18.712 Introduction to Representation Theory
Fall 2008

5 Quiver Representations

5.1 Problems

Problem 5.1. Field embeddings. *Recall that $k(y_1, ..., y_m)$ denotes the field of rational functions of $y_1, ..., y_m$ over a field k. Let $f : k[x_1, ..., x_n] \to k(y_1, ..., y_m)$ be an injective homomorphism. Show that $m \geq n$. (Look at the growth of dimensions of the spaces W_N of polynomials of degree N in x_i and their images under f as $N \to \infty$). Deduce that if $f : k(x_1, ..., x_n) \to k(y_1, ..., y_m)$ is a field embedding, then $m \geq n$.*

Problem 5.2. Some algebraic geometry.

Let k be an algebraically closed field, and $G = GL_n(k)$. Let V be a polynomial representation of G. Show that if G has finitely many orbits on V then $\dim(V) \le n^2$. Namely:

(a) Let $x_1, ..., x_N$ be linear coordinates on V. Let us say that a subset X of V is Zariski dense if any polynomial $f(x_1, ..., x_N)$ which vanishes on X is zero (coefficientwise). Show that if G has finitely many orbits on V then G has at least one dense orbit on V.

(b) Use (a) to construct a field embedding $k(x_1, ..., x_N) \to k(g_{pq})$, then use Problem 5.1.

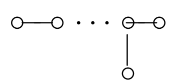
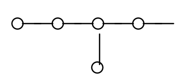
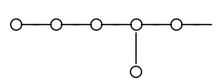
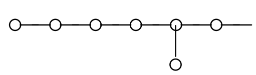
(c) generalize the result of this problem to the case when $G = GL_{n_1}(k) \times ... \times GL_{n_m}(k)$.

Problem 5.3. Dynkin diagrams.

Let Γ be a graph, i.e. a finite set of points (vertices) connected with a certain number of edges (we allow multiple edges). We assume that Γ is connected (any vertex can be connected to any other by a path of edges) and has no self-loops (edges from a vertex to itself). Suppose the vertices of Γ are labeled by integers $1, ..., N$. Then one can assign to Γ an $N \times N$ matrix $R_\Gamma = (r_{ij})$, where r_{ij} is the number of edges connecting vertices i and j. This matrix is obviously symmetric, and is called the adjacency matrix. Define the matrix $A_\Gamma = 2I - R_\Gamma$, where I is the identity matrix.

Main definition: Γ *is said to be a Dynkin diagram if the quadratic from on $\mathbb{R}^N$ with matrix A_Γ is positive definite. Dynkin diagrams appear in many areas of mathematics (singularity theory, Lie algebras, representation theory, algebraic geometry, mathematical physics, etc.) In this problem you will get a complete classification of Dynkin diagrams. Namely, you will prove*

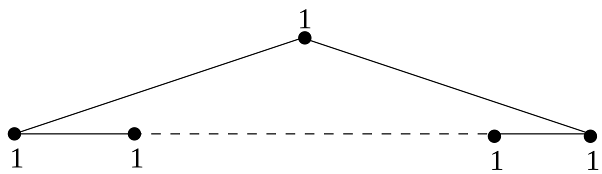
Theorem. Γ *is a Dynkin diagram if and only if it is one on the following graphs:*

- A_n :
- D_n: 
- E_6 : 
- E_7 : 
- E_8 : 

(a) Compute the determinant of A_Γ where $\Gamma = A_N, D_N$. (Use the row decomposition rule, and write down a recursive equation for it). Deduce by Sylvester criterion that A_N, D_N are Dynkin diagrams.[7]

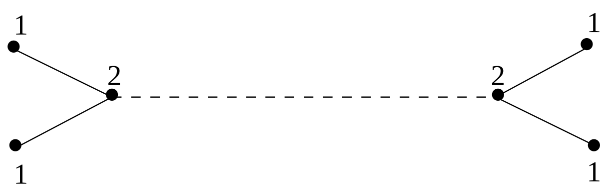
(b) Compute the determinants of A_Γ for E_6, E_7, E_8 (use row decomposition and reduce to (a)). Show they are Dynkin diagrams.

(c) Show that if Γ is a Dynkin diagram, it cannot have cycles. For this, show that $\det(A_\Gamma) = 0$ for a graph Γ below [8]

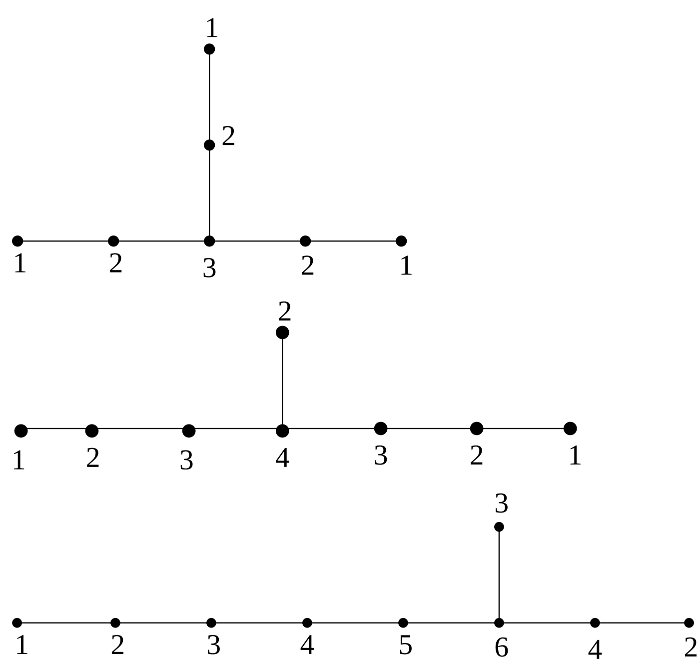


(show that the sum of rows is 0). Thus Γ has to be a tree.

(d) Show that if Γ is a Dynkin diagram, it cannot have vertices with 4 or more incoming edges, and that Γ can have no more than one vertex with 3 incoming edges. For this, show that $\det(A_\Gamma) = 0$ for a graph Γ below:



(e) Show that $\det(A_\Gamma) = 0$ for a graphs Γ below:



(f) Deduce from (a)-(e) the classification theorem for Dynkin diagrams.

(g) A (simply laced) affine Dynkin diagram is a connected graph without self-loops such that the

[7]The Sylvester criterion says that a symmetric bilinear form $(,)$ on $\mathbb{R}^N$ is positive definite iff for any $k \leq N$, $\det_{1 \leq i,j \leq k}(e_i, e_j) > 0$.

[8]Please ignore the numerical labels; they will be relevant for Problem 5.5 below.

quadratic form defined by A_Γ is positive semidefinite. Classify affine Dynkin diagrams. (Show that they are exactly the forbidden diagrams from (c)-(e)).

