Lecture XXI

Line Integrals; Conservative Fields

1 Line integrals

Let us recapitulate the basic notions reffering to line integrals. For both scalar and vector fields, we can define line integrals. For a scalar field f on a curve C, the line integral is denoted by $\int_C f ds$. For a vector field $\vec{F}$, it is denoted $\int_C \vec{F} \cdot d\vec{R}$. We can evaluate line integrals either by their definition as limits of Riemann sums, or by parameters. Let us consider a path $\vec{R}(t)$ for the curve C, t going from a to b. Then we can evaluate the line integral using parameter t:

$$\int_C f ds = \int_a^b f \left| \frac{d\vec{R}}{dt} \right| dt, \qquad \int_C \vec{F} \cdot d\vec{R} = \int_a^b \left(\vec{F} \cdot \frac{d\vec{R}}{dt} \right) dt.$$

For vector fields we can also define other line integrals as well. If $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$, then $\int_C F ds = \left(\int_C M ds\right)\hat{\imath} + \left(\int_C N ds\right)\hat{\jmath}$. Also, $\frac{\int_C \vec{F} ds}{\int_C ds}$ is the average value of $\vec{F}$ on C. We can also define the line integral $\int_C \vec{F} \times d\vec{R} = \int_C \left(\vec{F} \times \frac{d\vec{R}}{dt} \right) dt$.

2 Conservative fields

Let D be the domain of a vector field $\vec{F}$. We say that $\vec{F}$ has *independence of path on D* if for any loop C in D, $\int_C \vec{F} \cdot d\vec{R} = 0$. If $\vec{F}$ has independence of path, then $\vec{F}$ is called a *conservative field*. Recall that $\vec{F}$ is called a *gradient field* if there exists a scalar field f such that $\vec{\nabla} f = \vec{F}$.

Theorem 1 (Conservative-field Theorem) *Let $\vec{F}$ be a continuous vector field on a domain D. Then $\vec{F}$ is conservative if and only if $\vec{F}$ is a gradient field.*

Proof:

$\Leftarrow$ Since $\vec{F}$ is a gradient field, it has a scalar potential f. Hence

$$\int_C \vec{F} \cdot d\vec{R} = \int_C \vec{\nabla} f \cdot d\vec{R} = \int_a^b \frac{d}{dt} f(\vec{R}(t))dt = f(\vec{R}(b)) - f(\vec{R}(a)).$$

$\Rightarrow$ Let Q be a fixed point in D. For any point $P \in D$, we define $f(P) = \int_C \vec{F} \cdot d\vec{R}$, where C is a curve from Q to P. Clearly, since $\vec{F}$ is conservative, f is well defined. It follows that f is a scalar potential for $\vec{F}$, so $\vec{F}$ is a gradient field.

If we know that $\vec{F}$ is conservative, there are two methods which we can use to find a scalar potential for $\vec{F}$.

1. The vector line integral method uses the conservative field theorem. More precisely, if we fix a point $Q \in D$ and let $f(P) = \int_C \vec{F} \cdot d\vec{R}$, where C is a curve from Q to P, then f is a scalar potential for $\vec{F}$.

2. The indefinite integral method is often simpler than the line integral method. Let $\vec{F}(x, y) = M(x, y)\hat{\imath} + N(x, y)\hat{\jmath}$ and f be the scalar potential to be found. Then $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. Hence $f = \int M dx + C(y)$, so $f_y = N = \frac{d}{dy}(\int M dx) + C'(y)$, and from this last equation we find $C(y)$, hence finding f. For example, let us take $\vec{F} = (2x-3y-4)\hat{\imath}+(4y-3x+2)\hat{\jmath}$. Then $f = x^2 - 3xy - 4x + C(y)$, so $\frac{\partial f}{\partial y} = -3x + C'(y) = 4y - 3x + 2$. Hence $C'(y) = 4y + 2$, so $C(y) = 2y^2 + 2y + c$. It follows that $f(x, y) = x^2 - 3xy - 4x + 2y^2 + 2x + c$ is a scalar potential for $\vec{F}$.