18.014 Homework 2 - Solutions


\[ x < y \]
\[ 0 < y - x \]

By theorem I30, \( \exists n \in \mathbb{Z}_+ \) such that:

\[ 1 < n(y - x) \]
\[ 1 < ny - nx \]
\[ nx + 1 < ny \]

According to exercise 5, page 28 (solved in recitation) \( \exists m \in \mathbb{Z} \) such that:

\[ nx < m \leq nx + 1 < ny \]
\[ nx < m < ny \]
\[ x < \frac{m}{n} < y \]

Since \( m, n \in \mathbb{Z}, \frac{m}{n} \in \mathbb{Q} \).

Following the same argument, there must exist a rational number between \( x \) and \( \frac{m}{n} \). This process can be carried on indefinitely, hence there must be an infinite number of rational numbers between \( x \) and \( y \).


(a) From exercise 6, we know that the number of lattice points in \( S=\{(x, y) \) s.t. \( 0 < x < b, 0 < y \leq x \cdot \frac{a}{b} \} \) is equal to \( \sum_{n=1}^{b-1}[-\frac{na}{b}] \). \( S \) is the set of all the points inside the right triangle of sides \( a \) and \( b \). Note that no lattice points lie on the hypotenuse of the triangle, since \( \frac{na}{b} \) is an integer if and only if \( n \) is a multiple of \( b \) (because \( a \) and \( b \) have no common factor), and \( 1 \leq n \leq b - 1 \). Hence, the number of lattice points is equal to half the number of lattice points in the rectangle of sides \( a \) and \( b \) (since there are no lattice points on the diagonal), and that is \( \frac{(a-1)(b-1)}{2} \).

(b) Since \( a \) and \( b \) have no common factors, \( \frac{na}{b} \) is not an integer for \( 1 \leq n \leq b - 1 \). Hence, by exercise 4:

\[ [-\frac{na}{b}] = -[\frac{na}{b}] - 1 \]
Therefore:

\[
\begin{align*}
\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= \sum_{n=1}^{b-1} \frac{a(b-n)}{b} \\
&= \sum_{n=1}^{b-1} \left( a - \frac{na}{b} \right) \\
&= \sum_{n=1}^{b-1} \left( a - \left\lfloor \frac{na}{b} \right\rfloor - 1 \right) \\
&= (a - 1)(b - 1) - \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor \\
\Rightarrow 2 \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= (a - 1)(b - 1) \\
\Rightarrow \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= \frac{(a - 1)(b - 1)}{2}
\end{align*}
\]

**Problem 3.**

Let \( s(x) \) be a step function defined on the partition \( P = \{x_0, x_1, ..., x_n\} \), such that \( s(x) = s_k \forall x_{k-1} < x < x_k \). I will add a new subdivision point, \( y \), such that \( x_{k-1} < y < x_k \). Therefore, \( s(y) = s_k \). With the new partition the term \( s^3_k(x_k - x_{k-1}) \) is replaced by:

\[
\begin{align*}
s^3_k(x_k - y) + s^3_k(y - x_{k-1}) &= s^3_kx_k - s^3_ky + s^3_ky - s^3_kx_{k-1} \\
&= s^3_kx_k - s^3_kx_{k-1} \\
&= s^3_k(x_k - x_{k-1})
\end{align*}
\]

Hence, the addition of any subdivision point doesn’t alter the summation, so it doesn’t alter the integral, proving that the integral is independent of the partition.
(a)

\[
\int_a^b s(x)\,dx = \sum_{k=1}^m s_k^3(x_k - x_{k-1}) \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_m = b
\]

\[
\int_b^c s(x)\,dx = \sum_{k=1}^{n-m} s_k^3(y_k - y_{k-1}) \quad \text{with} \quad b = y_0 < y_1 < \cdots < y_{n-m} = c
\]

Let \( y_i = x_{m+i} \). Then \( P = \{x_0, x_1, \ldots, x_n\} \) is a partition for \([a, c]\), in which \( s(x) \) is a step function. Then:

\[
\int_a^b s(x)\,dx + \int_b^c s(x)\,dx = \sum_{k=1}^m s_k^3(x_k - x_{k-1}) + \sum_{k=m+1}^n s_k^3(x_k - x_{k-1})
\]

\[
= \sum_{k=1}^n s_k^3(x_k - x_{k-1})
\]

\[
= \int_a^c s(x)\,dx
\]

(b)

Let \( s(x) = 1, t(x) = 2 \ \forall x \in [0, 1] \). Then:

\[
\int_a^b (s + t) = (1 + 2)^3(1 - 0) = 27 \neq 9 = 1^3(1 - 0) + 2^3(1 - 0) = \int_a^b s + \int_a^b t
\]

(c)

Let \( s(x) = 1, c = 2 \ \forall x \in [0, 1] \). Then:

\[
\int_a^b cs = (1 \cdot 2)^3(1 - 0) = 8 \neq 2 = 2 \cdot (1)^3(1 - 0) = c \cdot \int_a^b s
\]