Problem 5.4. *Let Q be a quiver with set of vertices D. We say that Q is of finite type if it has finitely many indecomposable representations. Let b_{ij} be the number of edges from i to j in Q ($i, j \in D$).*

There is the following remarkable theorem, proved by P. Gabriel in the 1970-s.

Theorem. *A connected quiver Q is of finite type if and only if the corresponding unoriented graph (i.e. with directions of arrows forgotten) is a Dynkin diagram.*

In this problem you will prove the "only if" direction of this theorem (i.e. why other quivers are NOT of finite type).

(a) Show that if Q is of finite type then for any rational numbers $x_i \geq 0$ which are not simultaneously zero, one has $q(x_1, ..., x_r) > 0$, where

$$q(x_1, ..., x_r) := \sum_{i \in D} x_i^2 - \frac{1}{2} \sum_{i,j \in D} b_{ij} x_i x_j.$$

Hint. It suffices to check the result for integers: $x_i = n_i$. First assume that $n_i \geq 0$, and consider the space W of representations V of Q such that $\dim V_i = n_i$. Show that the group $\times_i GL_{n_i}(k)$ acts with finitely many orbits on $W \oplus k$, and use Problem 5.2 to derive the inequality. Then deduce the result in the case when n_i are arbitrary integers.

(b) Deduce that q is a positive definite quadratic form.

Hint. Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) Show that a quiver of finite type can have no self-loops. Then, using Problem 5.3, deduce the theorem.

Problem 5.5. *Let $G \neq 1$ be a finite subgroup of $SU(2)$, and V be the 2-dimensional representation of G coming from its embedding into $SU(2)$. Let V_i, $i \in I$, be all the irreducible representations of G. Let r_{ij} be the multiplicity of V_i in $V \otimes V_j$.*

(a) Show that $r_{ij} = r_{ji}$.

(b) The McKay graph of G, $M(G)$, is the graph whose vertices are labeled by $i \in I$, and i is connected to j by r_{ij} edges. Show that $M(G)$ is connected. (Use Problem 3.26)

(c) Show that $M(G)$ is an affine Dynkin graph (one of the "forbidden" graphs in Problem 5.3). For this, show that the matrix $a_{ij} = 2\delta_{ij} - r_{ij}$ is positive semidefinite but not definite, and use Problem 5.3.

Hint. Let $f = \sum x_i \chi_{V_i}$, where χ_{V_i} be the characters of V_i. Show directly that $((2-\chi_V)f, f) \geq 0$. When is it equal to 0? Next, show that $M(G)$ has no self-loops, by using that if G is not cyclic then G contains the central element $-Id \in SU(2)$.

(d) Which groups from Problem 3.24 correspond to which diagrams?

(e) Using the McKay graph, find the dimensions of irreducible representations of all finite $G \subset SU(2)$ (namely, show that they are the numbers labeling the vertices of the affine Dynkin diagrams on our pictures). Compare with the results on subgroups of $SO(3)$ we obtained in Problem 3.24.

5.2 Indecomposable representations of the quivers A_1, A_2, A_3

We have seen that a central question about representations of quivers is whether a certain quiver has only finitely many indecomposable representations. In the previous subsection it is shown that only those quivers whose underlying undirected graph is a Dynkin diagram may have this property. To see if they actually do have this property, we first explicitly decompose representations of certain easy quivers.

Remark 5.6. By an object of the type $1 \longrightarrow 0$ we mean a map from a one-dimensional vector space to the zero space. Similarly, an object of the type $0 \longrightarrow 1$ is a map from the zero space into a one-dimensional space. The object $1 \longrightarrow 1$ means an isomorphism from a one-dimensional to another one-dimensional space. The numbers in such diagrams always mean the dimension of the attached spaces and the maps are the canonical maps (unless specified otherwise)

Example 5.7 (A_1)**.** The quiver A_1 consists of a single vertex and has no edges. Since a representation of this quiver is just a single vector space, the only indecomposable representation is the ground field (=a one-dimensional space).

Example 5.8 (A_2)**.** The quiver A_2 consists of two vertices connected by a single edge.

$$\bullet \longrightarrow \bullet$$

A representation of this quiver consists of two vector spaces V, W and an operator $A : V \to W$.

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet}$$

To decompose this representation, we first let V' be a complement to the kernel of A in V and let W' be a complement to the image of A in W. Then we can decompose the representation as follows

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} \;=\; \underset{\ker V}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \;\oplus\; \underset{V'}{\bullet} \xrightarrow[\sim]{A} \underset{\mathrm{Im} A}{\bullet} \;\oplus\; \underset{0}{\bullet} \xrightarrow{0} \underset{W'}{\bullet},$$

The first summand is a multiple of the object $1 \longrightarrow 0$, the second a multiple of $1 \longrightarrow 1$, the third of $0 \longrightarrow 1$. We see that the quiver A_2 has three indecomposable representations, namely

$$1 \longrightarrow 0, \quad 1 \longrightarrow 1 \quad \text{and} \quad 0 \longrightarrow 1 .$$

Example 5.9 (A_3)**.** The quiver A_3 consists of three vertices and two connections between them. So we have to choose between two possible orientations.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{or} \quad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

1. We first look at the orientation

$$\bullet \longrightarrow \bullet \longrightarrow \bullet .$$

   Then a representation of this quiver looks like

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} \xrightarrow{B} \underset{Y}{\bullet} .$$

   Like in Example 5.8 we first split away

$$\underset{\ker A}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \xrightarrow{0} \underset{0}{\bullet} .$$

This object is a multiple of $1 \longrightarrow 0 \longrightarrow 0$. Next, let Y' be a complement of $\mathrm{Im}B$ in Y. Then we can also split away

$$\underset{0}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \xrightarrow{0} \underset{Y'}{\bullet},$$

which is a multiple of the object $0 \longrightarrow 0 \longrightarrow 1$. This results in a situation where the map A is injective and the map B is surjective (we rename the spaces to simplify notation):

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet}.$$

Next, let $X = \ker(B \circ A)$ and let X' be a complement of X in V. Let W' be a complement of $A(X)$ in W such that $A(X') \subset W'$. Then we get

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} \;=\; \underset{X}{\bullet} \xrightarrow{A} \underset{A(X)}{\bullet} \xrightarrow{B} \underset{0}{\bullet} \;\oplus\; \underset{X'}{\bullet} \overset{A}{\hookrightarrow} \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet}$$

