Lecture III
Vector Algebra in Cartesian Coordinates

Let us construct a Cartesian coordinates system in $\mathbb{E}^3$. First we choose a point $O$, called the origin. Then we chose three mutually perpendicular rays starting from $O$. These rays are called the positive $x$ axis, positive $y$ axis, and positive $z$ axis. Consider the lines containing these rays. For any of these lines, every point on it can be identified with a real number: if the point is on the ray, the real number is the distance to $O$, if it’s not on the ray, the number is the distance to $O$ times $-1$. Let us denote these three lines by $X, Y, Z$. These points give the Cartesian coordinates of $P$, denoted $x_P, y_P, z_P$. Any triplet of real numbers forms the coordinates for some point $P$. Different points have different coordinates.

The three unit vectors in the directions of the positive $x, y,$ and $z$ axes are customarily denoted by $\hat{i}, \hat{j},$ and $\hat{k}$. Let $\vec{A}$ be a vector in $\mathbb{E}^3$ and let $P$ be the point such that $\overrightarrow{OP} = \vec{A}$. Let $(a_1, a_2, a_3)$ be the coordinates of $P$. Consider the vectors $\vec{A}_1 = a_1 \hat{i}, \vec{A}_2 = a_2 \hat{j},$ and $\vec{A}_3 = a_3 \hat{k}$. by vector addition and multiplication with scalars, one obtains the following expression:

$$\vec{A} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

Then $a_1, a_2,$ and $a_3$ are called the scalar components of $\vec{A}$, and $a_1 \hat{i}, a_2 \hat{j},$ and $a_3 \hat{k}$ are called the vector components of $\vec{A}$. By using the $\hat{i}, \hat{j}, \hat{k}$ unit vectors, we obtain coordinate formulas for the four basic vector operations:

1. Multiplication by a scalar

$$c \vec{A} = (ca_1) \hat{i} + (ca_2) \hat{j} + (ca_3) \hat{k},$$

for any scalar $c.$
2. Addition of vectors

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{A} + \vec{B} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}. $$

3. Dot product

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, considering that $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$, we get that

$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3.$$ 

4. Cross product

Let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. To compute the cross product, we use the following equalities:

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0, \quad \text{and} \quad \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}, \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}, \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}.$$ 

We get the following formula:

$$\vec{A} \times \vec{B} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$ 

Using determinants one can easily remember the formula, since:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ 

Considering that $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$, we get that $\vec{A}$ can be expressed as

$$\vec{A} = (\vec{A} \cdot \hat{i})\hat{i} + (\vec{A} \cdot \hat{j})\hat{j} + (\vec{A} \cdot \hat{k})\hat{k},$$

which is known as the *frame identity*.

**Triple products**

1. Scalar triple product
Let \( \vec{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \), \( \vec{B} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} \), and \( \vec{C} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k} \). The triple product \((\vec{A} \times \vec{B}) \cdot \vec{C}\) is called a scalar triple product, since it is a scalar quantity. Its value is given by:

\[
(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix}.
\]

Hence we can simply write \((\vec{A} \times \vec{B}) \cdot \vec{C}\) as \([\vec{A}, \vec{B}, \vec{C}]\) without specifying the positions for the cross and dot signs. We will use the following equalities in computing quadruple products:

\[
[\vec{A}, \vec{B}, \vec{C}] = [\vec{B}, \vec{C}, \vec{A}] = [\vec{C}, \vec{A}, \vec{B}] = -[\vec{A}, \vec{C}, \vec{B}] = -[\vec{C}, \vec{B}, \vec{A}] = -[\vec{B}, \vec{A}, \vec{C}].
\]

2. Vector triple product

The cross product is not associative so we will give two formulas:

\[
(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{C} \cdot \vec{A})\vec{B} - (\vec{C} \cdot \vec{B})\vec{A}, \quad \text{and}
\]

\[
\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}.
\]

**Quadruple products**

1. Scalar quadruple product

The expression \((\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D})\) is called a quadruple scalar product, and by applying the formulas for triple products, we get the value:

\[
(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{C} \cdot \vec{A}) (\vec{B} \cdot \vec{D}) - (\vec{C} \cdot \vec{B}) (\vec{A} \cdot \vec{D}).
\]

2. Vector quadruple product

The expression \((\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})\) is called a quadruple vector product, and by applying the formulas for triple products, we get the value:

\[
(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{C}, \vec{D}, \vec{A}]\vec{B} - [\vec{C}, \vec{D}, \vec{B}]\vec{A}.
\]