to the orbifold FQH states,[22] except we take the anti-holomorphic part of the Z_2 orbifold as the non-Abelian part of the CFT instead of the holomorphic part; the "charge" part is the $c = 1$ chiral (holomorphic) scalar field. Thus the total central charge of the CFT is $c_{\text{tot}} = c + \bar{c} = 2$, while the difference in central charges is $c_{\text{rel}} = c - \bar{c} = 0$; this indicates that such a phase would have zero thermal Hall conductance, as expected from the fact that it does not have protected edge modes (see Sec. VII C).

We could also take the holomorphic part of the Z_2 orbifold as the non-Abelian part, and the "charge" part to be antiholomorphic. This would yield the time-reversed counterpart of this phase.

The operator content of the Z_2 orbifold CFT is reviewed in Appendix A. For the twisted Z_N states, we take the "electron" operator to be

$$V_e(z, \bar{z}) = \bar{\phi}_N^1(\bar{z}) e^{i\sqrt{\nu^{-1}}\varphi(z)}, \tag{30}$$

where $\nu = 2/N$. The quasiparticle operators V_q are those operators that are mutually local with respect to the electron operator:

$$V_q(z, \bar{z}) = \mathcal{O}(\bar{z}) e^{i Q \sqrt{\nu^{-1}}\varphi(z)}. \tag{31}$$

The OPE of V_q with V_e is

$$V_q(w, \bar{w}) V_e(z, \bar{z}) \sim (w - z)^{Q/\nu}(\bar{w} - \bar{z})^{h_{\mathcal{O}_2} - h_{\mathcal{O}} - h_{\bar{\phi}_N^1}} \mathcal{O}_2 + \cdots. \tag{32}$$

Thus for V_q to be local with respect to V_e, we require

$$Q/\nu - (h_{\mathcal{O}_2} - h_{\mathcal{O}} - h_{\bar{\phi}_N^1}) = \text{integer}. \tag{33}$$

Two quasiparticle operators are topologically equivalent if they can be related by the electron operator. Proceeding in this fashion, we find topological orders that agree with the results of the previous section. This construction allows us to obtain all of the topological information of the twisted Z_N phases. In the next section, we list examples of results that we obtain from this construction.

We note that this is an interesting non-trivial example of the CS/CFT correspondence[30] because the boundary CFT in this case contains both holomorphic and antiholomorphic parts that are glued together in a special way.

VI. EXAMPLES

In this section, we list results obtained from the CFT consideration for different twisted Z_N states. For $N = 3$, the results are summarized in Table II. We see that there are 15 types of quasiparticles. Those particles carry fractional angular momentum, which we call spin.[36] Note that the spin (or angular momentum) does not have to be multiples of $\hbar/2$ in $2 + 1$D.

TABLE II. Quasiparticle operators for CFT construction of twisted Z_3 phase.

	CFT Label	Quantum dimension	Spin
0	$\mathbb{I}$	1	0
1	$e^{i2/3\sqrt{3/2}\varphi}$	1	$0 + 1/3 = 1/3$
2	$\phi_N^2 e^{i1/3\sqrt{3/2}\varphi}$	1	$-3/4 + 1/12 \sim 1/3$
3	j	1	$-1 + 0 \sim 0$
4	$j e^{i2/3\sqrt{3/2}\varphi}$	1	$-1 + 1/3 \sim 1/3$
5	$\phi_N^1 e^{i1/3\sqrt{3/2}\varphi}$	1	$-3/4 + 1/12 \sim 1/3$
6	$\sigma_1 e^{i1/2\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16 + 3/16 = 1/8$
7	$\sigma_2 e^{i1/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16 + 1/48 = -1/24$
8	$\sigma_2 e^{i5/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16 + 25/48 = 11/24$
9	$\tau_1 e^{i1/2\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16 + 3/16 \sim 5/8$
10	$\tau_2 e^{i1/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16 + 1/48 \sim 11/24$
11	$\tau_2 e^{i5/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16 + 25/48 = -1/24$
12	$\phi_1 e^{i1/3\sqrt{3/2}\varphi}$	2	$-1/12 + 1/12 = 0$
13	$\phi_2 e^{i0\sqrt{3/2}\varphi}$	2	$-1/3 + 0 \sim 2/3$
14	$\phi_2 e^{i2/3\sqrt{3/2}\varphi}$	2	$1/3 - 1/3 = 0$

The spin of a quasiparticle can be measured by putting the system on a sphere or on other curved spaces.