The first of these summands is a multiple of $1 \xrightarrow{\sim} 1 \longrightarrow 0$. Looking at the second summand, we now have a situation where A is injective, B is surjective and furthermore $\ker(B \circ A) = 0$. To simplify notation, we redefine

$$V = X',\; W = W'.$$

Next we let $X = \mathrm{Im}(B \circ A)$ and let X' be a complement of X in Y. Furthermore, let $W' = B^{-1}(X')$. Then W' is a complement of $A(V)$ in W. This yields the decomposition

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \xrightarrow[\sim]{A} \underset{A(V)}{\bullet} \xrightarrow[\sim]{B} \underset{X}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{X'}{\bullet}$$

Here, the first summand is a multiple of $1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1$. By splitting away the kernel of B, the second summand can be decomposed into multiples of $0 \longrightarrow 1 \xrightarrow{\sim} 1$ and $0 \longrightarrow 1 \longrightarrow 0$. So, on the whole, this quiver has six indecomposable representations:

$$1 \longrightarrow 0 \longrightarrow 0, \quad 0 \longrightarrow 0 \longrightarrow 1, \quad 1 \xrightarrow{\sim} 1 \longrightarrow 0,$$

$$1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1, \quad 0 \longrightarrow 1 \xrightarrow{\sim} 1, \quad 0 \longrightarrow 1 \longrightarrow 0$$

2. Now we look at the orientation

$$\bullet \longrightarrow \bullet \longleftarrow \bullet.$$

Very similarly to the other orientation, we can split away objects of the type

$$1 \longrightarrow 0 \longleftarrow 0, \quad 0 \longrightarrow 0 \longleftarrow 1$$

which results in a situation where both A and B are injective:

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\hookleftarrow} \underset{Y}{\bullet}.$$

By identifying V and Y as subspaces of W, this leads to the problem of classifying pairs of subspaces of a given space W up to isomorphism (the **pair of subspaces problem**). To do so, we first choose a complement W' of $V \cap Y$ in W, and set $V' = W' \cap V$, $Y' = W' \cap Y$. Then we can decompose the representation as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V'}{\bullet} \hookrightarrow \underset{W'}{\bullet} \hookleftarrow \underset{Y'}{\bullet} \;\oplus\; \underset{V \cap Y}{\bullet} \xrightarrow{\sim} \underset{V \cap Y}{\bullet} \xleftarrow{\sim} \underset{V \cap Y}{\bullet}.$$

The second summand is a multiple of the object $1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1$. We go on decomposing the first summand. Again, to simplify notation, we let

$$V = V', \ W = W', \ Y = Y'.$$

We can now assume that $V \cap Y = 0$. Next, let W' be a complement of $V \oplus Y$ in W. Then we get

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \hookrightarrow \underset{V \oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{W'}{\bullet} \longleftarrow \underset{0}{\bullet}$$

The second of these summands is a multiple of the indecomposable object $0 \longrightarrow 1 \longleftarrow 0$. The first summand can be further decomposed as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{V \oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \xrightarrow{\sim} \underset{V}{\bullet} \longleftarrow \underset{0}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{Y}{\bullet} \xleftarrow{\sim} \underset{Y}{\bullet}$$

These summands are multiples of

$$1 \longrightarrow 1 \longleftarrow 0 \ , \quad 0 \longrightarrow 1 \longleftarrow 1$$

So - like in the other orientation - we get 6 indecomposable representations of A_3:

$$1 \longrightarrow 0 \longleftarrow 0 \ , \quad 0 \longrightarrow 0 \longleftarrow 1 \ , \quad 1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1 \ ,$$

$$0 \longrightarrow 1 \longleftarrow 0 \ , \quad 1 \longrightarrow 1 \longleftarrow 0 \ , \quad 0 \longrightarrow 1 \longleftarrow 1$$

5.3 Indecomposable representations of the quiver D_4

As a last - slightly more complicated - example we consider the quiver D_4.

Example 5.10 (D_4)**.** We restrict ourselves to the orientation

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \\ & & \uparrow & & \\ & & \bullet & & \end{array} .$$

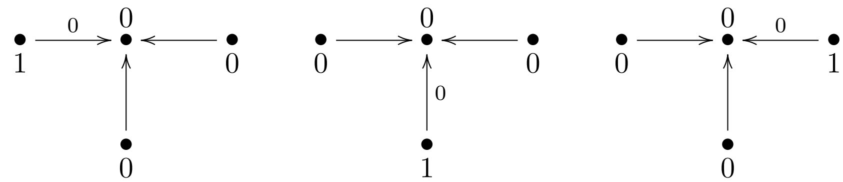
So a representation of this quiver looks like

$$\begin{array}{ccccc} & & V & & \\ \underset{V_1}{\bullet} & \xrightarrow{A_1} & \bullet & \xleftarrow{A_3} & \underset{V_3}{\bullet} \\ & & \uparrow {\scriptstyle A_2} & & \\ & & \underset{V_2}{\bullet} & & \end{array}$$

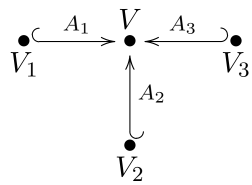
The first thing we can do is - as usual - split away the kernels of the maps A_1, A_2, A_3. More precisely, we split away the representations

$$\begin{array}{ccccc} & & 0 & & \\ \underset{\ker A_1}{\bullet} & \xrightarrow{0} & \bullet & \longleftarrow & \underset{0}{\bullet} \\ & & \uparrow & & \\ & & \underset{0}{\bullet} & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ \underset{0}{\bullet} & \longrightarrow & \bullet & \longleftarrow & \underset{0}{\bullet} \\ & & \uparrow {\scriptstyle 0} & & \\ & & \underset{\ker A_2}{\bullet} & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ \underset{0}{\bullet} & \longrightarrow & \bullet & \xleftarrow{0} & \underset{\ker A_3}{\bullet} \\ & & \uparrow & & \\ & & \underset{0}{\bullet} & & \end{array}$$

These representations are multiples of the indecomposable objects

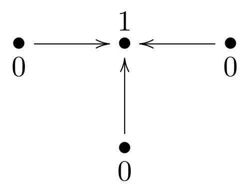


So we get to a situation where all of the maps A_1, A_2, A_3 are injective.



As in 2, we can then identify the spaces V_1, V_2, V_3 with subspaces of V. So we get to the **triple of subspaces problem** of classifying a triple of subspaces of a given space V.

