Lecture XXI
Line Integrals; Conservative Fields

1 Line integrals

Let us recapitulate the basic notions referring to line integrals. For both scalar and vector fields, we can define line integrals. For a scalar field $f$ on a curve $C$, the line integral is denoted by $\int_C f ds$. For a vector field $\vec{F}$, it is denoted $\int_C \vec{F} \cdot d\vec{R}$. We can evaluate line integrals either by their definition as limits of Riemann sums, or by parameters. Let us consider a path $\vec{R}(t)$ for the curve $C$, $t$ going from $a$ to $b$. Then we can evaluate the line integral using parameter $t$:

$$\int_C f ds = \int_a^b f \left| \frac{d\vec{R}}{dt} \right| dt, \quad \int_C \vec{F} \cdot d\vec{R} = \int_a^b \left( \vec{F} \cdot \frac{d\vec{R}}{dt} \right) dt.$$

For vector fields we can also define other line integrals as well. If $\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$, then $\int_C F ds = \left( \int_C M ds \right) \hat{i} + \left( \int_C N ds \right) \hat{j}$. Also, $\frac{\int_C \vec{F} ds}{\int_C ds}$ is the average value of $\vec{F}$ on $C$. We can also define the line integral $\int_C \vec{F} \times d\vec{R} = \int_C \left( \vec{F} \times \frac{d\vec{R}}{dt} \right) dt$.

2 Conservative fields

Let $D$ be the domain of a vector field $\vec{F}$. We say that $\vec{F}$ has independence of path on $D$ if for any loop $C$ in $D$, $\int_C \vec{F} \cdot d\vec{R} = 0$. If $\vec{F}$ has independence of path, then $\vec{F}$ is called a conservative field. Recall that $\vec{F}$ is called a gradient field if there exists a scalar field $f$ such that $\nabla f = \vec{F}$.

**Theorem 1 (Conservative-field Theorem)** Let $\vec{F}$ be a continuous vector field on a domain $D$. Then $\vec{F}$ is conservative if and only if $\vec{F}$ is a gradient field.

Proof:
\(\leftarrow\) Since \(\vec{F}\) is a gradient field, it has a scalar potential \(f\). Hence
\[
\int_C \vec{F} \cdot d\vec{R} = \int_C \nabla f \cdot d\vec{R} = \int_a^b \frac{d}{dt} f(\vec{R}(t)) dt = f(\vec{R}(b)) - f(\vec{R}(a)).
\]

\(\Rightarrow\) Let \(Q\) be a fixed point in \(D\). For any point \(P \in D\), we define \(f(P) = \int_C \vec{F} \cdot d\vec{R}\), where \(C\) is a curve from \(Q\) to \(P\). Clearly, since \(\vec{F}\) is conservative, \(f\) is well defined. It follows that \(f\) is a scalar potential for \(\vec{F}\), so \(\vec{F}\) is a gradient field.

If we know that \(\vec{F}\) is conservative, there are two methods which we can use to find a scalar potential for \(\vec{F}\).

1. The vector line integral method uses the conservative field theorem. More precisely, if we fix a point \(Q \in D\) and let \(f(P) = \int_C \vec{F} \cdot d\vec{R}\), where \(C\) is a curve from \(Q\) to \(P\), then \(f\) is a scalar potential for \(\vec{F}\).

2. The indefinite integral method is often simpler than the line integral method. Let \(\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}\) and \(f\) be the scalar potential to be found. Then \(M = \frac{\partial f}{\partial x}\) and \(N = \frac{\partial f}{\partial y}\). Hence \(f = \int M\, dx + C(y)\), so \(f_y = N = \frac{\partial}{\partial y}(\int M\, dx) + C'(y)\), and from this last equation we find \(C(y)\), hence finding \(f\). For example, let us take \(\vec{F} = (2x - 3y - 4)\hat{i} + (4y - 3x + 2)\hat{j}\). Then \(f = x^2 - 3xy - 4x + C(y)\), so \(\frac{\partial f}{\partial y} = -3x + C'(y) = 4y - 3x + 2\). Hence \(C'(y) = 4y + 2\), so \(C(y) = 2y^2 + 2y + c\). It follows that \(f(x, y) = x^2 - 3xy - 4x + 2y^2 + 2x + c\) is a scalar potential for \(\vec{F}\).