For $N = 2$, we have nine quasiparticles, as summarized in Table III. It appears that this coincides with the Ising $\times$ $\overline{\text{Ising}}$ topological order. Condensation of the boson $\psi \otimes \bar{\psi} = j$ yields the Z_2 topological order.

VII. DISCUSSION

A. Transition to twisted Z_N topological phases

Let γ denote an anyon with statistical angle $\theta = 2\pi/N$ in a topological phase, and let m control the mass of, or energy gap to creating, γ. As we tune m, γ may condense and drive a phase transition to a new phase. This transition can be described by the $\langle\phi\rangle = 0 \to \langle\phi\rangle \neq 0$ transition in a Chern-Simons Ginzburg-Landau theory:

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - f|\phi|^2 - g|\phi|^4$$
$$- \frac{\pi}{\theta}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}. \quad (34)$$

TABLE III. Quasiparticle operators for CFT construction of twisted Z_2 phase. Note that this is equivalent to Ising $\times$ $\overline{\text{Ising}}$.

	CFT Label	Quantum dimension	spin	Ising $\times$ $\overline{\text{Ising}}$ fields
0	$\mathbb{I}$	1	$0 + 0 = 0$	$\mathbb{I} \otimes \mathbb{I}$
1	ϕ_N^1	1	$-1/2 + 0 = 1/2$	$\psi \otimes \mathbb{I}$
2	j	1	$-1 + 0 \sim 0$	$\psi \otimes \bar{\psi}$
3	ϕ_N^2	1	$-1/2 + 0 \sim 1/2$	$\mathbb{I} \otimes \bar{\psi}$
4	$\sigma_1 e^{i1/2\sqrt{2}\varphi}$	$\sqrt{2}$	$-1/16 + 1/8 = 1/16$	$\sigma \otimes \mathbb{I}$
5	σ_2	$\sqrt{2}$	$-1/16 + 0 = -1/16$	$\sigma \otimes \bar{\psi}$
6	τ_2	$\sqrt{2}$	$-9/16 + 0 = -9/16$	$\mathbb{I} \otimes \bar{\sigma}$
7	$\tau_1 e^{i1/2\sqrt{2}\varphi}$	$\sqrt{2}$	$-9/16 + 1/8 \sim 9/16$	$\psi \otimes \bar{\sigma}$
8	$\phi_1 e^{i1/2\sqrt{2}\varphi}$	2	$-1/8 + 1/8 = 0$	$\sigma \otimes \bar{\sigma}$

In the above Lagrangian, the anyon number is conserved in this case, the anyon condensation induces a transition between Abelian states described by different K matrices.

In the case where γ is only conserved modulo N, there will be an additional term in the Lagrangian:

$$\delta\mathcal{L} = t(\phi\hat{M})^N + \text{H.c.} \quad (35)$$

In this case, the anyon condensation may induce a transition between Abelian and non-Abelian states.

In our study of bilayer quantum Hall phase transitions in Ref. 21, it was suggested that this transition, in the presence of the $\delta\mathcal{L}$ term, may be dual to a 3D Ising transition. In those cases, one starts from an Abelian bilayer FQH phase and obtains the non-Abelian orbifold FQH states by tuning the interlayer tunneling and/or interlayer repulsion. We may obtain a similar situation in the context of Z_N gauge theory if we reduce the energy gap to the $(1,1)$ quasiparticles (the bound state of a single electric and a single magnetic quasiparticle). The $(1,1)$ quasiparticles are conserved only modulo N, because there is no additional conserved $U(1)$ charge as in the FQH phases. This implies the possibility of an analog of the bilayer $(NN0)$ FQH phase transitions studied earlier but for a system in the absence of a magnetic field and with no protected edge modes. The $(1,1)$ quasiparticle plays the role of the f-exciton, both of which have statistical angle $\theta = 2\pi/N$. Tuning the interlayer repulsion is equivalent to tuning the attraction between the minimal electric and magnetic quasiparticles.