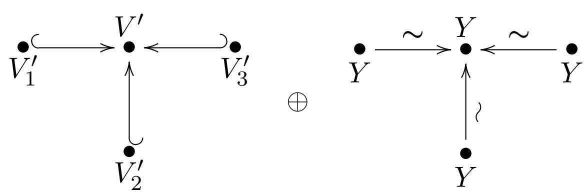
The next step is to split away a multiple of



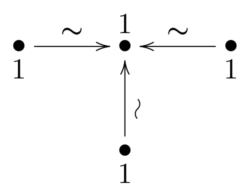
to reach a situation where

$$V_1 + V_2 + V_3 = V.$$

By letting $Y = V_1 \cap V_2 \cap V_3$, choosing a complement V' of Y in V, and setting $V_i' = V' \cap V_i$, $i = 1, 2, 3$, we can decompose this representation into



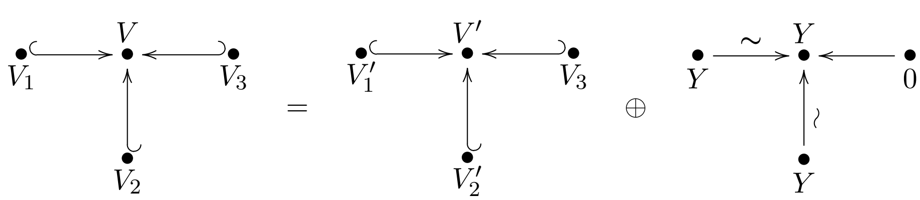
The last summand is a multiple of the indecomposable representation



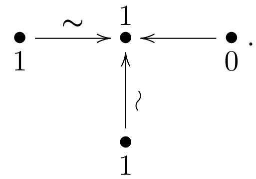
So - considering the first summand and renaming the spaces to simplify notation - we are in a situation where

$$V = V_1 + V_2 + V_3, \quad V_1 \cap V_2 \cap V_3 = 0.$$

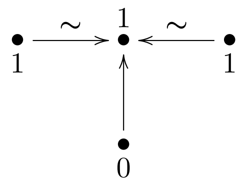
As a next step, we let $Y = V_1 \cap V_2$ and we choose a complement V' of Y in V such that $V_3 \subset V'$, and set $V_1' = V' \cap V_1$, ${}_2' = V' \cap V_2$ of Y. This yields the decomposition



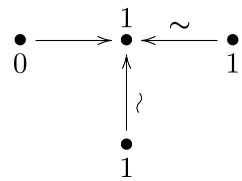
The second summand is a multiple of the indecomposable object



In the resulting situation we have $V_1 \cap V_2 = 0$. Similarly we can split away multiples of



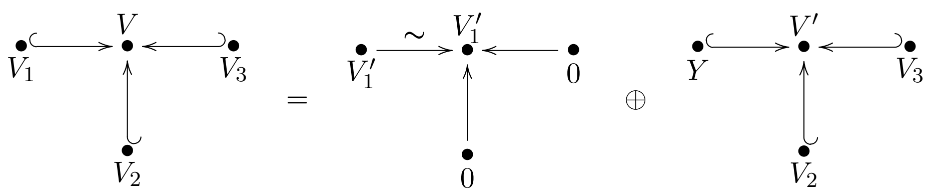
and



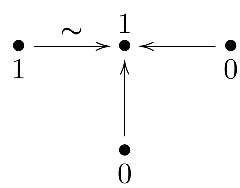
to reach a situation where the spaces V_1, V_2, V_3 do not intersect pairwise

$$V_1 \cap V_2 \;=\; V_1 \cap V_3 \;=\; V_2 \cap V_3 \;=\; 0.$$

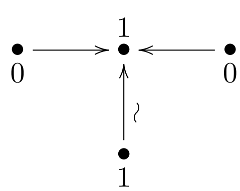
If $V_1 \not\subseteq V_2 \oplus V_3$ we let $Y = V_1 \cap (V_2 \oplus V_3)$. We let V_1' be a complement of Y in V_1. Since then $V_1' \cap (V_2 \oplus V_3) = 0$, we can select a complement V' of V_1' in V which contains $V_2 \oplus V_3$. This gives us the decomposition



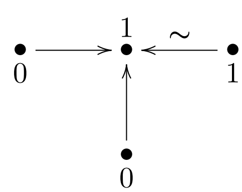
The first of these summands is a multiple of



By splitting these away we get to a situation where $V_1 \subseteq V_2 \oplus V_3$. Similarly, we can split away objects of the type



and



to reach a situation in which the following conditions hold

1. $V_1 + V_2 + V_3 = V$.
2. $V_1 \cap V_2 = 0, \quad V_1 \cap V_3 = 0, \quad V_2 \cap V_3 = 0.$
3. $V_1 \subseteq V_2 \oplus V_3, \quad V_2 \subseteq V_1 \oplus V_3, \quad V_3 \subseteq V_1 \oplus V_2.$

But this implies that

$$V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3 = V.$$

So we get

$$\dim V_1 = \dim V_2 = \dim V_3 = n$$

and

$$\dim V = 2n.$$

Since $V_3 \subseteq V_1 \oplus V_2$ we can write every element of V_3 in the form

$$x \in V_3, \quad x = (x_1, x_2),\ x_1 \in V_1,\ x_2 \in V_2.$$

We then can define the projections

$$B_1 : V_3 \to V_1, \quad (x_1, x_2) \mapsto x_1,$$

$$B_2 : V_3 \to V_2, \quad (x_1, x_2) \mapsto x_2.$$

Since $V_3 \nsubseteq V_1, V_3 \nsubseteq V_2$, these maps have to be injective and therefore are isomorphisms. We then define the isomorphism

$$A = B_2 \circ B_1^{-1} : V_1 \to V_2.$$

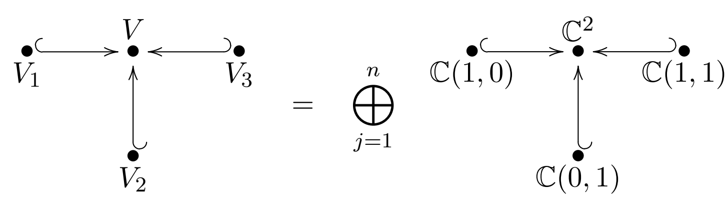
Let $e_1, \ldots, e_n$ be a basis for V_1. Then we get

$$V_1 = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2 \oplus \cdots \oplus \mathbb{C}\, e_n$$

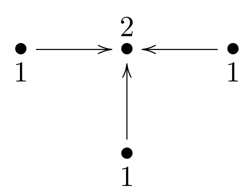
$$V_2 = \mathbb{C}\, Ae_1 \oplus \mathbb{C}\, Ae_2 \oplus \cdots \oplus \mathbb{C}\, Ae_n$$

$$V_3 = \mathbb{C}\,(e_1 + Ae_1) \oplus \mathbb{C}\,(e_2 + Ae_2) \oplus \cdots \oplus \mathbb{C}\,(e_n + Ae_n).$$

So we can think of V_3 as the graph of an isomorphism $A : V_1 \to V_2$. From this we obtain the decomposition



These correspond to the indecomposable object



Thus the quiver D_4 with the selected orientation has 12 indecomposable objects. If one were to explicitly decompose representations for the other possible orientations, one would also find 12 indecomposable objects.

