Problem 1.

a) Let \( a_n \) denote the length of the \( n \)-th segment. Let \( d \) be the spacing between two consecutive segments. We want to know whether \( \Sigma a_n \) converges or not.

First, we will define \( a_n \):

\[
a_n = \frac{1}{(1 + nd)^2} - \frac{1}{(1 + nd)^3} = \frac{nd}{(1 + nd)^3}
\]

Now, we will compare \( a_n \) with \( \frac{1}{(nd)^2} \).

\[
\lim_{n \to \infty} \frac{\frac{nd}{(1 + nd)^3}}{\frac{1}{(nd)^2}} = \lim_{n \to \infty} \frac{(nd)^3}{(1 + nd)^3} = \lim_{n \to \infty} \left( \frac{1 + \frac{1}{nd}}{1} \right)^3 = 1
\]

Since \( \frac{1}{d^2} \Sigma \frac{1}{n^2} \) converges, then \( \Sigma a_n \) converges. Hence the sum is finite.

b) \[\]

\[
a_n = \frac{1}{(1 + nd)} - \frac{1}{(1 + nd)^3} = \frac{(nd)^2 + 2nd}{(1 + nd)^3}
\]

Now, we will compare \( a_n \) with \( \frac{1}{nd} \).

\[
\lim_{n \to \infty} \frac{\frac{(nd)^2 + 2nd}{(1 + nd)^3}}{\frac{1}{nd}} = \lim_{n \to \infty} \frac{(nd)^3 + 2(nd)^2}{(1 + nd)^3} = \lim_{n \to \infty} \left( \frac{1 + \frac{2}{nd}}{1} \right)^3 = 1
\]

Since \( \frac{1}{d} \Sigma \frac{1}{n} \) diverges, then \( \Sigma a_n \) diverges too. The sum is infinite.

Problem 2.

a) P. 415: 8.

Compare to \( \Sigma n^{s+\frac{1}{2}} \). You will get that the series converges for \( s < \frac{1}{2} \) and diverges otherwise.

b) P. 415: 9.

Compare to \( \Sigma n^{-2} \). You will get that the series converges.

Problem 3. Solve the integral by partial fractions. You will get:

\[
\int_{1}^{y} \left( \frac{2x^2 + bx + a}{x(2x + a)} - 1 \right) dx = \ln y + \frac{b - a - 2}{2} (\ln(2y - a) - \ln(2 + a))
\]
Note that if $a > b$ the limit as $y \to \infty$ will go to 0, and if $b > a$ the limit will go to infinity. So $a = b$ and solving gives $a = b = 2e - 2$.

Problem 4. Using the root test, you get that the radius of convergence is $\frac{1}{e}$. Simple inspection shows that for $z = \pm \frac{1}{e}$ it diverges, since the limit is not 0.

Problem 5. Ratio test shows that the radius of convergence is $\frac{1}{3}$. Inspection shows that for $x = \pm \frac{1}{2}$ it diverges. Using the integral, you get that the series converges to $\frac{1}{1+2x} + \frac{\ln(1+2x)}{2x}$. 