Note that while the Z_N phase can be obtained in a time-reversal invariant system, condensing the $(1,1)$ quasiparticle breaks time reversal for $N > 2$. Therefore consider starting with the Hamiltonian that gives deconfined Z_N and adding a term that can tune the attraction between the minimal electric and magnetic quasiparticles. This will reduce the energy gap to their bound state, and may be used to tune through a 3D Ising phase transition. The phase that appears after the transition, in analogy to the bilayer FQH cases, may be the twisted Z_N gauge theory, described by $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory.

B. Time-reversal invariance

We see that for $N > 2$, the topological quantum numbers break time reversal symmetry; there is no way that a topological phase with these quantum numbers can preserve time-reversal symmetry. In fact, we saw that we had a choice of whether to pick the holomorphic part of the Z_2 orbifold and the antiholomorphic part of the $U(1)$ sector, or vice versa. This fact at first appears worrisome, because these phases are separated from the Z_N Abelian phases through a 3D Ising transition, and the Z_N phases are time-reversal invariant phases. In the following, we outline reasons to believe that indeed these twisted Z_N phases are not time-reversal invariant for $N > 2$.

First, observe that for $N > 2$, the number of quasiparticles in these phases is not a perfect square. Typically, almost all time-reversal invariant topological phases are "doubled" theories in the sense that mathematically they are described by $G \otimes \bar{G}$ modular tensor categories, where G is itself a modular tensor category and $\bar{G}$ is its time-reversed partner. More in depth considerations also suggest that for $N > 2$, there is no consistent topological phase that is time-reversal invariant

and that has $N(N + 7)/2$ quasiparticles with the quantum dimensions described in Sec. IV.[37]

In addition to general considerations of what mathematically consistent time-reversal invariant topological phases can exist, also note that the only way that we currently know how to describe the $U(1) \times U(1) \rtimes Z_2$ CS theory from a microscopic starting point is through a slave-Ising/parton construction, where partons are put into $\nu = \pm 1$ IQH states. Such a UV-completion necessarily breaks time-reversal symmetry, so it is consistent to find phases that cannot exist in the presence of time-reversal symmetry. In the case of the Z_N Abelian phase, there are other microscopic realizations of such topological order that do preserve time-reversal symmetry.

Finally, note that the picture that we developed for the transition from the Z_N phase to the twisted Z_N phase involved the condensation of a particular anyon that has spin $1/N$. Thus for $N > 2$, putting this anyon into some collective state will necessarily break time-reversal symmetry, unless the anyon with spin $-1/N$ is treated on exactly the same footing.

C. Protected edge modes

The Z_N Abelian phase does not have protected gapless edge modes in the absence of any symmetries, and here we have seen that it is separated from the twisted Z_N non-Abelian phases by a Z_2 transition. Viewed from the twisted phase, the transition can be thought of as the condensation of a boson j that squares to a topologically trivial excitation. On general grounds, we expect that the boundary between two topological phases will not have protected gapless edge modes if the two phases are related by a Z_N boson condensation transition.[37] Since the Z_N phase does not have protected gapless edge modes at a boundary with the vacuum, this means that the twisted Z_N phase will also not have protected gapless edge modes at a boundary with the vacuum.

We expect that the above discussion can be made more concrete by studying the edge through the $U(1) \times U(1) \rtimes Z_2$ CS theory and the slave-Ising theory and showing that all possible gapless edge modes can be gapped out by allowed perturbations.

VIII. SUMMARY, CONCLUSION, AND OUTLOOK

We have seen that the deconfined phase of Z_N gauge theories has a neighboring non-Abelian phase, the twisted Z_N states. These two phases are separated by a continuous quantum phase transition and the non-Abelian states can be accessed, for $N > 2$, only by breaking time-reversal symmetry.

In this paper, we have studied the full topological order of these non-Abelian states. Much of the topological order can be deduced directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory and the fact that it is separated from the conventional Z_N states by the condensation of a Z_2-charged boson. We found a way to compute the rest of the topological properties that we could not calculate directly, although those results rely on additional assumptions.