It appears as if the number of indecomposable representations does not depend on the orientation of the edges, and indeed - Gabriel's theorem will generalize this observation.

5.4 Roots

From now on, let Γ be a fixed graph of type A_n, D_n, E_6, E_7, E_8. We denote the adjacency matrix of Γ by R_Γ.

Definition 5.11 (Cartan Matrix)**.** We define the Cartan matrix as

$$A_\Gamma = 2\mathrm{Id} - R_\Gamma.$$

On the lattice $\mathbb{Z}^n$ (or the space $\mathbb{R}^n$) we then define an inner product

$$B(x, y) = x^T A_\Gamma y$$

corresponding to the graph Γ.

Lemma 5.12. *1. B is positive definite.*

2. $B(x, x)$ takes on only even values for $x \in \mathbb{Z}^n$.

Proof. 1. This follows by definition, since Γ is a Dynkin diagram.

2. By the definition of the Cartan matrix we get

$$B(x, x) = x^T Ay = \sum_{i,j} x_i \, a_{ij} \, x_j = 2 \sum_i x_i^2 + \sum_{i,j,\, i \neq j} x_i \, a_{ij} \, x_j.$$

But since A is symmetric, we obtain

$$B(x, x) = 2 \sum_i x_i^2 + \sum_{i,j,\, i \neq j} x_i \, a_{ij} \, x_j = 2 \sum_i x_i^2 + 2 \cdot \sum_{i<j} a_{ij} \, x_i x_j$$

which is even.

□

Definition 5.13 (Root)**.** A root with respect to a certain positive inner product is a shortest (with respect to this inner product), nonzero vector in $\mathbb{Z}^n$.

So for the inner product B, a root is a nonzero vector $x \in \mathbb{Z}^n$ such that

$$B(x, x) = 2.$$

Remark 5.14. There can be only finitely many roots, since all of them have to lie in a ball of some radius.

Definition 5.15. We call vectors of the form

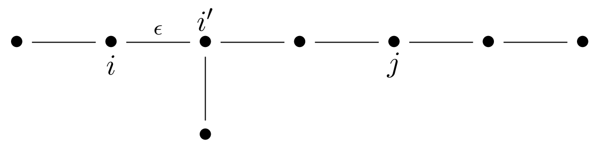
$$\alpha_i = (0, \ldots, \overbrace{1}^{i-\text{th}}, \ldots, 0)$$

simple roots.

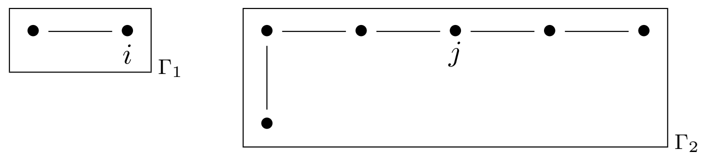
The α_i naturally form a basis of the lattice $\mathbb{Z}^n$.

Lemma 5.16. *Let α be a root, $\alpha = \sum_{i=1}^n k_i \alpha_i$. Then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i.*

Proof. Assume the contrary, i.e. $k_i > 0$, $k_j < 0$. Without loss of generality, we can also assume that $k_s = 0$ for all s between i and j. We can identify the indices i, j with vertices of the graph Γ.



Next, let ϵ be the edge connecting i with the next vertex towards j and i' be the vertex on the other end of ϵ. We then let Γ_1, Γ_2 be the graphs obtained from Γ by removing ϵ. Since Γ is supposed to be a Dynkin diagram - and therefore has no cycles or loops - both Γ_1 and Γ_2 will be connected graphs, which are not connected to each other.



Then we have $i \in \Gamma_1$, $j \in \Gamma_2$. We define

$$\beta = \sum_{m \in \Gamma_1} k_m \alpha_m, \quad \gamma = \sum_{m \in \Gamma_2} k_m \alpha_m.$$

With this choice we get

$$\alpha = \beta + \gamma.$$

Since $k_i > 0, k_j < 0$ we know that $\beta \neq 0, \gamma \neq 0$ and therefore

$$B(\beta, \beta) \geq 2, \quad B(\gamma, \gamma) \geq 2.$$

Furthermore,

$$B(\beta, \gamma) = -k_i k_{i'},$$

since Γ_1, Γ_2 are only connected at ϵ. But this has to be a nonnegative number, since $k_i > 0$ and $k_{i'} \leq 0$. This yields

$$B(\alpha, \alpha) = B(\beta + \gamma, \beta + \gamma) = \underbrace{B(\beta, \beta)}_{\geq 2} + 2 \underbrace{B(\beta, \gamma)}_{\geq 0} + \underbrace{B(\gamma, \gamma)}_{\geq 2} \geq 4.$$

But this is a contradiction, since α was assumed to be a root. □

Definition 5.17 (positive and negative roots)**.** We call a root $\alpha = \sum_i k_i \alpha_i$ a positive root, if all $k_i \geq 0$. A root for which $k_i \leq 0$ for all i is called a negative root.

Remark 5.18. Lemma 5.16 states that every root is either positive or negative.

Example 5.19. 1. Let Γ be of the type A_{n-1}. Then the lattice $L = \mathbb{Z}^{n-1}$ can be realized as a subgroup of the lattice $\mathbb{Z}^n$ by letting $L \subseteq \mathbb{Z}^n$ be the subgroup of all vectors $(x_1, \ldots, x_n)$ such that

$$\sum_i x_i = 0.$$

The vectors

$$\begin{aligned} \alpha_1 &= (1, -1, 0, \ldots, 0) \\ \alpha_2 &= (0, 1, -1, 0, \ldots, 0) \\ &\vdots \\ \alpha_{n-1} &= (0, \ldots, 0, 1, -1) \end{aligned}$$

naturally form a basis of L. Furthermore, the standard inner product

$$(x, y) = \sum x_i y_i$$

on $\mathbb{Z}^n$ restricts to the inner product B given by Γ on L, since it takes the same values on the basis vectors:

$$(\alpha_i, \alpha_i) = 2$$

$$(\alpha_i, \alpha_j) = \begin{cases} -1 & i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

This means that vectors of the form

$$(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

and

$$(0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0) = -(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1})$$

are the roots of L. Therefore the number of positive roots in L equals

$$\frac{n(n-1)}{2}.$$

2. As a fact we also state the number of positive roots in the other Dynkin diagrams:

D_n	$n(n-1)$
E_6	36 roots
E_7	63 roots
E_8	120 roots

Definition 5.20 (Root reflection)**.** Let $\alpha \in \mathbb{Z}^n$ be a positive root. The reflection s_α is defined by the formula

$$s_\alpha(v) = v - B(v, \alpha)\alpha.$$

We denote s_{α_i} by s_i and call these **simple reflections**.

