Lecture XVI

Integrals in Polar, Cylindrical, or Spherical Coordinates

Usually, we write functions in the Cartesian coordinate system. Hence we write and compute multiple integrals in Cartesian coordinates. But there are other coordinate systems that can help us compute iterated integrals faster. We analyze bellow three such coordinate systems.

1 Polar coordinates

In $\mathbf{E^2}$, the *polar coordinates system* is often used along with a Cartesian system. The polar coordinates of a point $P(x, y)$ are r and θ where r is the distance from the origin O and θ is the angle done by $\vec{OP}$ and $\vec{Ox}$ measured counter-clockwise from $\vec{Ox}$. The equations we use to switch from one system to the other are:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}.$$

To transform an integral $\int\int_R f(x, y)dA$ from Cartesian coordinates to polar coordinates, we take a look at Reimann sums. By consider a particular type of subregions, we get that $dA = rdrd\theta$, so we can write:

$$\int\int_R f(x, y)dA = \int\int_{\hat{R}} f(r\cos\theta, r\sin\theta)rdrd\theta,$$

where $\hat{R}$ is the region corresponding to R in a plane with Cartesian coordinates where the axes are r and θ. This plane is called *the $r\theta$ plane*. Now if $\hat{R}$ is $r\theta$-simple, we can compute the iterated integral in polar coordinates.

For example, let us compute the area of the region R enclosed by the curve $r = 1 + \cos\theta$, where θ goes from 0 to 2π. By looking at the graph of the curve, we can see that the area is equal to twice the area enclosed when θ goes from 0 to π. In polar coordinates, we can see that $\hat{R}$ is a simple region, so we can write:

$$\int\int_R dA = 2\int_0^{\pi}\int_0^{1+\cos\theta} rdrd\theta$$

$$= \int_0^{\pi} (r^2|_0^{1+\cos\theta}) d\theta$$

$$= \int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta d\theta$$

$$= \pi + \int_0^{\pi} \cos^2\theta d\theta$$

Knowing that for any $n \in \mathbb{Z}$, the following equality holds

$$\int_0^{n\pi} \cos^2\theta d\theta = \int_0^{n\pi} \sin^2\theta d\theta,$$

and that $\cos^2\theta + \sin^2\theta = 1$ for all θ, we get that

$$\int_0^{\pi} \cos^2\theta d\theta = \frac{\pi}{2}, \qquad so$$

$$\int\int_R dA = \pi + \int_0^{\pi} \cos^2\theta d\theta = \frac{3\pi}{2}.$$

2 Cylindrical coordinates

In $\mathbf{E^3}$, it is easier sometimes to use the *cylindrical coordinates system*. We can view this system as the extension of the polar coordinates system in $\mathbf{E^3}$. Let $P(x, y, z)$ be a point in $\mathbf{E^3}$ and let P' be its projection on the xy plane. The cylindrical coordinates of P are r, θ, and z, where r and θ are the polar coordinates of P' in the xy plane, and z is the same as in Cartesian coordinates. Hence the equations that link the two systems are the same as for polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}.$$

Hence for integrals the following equality holds:

$$\int\int\int_R f(x, y, z) dV = \int\int\int_{\hat{R}} f(r\cos\theta, r\sin\theta, z) r dr d\theta dz,$$

where $\hat{R}$ is the region corresponding to R in the $r\theta z$ space.

3 Spherical coordinates

In $\mathbf{E^3}$, we can also use the *spherical coordinates system*. Let x, y, z be the Cartesian coordinates of a point P and let us denote its spherical coordinates

by ρ, φ, and θ. Then ρ is the distance from P to the origin O, φ is the angle made by $\vec{OP}$ and $\vec{Oz}$ that is not greater than π, and θ is defined exactly as in the cylindrical coordinates system. The equations that link the spherical and Cartesian systems are:

$$x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi,$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

To move integrals from the Cartesian system to the spherical system, we use the following equality:

$$dV = \rho^2 \sin\varphi d\rho d\varphi d\theta.$$

Hence in terms of integrals, the following equality holds

$$\int\int\int_R f(x, y, z) dV =$$

$$= \int\int\int_{\hat{R}} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \rho^2 \sin\varphi d\rho d\varphi d\theta,$$

where $\hat{R}$ is the region coresponding to R in the $\rho\varphi\theta$ space.