In addition to deriving the topological order of these states, we presented a parton construction that allows us to describe the Z_N topological order in terms of fermions in band insulators with Chern number ± 1. This description of the Z_N

states then allowed us to describe the non-Abelian twisted Z_N through a slave Ising theory of Z_2 fractionalization. The parton construction provides trial projected wave functions and provides further evidence to establish that these phases are physical in that they can be realized in bosonic systems with local interactions. While we have not provided an explicit microscopic Hamiltonian that can realize these phases, all known fractionalized phases can be described through such slave-particle constructions, and there is a large classification of such fractionalized phases in terms of microscopic models with local interactions.[33] We believe that the existence of a stable slave-particle construction for a fractionalized phase is therefore strong evidence in favor of the fact that the proposed phase can be stabilized by a microscopic Hamiltonian with local interactions.

There are two main conceptual issues lacking in our understanding of these states. First, we should be able to prove more rigorously that the full topological quantum numbers presented here coincide with those of the $U(1) \times U(1) \rtimes Z_2$ CS theory and the associated slave Ising description. Second, and more importantly, we would like to understand better how to access these non-Abelian states by starting from the Abelian Z_N states. We know little besides the fact that the energy gap of the $(1, 1)$ quasiparticles should probably be tuned through zero.

In the case of the Z_N topological order, we found a way through field theoretic and slave-particle constructions to essentially gauge the electric-magnetic symmetry of the topological quantum numbers. However, conceptually we do not know how to extend these ideas to other discrete gauge theories. It would be interesting to develop more general theoretical, physical descriptions that allow us to "twist" the symmetries of the topological quantum numbers of a phase. In CFT, such a procedure is referred to as orbifolding. In the context of bulk $2 + 1$-dimensional states of matter, we do not have any physical understanding of how this can be done more generally. One starting point would obviously be to try to develop Chern-Simons descriptions of discrete gauge theories, in the way that the mutual $U(1) \times U(1)$ CS theory describes Z_N gauge theory.

Recently, another series of topological phase transitions was found involving the non-Abelian $SU(2)_N \times \overline{SU(2)}_N$ states, where the transition involves the condensation of a boson with Z_2 fusion rules.[38] By explicitly constructing a lattice model, it was found that the condensation of the boson yields a continuous phase transition in the 3D Ising universality class. For $N = 2$, the results of Ref. 38 coincide with our results. However, the generalization to $N > 2$ is different; in our case, the $N > 2$ twisted Z_N states break time-reversal symmetry though they have no topologically protected edge modes, and the states on the other side of the transition are described by Z_N gauge theory. On the other hand, the $SU(2)_N \times \overline{SU(2)}_N$ states can always exist in time-reversal invariant systems, and the states on the other side of the phase transition are not describable by Z_N gauge theory for $N > 2$.

Note added in proof. Recently, it has been shown that the construction in Sec. II and the many-body wave function of Eq. (6) describe a boson integer quantum Hall state when $N = 1$ and there is a $U(1)$ number conservation symmetry for the bosons.[39]

APPENDIX A: OPERATOR CONTENT OF $U(1)/Z_2$ ORBIFOLD CFT

Since the $U(1)/Z_2$ orbifold at $c = 1$ plays an important role in understanding the topological properties of the twisted Z_N states, here, we will give a brief account of some of its properties. The information here is taken from Ref. 40, where a more complete discussion can be found.

The $U(1)/Z_2$ orbifold CFT, at central charge $c = 1$, is the theory of a scalar boson φ, compactified at a radius R, so that $\varphi \sim \varphi + 2\pi R$, and with an additional Z_2 gauge symmetry: $\varphi \sim -\varphi$. When $\frac{1}{2} R^2$ is rational, i.e., $\frac{1}{2} R^2 = p/p'$, with p and p' coprime, then it is useful to consider an algebra generated by the fields $j = i\partial\varphi$, and $e^{\pm i\sqrt{2N}\varphi}$, for $N = pp'$. This algebra is referred to as an extended chiral algebra. The infinite number of Virasoro primary fields in the $U(1)$ CFT can now be organized into a finite number of representations of this extended algebra $\mathcal{A}_N$. There are $2N$ of these representations, and the primary fields are written as $V_k = e^{ik\varphi/\sqrt{2N}}$, with $k = 0,1,\ldots,2N-1$. The Z_2 action takes $V_k \to V_{2N-k}$.