Remark 5.21. As a linear operator of $\mathbb{R}^n$, s_α fixes any vector orthogonal to α and

$$s_\alpha(\alpha) = -\alpha$$

Therefore s_α is the reflection at the hyperplane orthogonal to α, and in particular fixes B. The s_i generate a subgroup $W \subseteq O(\mathbb{R}^n)$, which is called *the Weyl group* of Γ. Since for every $w \in W$, $w(\alpha_i)$ is a root, and since there are only finitely many roots, W has to be finite.

5.5 Gabriel's theorem

Definition 5.22. Let Q be a quiver with any labeling $1, \ldots, n$ of the vertices. Let $V = (V_1, \ldots, V_n)$ be a representation of Q. We then call

$$d(V) = (\dim V_1, \ldots, \dim V_n)$$

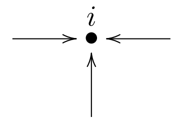
the dimension vector of this representation.

We are now able to formulate Gabriel's theorem using roots.

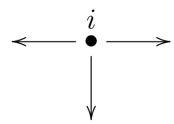
Theorem 5.23 (Gabriel's theorem)**.** *Let Q be a quiver of type A_n, D_n, E_6, E_7, E_8. Then Q has finitely many indecomposable representations. Namely, the dimension vector of any indecomposable representation is a positive root (with respect to B_Γ) and for any positive root α there is exactly one indecomposable representation with dimension vector α.*

5.6 Reflection Functors

Definition 5.24. Let Q be any quiver. We call a vertex $i \in Q$ a sink, if all edges connected to i point towards i.



We call a vertex $i \in Q$ a source, if all edges connected to i point away from i.



Definition 5.25. Let Q be any quiver and $i \in Q$ be a sink (a source). Then we let $\overline{Q_i}$ be the quiver obtained from Q by reversing all arrows pointing into (pointing out of) i.

We are now able to define the reflection functors (also called *Coxeter functors*).

Definition 5.26. Let Q be a quiver, $i \in Q$ be a sink. Let V be a representation of Q. Then we define the reflection functor

$$F_i^+ : \mathrm{Rep}Q \to \mathrm{Rep}\overline{Q_i}$$

by the rule

$$F_i^+(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^+(V)_i = \ker\left(\varphi : \bigoplus_{j \to i} V_j \to V_i\right).$$

Also, all maps stay the same but those now pointing out of i; these are replaced by compositions of the inclusion of $\ker\varphi$ into $\oplus V_j$ with the projections $\oplus V_j \to V_k$.

Definition 5.27. Let Q be a quiver, $i \in Q$ be a source. Let V be a representation of Q. Let ψ be the canonical map

$$\psi : V_i \to \bigoplus_{i \to j} V_j.$$

Then we define the reflection functor

$$F_i^- : \mathrm{Rep}Q \to \mathrm{Rep}\overline{Q_i}$$

by the rule

$$F_i^-(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^-(V)_i = \mathrm{Coker}\,(\psi) = \left(\bigoplus_{i \to j} V_j\right) / \mathrm{Im}\psi.$$

Again, all maps stay the same but those now pointing into i; these are replaced by the compositions of the inclusions $V_k \to \oplus_{i \to j} V_j$ with the natural map $\oplus V_j \to \oplus V_j / \mathrm{Im}\psi$.

Proposition 5.28. *Let Q be a quiver, V an indecomposable representation of Q.*

1. *Let $i \in Q$ be a sink. Then either* $\dim V_i = 1$, $\dim V_j = 0$ *for* $j \neq i$ ***or***

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

is surjective.

2. *Let $i \in Q$ be a source. Then either* $\dim V_i = 1$, $\dim V_j = 0$ *for* $j \neq i$ ***or***

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

is injective.

Proof. 1. Choose a complement W of $\mathrm{Im}\varphi$. Then we get

$$V \quad = \quad \begin{array}{ccccc} & & W & & \\ \underset{0}{\bullet} & \longrightarrow & \bullet & \longleftarrow & \underset{0}{\bullet} \\ & & \uparrow & & \\ & & \underset{0}{\bullet} & & \end{array} \quad \oplus \quad V'$$

Since V is indecomposable, one of these summands has to be zero. If the first summand is zero, then φ has to be surjective. If the second summand is zero, then the first one has to be of the desired form, because else we could write it as a direct sum of several objects of the type

$$\begin{array}{ccccc} & & 1 & & \\ \underset{0}{\bullet} & \longrightarrow & \bullet & \longleftarrow & \underset{0}{\bullet} \\ & & \uparrow & & \\ & & \underset{0}{\bullet} & & \end{array}$$

which is impossible, since V was supposed to be indecomposable.

2. Follows similarly by splitting away the kernel of ψ.

□

Proposition 5.29. *Let Q be a quiver, V be a representation of Q.*

1. *If*

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

is surjective, then

$$F_i^- F_i^+ V = V.$$

2. *If*

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

is injective, then

$$F_i^+ F_i^- V = V.$$

Proof. In the following proof, we will always mean by $i \to j$ that i points into j in the original quiver Q. We only establish the first statement and we also restrict ourselves to showing that the spaces of V and $F_i^- F_i^+ V$ are the same. It is enough to do so for the i-th space. Let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective and let

$$K = \ker \varphi.$$

When applying F_i^+, the space V_i gets replaced by K. Furthermore, let

$$\psi : K \to \bigoplus_{j \to i} V_j.$$

After applying F_i^-, K gets replaced by

$$K' = \left(\bigoplus_{j \to i} V_j \right) / (\mathrm{Im}\psi).$$

But

$$\mathrm{Im}\psi = K$$

and therefore

$$K' = \left(\bigoplus_{j \to i} V_j \right) / \left(\ker(\varphi : \bigoplus_{j \to i} V_j \to V_i) \right) = \mathrm{Im}(\psi : \bigoplus_{j \to i} V_j \to V_i)$$

by the homomorphism theorem. Since φ was assumed to be surjective, we get

$$K' = V_i.$$

□

Proposition 5.30. *Let Q be a quiver, and V be an indecomposable representation of Q. Then F_i^+V and F_i^-V (whenever defined) are either indecomposable or 0.*

Proof. We prove the proposition for F_i^+V - the case F_i^-V follows similarly. By Proposition 5.28 it follows that either

