Lecture XXIV

Measures

Remember that an elementary region R in $\mathbf{E^2}$ is a region that has a simple, closed, piecewise smooth curve C as its boundary. A regular region R is a region that is either regular or can be divided into finitely many regular regions. In the second case, the boundary of R consists of two or more simple, closed, piecewise smooth curves, of which one is the outer boundary curve, denoted by C, and the others are interior boundary curves $C_1, \ldots, C_n$. Then R consists of the points in the interior of C, excluding those in the interior of $C_1, \ldots, C_n$.

Similarly, in $\mathbf{E^3}$ a region is regular if its boundary is a simple, closed, piecewise smooth surface S. An elementary region in $\mathbf{E^3}$ is either regular or can be divided into finitely many regular regions. The boundary of an elementary region that is not regular is formed by one or more closed, but not simple piecewise smooth surfaces. In the case that the boundary is formed by two or more surfaces, one of them is the outer boundary surface S, and the other are interior boundary surfaces $S_1, \ldots, S_n$. Then R consists of the points in the interior of S, excluding those in the interior of $S_1, \ldots, S_n$.

Definition 1 *Let D be a region in $\mathbf{E^2}$ or $\mathbf{E^3}$. Let μ be a real-valued function that has as inputs the regular subregions of D. The function μ is called a* finite measure *on D if for every regular subregion R and for every division of R into regular subregions R_1 and R_2 we have*

$$\mu(R_1) + \mu(R_2) = \mu(R).$$

In $\mathbf{E^2}$, an example of a finite measure is the function that assigns to every regular region its area. Similarly, in $\mathbf{E^3}$, the function that assignes to every regular region its volume is a finite measure. More generally, in $\mathbf{E^2}$, if f is a continuous scalar field on a region D, for every regular subregion R of D we define

$$\mu_f(R) = \int\int_R f dA.$$

Then μ_f is a measure on D. Similarly, in $\mathbf{E^3}$, $\mu_f(R) = \int\int\int_R f dV$ is a measure for every continuous scalar field f.

Definition 2 *Let $\vec{F}$ be a continuous vector field on a region D in $\mathbf{E^2}$. We define the finite measure $\mu_{\vec{F}}$ in the following manner:*

1. *if R is elementary, $\mu_{\vec{F}} = \oint_C \vec{F} \cdot d\vec{R}$, where C is the counterclockwise directed boundary of R.*

2. *if R is not elementary, $\mu_{\vec{F}} = \oint_C \vec{F} \cdot d\vec{R} + \oint_{C_1} \vec{F} \cdot d\vec{R} + \cdots + \oint_{C_n} \vec{F} \cdot d\vec{R}$, where C is the counterclockwise directed outer boundary of R and $C_1, \ldots, C_n$ are the clockwise directed inner boundaries of R.*

It is easily verifiable that $\mu_{\vec{F}}$ is a finite measure. It is called the circulation measure on D given by $\vec{F}$.

Definition 3 *Let $\vec{F}$ be a continuous vector field on D in $\mathbf{E^3}$. We define the finite measure $\mu^f_{\vec{F}}$ in the following way:*

1. *if R has a single boundary surface S, then $\mu^f_{\vec{F}}(R) = \oiint_S \vec{F} \cdot d\vec{\sigma}$, where S is directed outward, away from R.*

2. *if R has boundary surfaces S, $S_1, \cdots, S_n$, then $\mu^f_{\vec{F}}(R) = \oiint_S \vec{F} \cdot d\vec{\sigma} + \oiint_{S_1} \vec{F} \cdot d\vec{\sigma} + \cdots + \oiint_{S_n} \vec{F} \cdot d\vec{\sigma}$, where S is directed outward, and $S_1, \ldots, S_n$ are directed inward.*

It is easy to prove that $\mu^f_{\vec{F}}$ is a finite measure. It is called the flux measure on D given by $\vec{F}$.

Definition 4 *Let D be a region in $\mathbf{E^3}$. Let μ^s be a real-valued function that has as inputs the finite, piecewise smooth, directed surfaces in D. The function μ^s is called a* finite surface measure *on D if for every finite, piecewise smooth, directed surface S and for every division of S into finite, piecewise smooth, directed surfaces S_1 and S_2 we have*

$$\mu^s(S_1) + \mu^s(S_2) = \mu^s(S).$$

Definition 5 *Let $\vec{F}$ be a continuous vector field on D in $\mathbf{E^3}$. We define the finite surface measure $\mu^c_{\vec{F}}$ in the following way:*

1. *if S is closed, then $\mu^c_{\vec{F}} = 0$.*

2. *if S is not closed, let $C_1, \ldots, C_n$ be the piecewise smooth, simple, closed boundary curves of S, with directions coherent with the direction of S. Then $\mu^c_{\vec{F}} = \oint_{C_1} \vec{F} \cdot d\vec{R} + \cdots + \oint_{C_n} \vec{F} \cdot d\vec{R}$.*

It is easily verifiable that $\mu^c_{\vec{F}}$ is a finite surface measure. It is called the circulation measure on D given by $\vec{F}$.