In the Z_2 orbifold, one now considers representations of the smaller algebra $\mathcal{A}_N/Z_2$. This includes the Z_2 invariant combinations of the original primary fields, which are of the form $\phi_k = \cos(k\varphi/\sqrt{2N})$; there are $N+1$ of these. In addition, there are six new primary fields. The gauging of the Z_2 allows for twist operators that are not local with respect to the fields in the algebra $\mathcal{A}_N/Z_2$, but rather local up to an element of Z_2. It turns out that there are two of these twisted sectors, and each sector contains one field that lies in the trivial representation of the Z_2, and one field that lies in the nontrivial representation of Z_2. These twist fields are labeled σ_1, τ_1, σ_2, and τ_2. In addition to these, an in-depth analysis[40] shows that the fixed points of the Z_2 action in the original $U(1)$ theory split into a Z_2 invariant and a noninvariant field. We have already counted the invariant ones in our $N+1$ invariant fields, which leaves two new fields. One fixed point is the identity sector, corresponding to V_0, which splits into two sectors: 1 and $j = i\partial\varphi$. The other fixed point corresponds to V_N. This splits into two primary fields, which are labeled as ϕ_N^i for $i = 1,2$ and which have scaling dimension $N/4$. In total, there are $N+7$ primary fields in the Z_2 rational orbifold at "level" $2N$. These fields and their properties are summarized in Table IV.

This spectrum for the Z_2 orbifold is obtained by first computing the partition function of the full Z_2 orbifold CFT defined on a torus, including both holomorphic and antiholomorphic parts. Then, the partition function is decomposed into holomorphic blocks, which are conjectured to be the generalized characters of the $\mathcal{A}_N/Z_2$ chiral algebra. This leads to the spectrum listed in Table IV. The fusion rules and scaling dimensions for these primary fields are obtained by studying the modular transformation properties of the characters.

The fusion rules are as follows. For N even,

$$j \times j = 1, \quad \phi_N^i \times \phi_N^i = 1, \quad \phi_N^1 \times \phi_N^2 = j. \quad (A1)$$

TABLE IV. Primary fields in the $U(1)_{2N}/Z_2$ orbifold CFT. The label k runs from 1 to $N-1$.

Label	Scaling dimension	Quantum dimension
$\mathbb{I}$	0	1
j	1	1
ϕ_N^1	$N/4$	1
ϕ_N^2	$N/4$	1
σ_1	1/16	$\sqrt{N}$
σ_2	1/16	$\sqrt{N}$
τ_1	9/16	$\sqrt{N}$
τ_2	9/16	$\sqrt{N}$
ϕ_k	$k^2/4N$	2

As mentioned in Ref. 40, the vertex operators ϕ_k have a fusion algebra consistent with their interpretation as $\cos\frac{k}{\sqrt{2N}}\varphi$,

$$\phi_k \times \phi_{k'} = \phi_{k+k'} + \phi_{k-k'} \quad (k' \neq k, N-k),$$
$$\phi_k \times \phi_k = 1 + j + \phi_{2k}, \quad (A2)$$
$$\phi_{N-k} \times \phi_k = \phi_{2k} + \phi_N^1 + \phi_N^2, \quad j \times \phi_k = \phi_k,$$

$$\sigma_i \times \sigma_i = 1 + \phi_N^i + \sum_{k \text{ even}} \phi_k, \quad \sigma_1 \times \sigma_2 = \sum_{k \text{ odd}} \phi_k,$$

$$j \times \sigma_i = \tau_i. \quad (A3)$$

For N odd, the fusion algebra of 1, j, and ϕ_N^i is Z_4:

$$j \times j = 1, \quad \phi_N^1 \times \phi_N^2 = 1, \quad \phi_N^i \times \phi_N^i = j. \quad (A4)$$

The fusion rules for the twist fields become

$$\sigma_i \times \sigma_i = \phi_N^i + \sum_{k \text{ odd}} \phi_k, \quad \sigma_1 \times \sigma_2 = 1 + \sum_{k \text{ even}} \phi_k. \quad (A5)$$

The fusion rules for the operators ϕ_k are unchanged.

For $N = 1$, it was observed that the Z_2 orbifold is equivalent to the $U(1)_8$ Gaussian theory. For $N = 2$, it was observed that the Z_2 orbifold is equivalent to two copies of the Ising CFT. For $N = 3$, it was observed that the Z_2 orbifold is equivalent to the Z_4 parafermion CFT of Zamolodchikov and Fateev.[41] In Tables V and VI, we list the fields from the Z_2 orbifold for

TABLE V. Primary fields in the Z_2 orbifold for $N = 3$, their scaling dimensions, and the Z_4 parafermion fields that they correspond to.