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

is surjective or $\dim V_i = 1, \dim V_j = 0, \quad j \neq i$. In the last case

$$F_i^+V = 0.$$

So we can assume that φ is surjective. In this case, assume that F_i^+V is decomposable as

$$F_i^+V = X \oplus Y$$

with $X, Y \neq 0$. But F_i^+V is injective at i, since the maps are canonical projections, whose direct sum is the tautological embedding. Therefore X and Y also have to be injective at i and hence (by 5.29)

$$F_i^+F_i^-X = X, \quad F_i^+F_i^-Y = Y$$

In particular

$$F_i^-X \neq 0, \quad F_i^-Y \neq 0.$$

Therefore

$$V = F_i^-F_i^+V = F_i^-X \oplus F_i^-Y$$

which is a contradiction, since V was assumed to be indecomposable. So we can infer that

$$F_i^+V$$

is indecomposable. □

Proposition 5.31. *Let Q be a quiver and V a representation of Q.*

1. *Let $i \in Q$ be a sink and let V be surjective at i. Then*

$$d(F_i^+ V) = s_i(d(V)).$$

2. *Let $i \in Q$ be a source and let V be injective at i. Then*

$$d(F_i^- V) = s_i(d(V)).$$

Proof. We only prove the first statement, the second one follows similarly. Let $i \in Q$ be a sink and let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective. Let $K = \ker \varphi$. Then

$$\dim K = \sum_{j \to i} \dim V_j - \dim V_i.$$

Therefore we get

$$\left(d(F_i^+ V) - d(V)\right)_i = \sum_{j \to i} \dim V_j - 2 \dim V_i = -B\left(d(V), \alpha_i\right)$$

and

$$\left(d(F_i^+ V) - d(V)\right)_j = 0, \quad j \neq i.$$

This implies

$$\begin{aligned} & d(F_i^+ V) - d(V) = -B\left(d(V), \alpha_i\right) \alpha_i \\ \Leftrightarrow \quad & d(F_i^+ V) \ = \ d(V) - B\left(d(V), \alpha_i\right) \alpha_i \ = \ s_i\left(d(V)\right). \end{aligned}$$

□

5.7 Coxeter elements

Definition 5.32. Let Q be a quiver and let Γ be the underlying graph. Fix any labeling $1, \ldots, r$ of the vertices of Γ. Then the Coxeter element c of Q corresponding to this labeling is defined as

$$c = s_1 s_2 \ldots s_r.$$

Lemma 5.33. *Let*

$$\beta = \sum_i k_i \alpha_i$$

with $k_i \geq 0$ for all i but not all $k_i = 0$. Then there is $N \in \mathbb{N}$, such that

$$c^N \beta$$

has at least one strictly negative coefficient.

Proof. c belongs to a finite group W. So there is $M \in \mathbb{N}$, such that

$$c^M = 1.$$

We claim that

$$1 + c + c^2 + \cdots + c^{M-1} = 0$$

as operators on $\mathbb{R}^n$. This implies what we need, since β has at least one strictly positive coefficient, so one of the elements

$$c\beta, c^2\beta, \ldots, c^{M-1}\beta$$

must have at least one strictly negative one. Furthermore, it is enough to show that 1 is not an eigenvalue for c, since

$$(1 + c + c^2 + \cdots + c^{M-1})v = w \neq 0$$

$$\Rightarrow \quad cw = c\left(1 + c + c^2 + \cdots + c^{M-1}\right)v = (c + c^2 + c^3 + \cdots + c^{M-1} + 1)v = w.$$

Assume the contrary, i.e. 1 is a eigenvalue of c and let v be a corresponding eigenvector.

$$cv = v \quad \Rightarrow \quad s_1 \ldots s_r v = v$$

$$\Leftrightarrow \quad s_2 \ldots s_r v = s_1 v.$$

But since s_i only changes the i-th coordinate of v, we get

$$s_1 v = v \quad \text{and} \quad s_2 \ldots s_r v = v.$$

Repeating the same procedure, we get

$$s_i v = v$$

for all i. But this means

$$B(v, \alpha_i) = 0.$$

for all i, and since B is nondegenerate, we get $v = 0$. But this is a contradiction, since v is an eigenvector. □

5.8 Proof of Gabriel's theorem

Let V be an indecomposable representation of Q. We introduce a fixed labeling $1, \ldots r$ on Q, such that $i < j$ if one can reach j from i. This is possible, since we can assign the highest label to any sink, remove this sink from the quiver, assign the next highest label to a sink of the remaining quiver and so on. This way we create a labeling of the desired kind.

We now consider the sequence

$$V^{(0)} = V,\ V^{(1)} = F_r^+ V,\ V^{(2)} = F_{r-1}^+ F_r^+ V, \ldots$$

This sequence is well defined because of the selected labeling: r has to be a sink of Q, $r-1$ has to be a sink of $\overline{Q_r}$ and so on. Furthermore we note that $V^{(r)}$ is a representation of Q again, since every arrow has been reversed twice (since we applied a reflection functor to every vertex). This implies that we can define

$$V^{(r+1)} = F_r^+ V^{(r)}, \ldots$$

and continue the sequence to infinity.

Theorem 5.34. *There is $m \in \mathbb{N}$, such that*

$$d\left(V^{(m)}\right) = \alpha_p$$

for some p.

Proof. If $V^{(i)}$ is surjective at the appropriate vertex k, then

$$d\left(V^{(i+1)}\right) = d\left(F_k^+ V^{(i)}\right) = s_k d\left(V^{(i)}\right).$$

This implies, that if $V^{(0)}, \ldots, V^{(i-1)}$ are surjective at the appropriate vertices, then

$$d\left(V^{(i)}\right) = \ldots s_{r-1} s_r d(V).$$

By Lemma 5.33 this cannot continue indefinitely - since $d\left(V^{(i)}\right)$ may not have any negative entries. Let i be smallest number such that $V^{(i)}$ is not surjective at the appropriate vertex. By Proposition 5.30 it is indecomposable. So, by Proposition 5.28, we get

$$d(V^{(i)}) = \alpha_p$$

for some p. □

We are now able to prove Gabriel's theorem. Namely, we get the following corollaries.

Corollary 5.35. *Let Q be a quiver, V be any indecomposable representation. Then $d(V)$ is a positive root.*

Proof. By Theorem 5.34

$$s_{i_1} \ldots s_{i_n} (d(V)) = \alpha_p.$$

Since the s_i preserve B, we get

$$B(d(V), d(V)) = B(\alpha_p, \alpha_p) = 2.$$

□

Corollary 5.36. *Let V, V' be indecomposable representations of Q such that $d(V) = d(V')$. Then V and V' are isomorphic.*

Proof. Let i be such that

$$d\left(V^{(i)}\right) = \alpha_p.$$

Then we also get $d\left(V'^{(i)}\right) = \alpha_p$. So

$$V'^{(i)} = V^{(i)} =: V^i.$$

Furthermore we have

$$V^{(i)} = F_k^+ \ldots F_{r-1}^+ F_r^+ V^{(0)}$$
$$V'^{(i)} = F_k^+ \ldots F_{r-1}^+ F_r^+ V'^{(0)}.$$

But both $V^{(i-1)}, \ldots, V^{(0)}$ and $V'^{(i-1)}, \ldots, V'^{(0)}$ have to be surjective at the appropriate vertices. This implies

$$F_r^- F_{r-1}^- \ldots F_k^- V^i = \begin{cases} F_r^- F_{r-1}^- \ldots F_k^- F_k^+ \ldots F_{r-1}^+ F_r^+ V^{(0)} & = V^{(0)} & = V \\ F_r^- F_{r-1}^- \ldots F_k^- F_k^+ \ldots F_{r-1}^+ F_r^+ V'^{(0)} & = V'^{(0)} & = V' \end{cases}$$

□

These two corollaries show that there are only finitely many indecomposable representations (since there are only finitely many roots) and that the dimension vector of each of them is a positive root. The last statement of Gabriel's theorem follows from

Corollary 5.37. *For every positive root α, there is an indecomposable representation V with*

$$d(V) = \alpha.$$

Proof. Consider the sequence

$$s_r\alpha, s_{r-1}s_r\alpha, \ldots$$

Consider the first element of this sequence which is a negative root (this has to happen by Lemma 5.33) and look at one step before that, calling this element β. So β is a positive root and $s_i\beta$ is a negative root for some i. But since the s_i only change one coordinate, we get

$$\beta = \alpha_i$$

and

$$(s_q \ldots s_{r-1}s_r)\alpha = \alpha_i.$$

We let $\mathbb{C}_{(i)}$ be the representation having dimension vector α_i. Then we define

$$V = F_r^- F_{r-1}^- \ldots F_q^- \mathbb{C}_{(i)}.$$

This is an indecomposable representation and

$$d(V) = \alpha.$$

□

Example 5.38. Let us demonstrate by example how reflection functors work. Consider the quiver D_4 with the orientation of all arrows towards the node (which is labeled by 4). Start with the 1-dimensional representation V_{α_4} sitting at the 4-th vertex. Apply to V_{α_4} the functor $F_3^- F_2^- F_1^-$. This yields

$$F_1^- F_2^- F_3^- V_{\alpha_4} = V_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.$$

Now applying F_4^- we get

$$F_4^- F_1^- F_2^- F_3^- V_{\alpha_4} = V_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4}.$$

Note that this is exactly the inclusion of 3 lines into the plane, which is the most complicated indecomposable representation of the D_4 quiver.

5.9 Problems

Problem 5.39. *Let Q_n be the cyclic quiver of length n, i.e. n vertices connected by n oriented edges forming a cycle. Obviously, the classification of indecomposable representations of Q_1 is given by the Jordan normal form theorem. Obtain a similar classification of indecomposable representations of Q_2. In other words, classify pairs of linear operators $A : V \to W$ and $B : W \to V$ up to isomorphism. Namely:*

(a) Consider the following pairs (for $n \geq 1$):

1) $E_{n,\lambda}$: $V = W = \mathbb{C}^n$, A is the Jordan block of size n with eigenvalue λ, $B = 1$ ($\lambda \in \mathbb{C}$).

2) $E_{n,\infty}$: is obtained from $E_{n,0}$ by exchanging V with W and A with B.

3) H_n*:* $V = \mathbb{C}^n$ *with basis* v_i*,* $W = \mathbb{C}^{n-1}$ *with basis* w_i*,* $Av_i = w_i$*,* $Bw_i = v_{i+1}$ *for* $i < n$*, and* $Av_n = 0$*.*

4) K_n *is obtained from* H_n *by exchanging* V *with* W *and* A *with* B*.*

Show that these are indecomposable and pairwise nonisomorphic.

(b) Show that if E *is a representation of* Q_2 *such that* AB *is not nilpotent, then* $E = E' \oplus E''$*, where* $E'' = E_{n,\lambda}$ *for some* $\lambda \neq 0$*.*

(c) Consider the case when AB *is nilpotent, and consider the operator* X *on* $V \oplus W$ *given by* $X(v, w) = (Bw, Av)$*. Show that* X *is nilpotent, and admits a basis consisting of chains (i.e. sequences* $u, Xu, X^2u, ...X^{l-1}u$ *where* $X^l u = 0$*) which are compatible with the direct sum decomposition (i.e. for every chain* $u \in V$ *or* $u \in W$*). Deduce that (1)-(4) are the only indecomposable representations of* Q_2*.*

(d) generalize this classification to the Kronecker quiver, which has two vertices 1 *and* 2 *and two edges both going from* 1 *to* 2*.*

(e)(harder!) can you generalize this classification to Q_n*,* $n > 2$*?*

Problem 5.40. *Let* $L \subset \frac{1}{2}\mathbb{Z}^8$ *be the lattice of vectors where the coordinates are either all integers or all half-integers (but not integers), and the sum of all coordinates is an even integer.*

(a) Let $\alpha_i = e_i - e_{i+1}$*,* $i = 1, ..., 6$*,* $\alpha_7 = e_6 + e_7$*,* $\alpha_8 = -1/2\sum_{i=1}^{8} e_i$*. Show that* α_i *are a basis of* L *(over* $\mathbb{Z}$*).*

(b) Show that roots in L *(under the usual inner product) form a root system of type* E_8 *(compute the inner products of* α_i*).*

(c) Show that the E_7 *and* E_6 *lattices can be obtained as the sets of vectors in the* E_8 *lattice* L *where the first two, respectively three, coordinates (in the basis* e_i*) are equal.*

(d) Show that E_6, E_7, E_8 *have 72,126,240 roots, respectively (enumerate types of roots in terms of the presentations in the basis* e_i*, and count the roots of each type).*

Problem 5.41. *Let* V_α *be the indecomposable representation of a Dynkin quiver* Q *which corresponds to a positive root* α*. For instance, if* α_i *is a simple root, then* V_{α_i} *has a 1-dimensional space at* i *and 0 everywhere else.*

(a) Show that if i *is a source then* $\mathrm{Ext}^1(V, V_{\alpha_i}) = 0$ *for any representation* V *of* Q*, and if* i *is a sink, then* $\mathrm{Ext}^1(V_{\alpha_i}, V) = 0$*.*

(b) Given an orientation of the quiver, find a Jordan-Holder series of V_α *for that orientation.*