Z_2 orbifold field	Scaling dimension h	Z_4 parafermion field
1	0	Φ_0^0
j	1	Φ_4^0
ϕ_N^1	3/4	Φ_2^0
ϕ_N^2	3/4	Φ_6^0
ϕ_1	1/12	Φ_2^2
ϕ_2	1/3	Φ_0^2
σ_1	1/16	Φ_1^1
σ_2	1/16	Φ_{-1}^1
τ_1	9/16	Φ_3^1
τ_2	9/16	Φ_5^1

TABLE VI. Primary fields in the Z_2 orbifold for $N = 2$, their scaling dimensions, and the fields from Ising^2 to which they correspond.

Z_2 orbifold field	Scaling dimension h	Ising^2 fields
1	0	$\mathbb{I} \otimes \mathbb{I}$
j	1	$\psi \otimes \psi$
ϕ_N^1	1/2	$\mathbb{I} \otimes \psi$
ϕ_N^2	1/2	$\psi \otimes \mathbb{I}$
ϕ_1	1/8	$\sigma \otimes \sigma$
σ_1	1/16	$\sigma \times \mathbb{I}$
σ_2	1/16	$\mathbb{I} \otimes \sigma$
τ_1	9/16	$\sigma \otimes \psi$
τ_2	9/16	$\psi \otimes \sigma$

$N = 2$ and 3, their scaling dimensions, and the fields in the Ising^2 or Z_4 parafermion CFTs that they correspond to.

APPENDIX B: GROUND-STATE DEGENERACY ON A TORUS FOR $SU(N)^3 \times U(1)$ GAUGE THEORY

A procedure for calculating the ground state degeneracy on a torus for states obtained through the projective construction was described in Ref. 42. This procedure works for gauge groups that are connected, while gauge groups of the form $G \rtimes H$, where G is connected and H is a discrete group, require further analysis.

The classical configuration space of CS theory consists of flat connections, for which the magnetic field vanishes: $\epsilon_{ij} \partial_i a_j = 0$. This configuration space is completely characterized by holonomies of the gauge field along the noncontractible loops of the torus:

$$W(\alpha) = \mathcal{P} e^{i \oint_\alpha a \cdot dl}. \tag{B1}$$

More generally, for a manifold M, the gauge-inequivalent set of $W(\alpha)$ form a group [Hom: $\pi_1(M) \to G$]$/G$, which is the group of homomorphisms of the fundamental group of M to the gauge group G, modulo G. For a torus, $\pi_1(T^2)$ is Abelian, which means that $W(\alpha)$ and $W(\beta)$, where α and β are the two distinct noncontractible loops of the torus, commute with each other and we can always perform a global gauge transformation so that $W(\alpha)$ and $W(\beta)$ lie in the maximal Abelian subgroup, G_{abl}, of G (this subgroup is called the maximal torus). The maximal torus is generated by the Cartan subalgebra of the Lie algebra of G; in the case at hand, this Cartan subalgebra is composed of $3N - 2$ matrices, $3(N - 1)$ of which lie in the Cartan subalgebra of $SU(N) \times SU(N) \times SU(N)$, in addition to $diag(1, 0, \ldots, 1, 0, \ldots, -1, 0, \ldots)$. Since we only need to consider components of the gauge field a^I that lie in the Cartan subalgebra, the CS Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I \partial a^J, \tag{B2}$$

where $K_{IJ} = \text{Tr}(p^I p^J)$ and p^I, $I = 1, \ldots, k+1$, are the generators that lie in the Cartan subalgebra.

There are large gauge transformations $U = e^{2\pi x_i p^I / L}$, where x_1 and x_2 are the two coordinates on the torus and

L is the length of each side. These act on the partons as

$$\psi \to U\psi, \tag{B3}$$

where $\psi^T = (\psi_1, \ldots, \psi_{3N})$, and they take $a_i^I \to a_i^I + 2\pi/L$. These transformations will be the minimal large gauge transformations if we normalize the generators as follows:

$$p_{ij}^I = \delta_{ij}(\delta_{i,I} - \delta_{i,I+1}), \quad I = 1, \ldots, N-1,$$
$$p_{ij}^I = \delta_{ij}(\delta_{i,I+1} - \delta_{i,I+2}), \quad I = N, \ldots, 2N-2,$$
$$p_{ij}^I = \delta_{ij}(\delta_{i,I+2} - \delta_{i,I+3}), \quad I = 2N-1, \ldots, 3N-3,$$
$$p_{ij}^{3N-2} = \delta_{ij}(\delta_{i,1} + \delta_{i,N+1} - \delta_{i,2N+1}). \tag{B4}$$

The effective K matrix is of the form

$$K = \begin{pmatrix} A & 0 & 0 & v \\ 0 & A & 0 & v \\ 0 & 0 & -A & v \\ v^T & v^T & v^T & 1 \end{pmatrix}, \tag{B5}$$

where A is the Cartan matrix of $SU(N)$ (an $N - 1 \times N - 1$ matrix) and v is an $(N - 1) \times 1$ column vector with 1 on the first entry and 0s everywhere else: $v^T = (1, 0, \ldots, 0)$. For example, for $N = 2$ the above K matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{B6}$$

For $N = 4$, it is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{B7}$$

In addition to the large gauge transformations, there are discrete gauge transformations $W \in SU(N) \times SU(N) \times SU(N) \times U(1)$, which keep the Abelian subgroup unchanged but interchanging the a^I's amongst themselves. These satisfy

$$W^\dagger G_{abl} W = G_{abl}, \tag{B8}$$

or, alternatively,

$$W^\dagger p^I W = T_{IJ} p^J, \tag{B9}$$

for some $(3N - 2) \times (3N - 2)$ matrix T. These discrete transformations correspond to the independent ways of interchanging the partons and they correspond to the Weyl group of the gauge group. The Weyl group for $SU(N)$ is S_N. These can be generated by pairwise interchanges of the partons.

Picking the gauge $a_0^I = 0$ and parametrizing the gauge field as

$$a_1^I = \frac{2\pi}{L} X_1^I \quad a_2^I = \frac{2\pi}{L} X_2^I, \tag{B10}$$

we have

$$L = 2\pi K_{IJ} X_1^I \dot{X}_2^J. \tag{B11}$$

The Hamiltonian vanishes. The conjugate momentum to X_2^J is

$$p_2^J = 2\pi K_{IJ} X_1^I. \tag{B12}$$

Since $X_2^J \sim X_2^J + 1$ as a result of the large gauge transformations, we can write the wave functions as

$$\psi(\vec{X}_2) = \sum_{\vec{n}} c_{\vec{n}} e^{2\pi \vec{n} \cdot \vec{X}_2}, \tag{B13}$$

where $\vec{X}_2 = (X_2^1, \ldots X_2^{2N-3})$ and $\vec{n}$ is a $(2N - 3)$-dimensional vector of integers. In momentum space, the wave function is

$$\phi(\vec{p}_2) = \sum_{\vec{n}} c_{\vec{n}} \delta^{(2N-3)}(\vec{p}_2 - 2\pi \vec{n})$$

$$\sim \sum_{\vec{n}} c_{\vec{n}} \delta^{(2N-3)}(K \vec{X}_1 - \vec{n}), \tag{B14}$$

where $\delta^{(2N-3)}(\vec{x})$ is a $(2N - 3)$-dimensional delta function. Since $X_1^J \sim X_1^J + 1$, it follows that $c_{\vec{n}} = c_{\vec{n}'}$, where $(\vec{n}')^I = n^I + K_{IJ}$, for any J. Furthermore, each discrete gauge transformation W_i that keeps the Abelian subgroup G_{abl} invariant corresponds to a matrix T_i [see Eq. (B9)], which acts on the diagonal generators. These lead to the equivalences $c_{\vec{n}} = c_{T_i \vec{n}}$.

Carrying out the result on the computer, we find that Det K is always equal to N^2, and, remarkably, we find that the Weyl group, i.e., the group of discrete transformations that keeps the Abelian subgroup unchanged, acts trivially in the sense that it does not lead to any identifications among the states. This suggests that the K matrix is a complete description of the theory on a torus!

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