In this lecture, we will define a new type of derivative for vector fields on \( \mathbb{E}^3 \), called divergence. Let \( \vec{F} \) be a vector field defined on a domain \( D \). Let us start by defining the divergence of \( \vec{F} \) on interior points of \( D \), i.e. points \( P \) such that there exists a sphere of center \( P \) and radius \( a > 0 \) with its interior contained in \( D \).

**Definition 1** Let \( \vec{F} \) be a continuous vector field on \( D \) in \( \mathbb{E}^3 \). Let \( P \) be an interior point of \( D \), and let \( S(P,a) \) be the sphere of center \( P \) and radius \( a \), for all \( a > 0 \). The volume of \( S(P,a) \) is \( \frac{4\pi a^3}{3} \) and the flux of \( \vec{F} \) through \( S(P,a) \) is \( \iint_{S(P,a)} \vec{F} \cdot d\vec{\sigma} \). Consider the limit

\[
\lim_{a \to 0} \frac{3}{4\pi a^3} \iint_{S(P,a)} \vec{F} \cdot d\vec{\sigma}.
\]

If this limit exists, then it is called the divergence of \( \vec{F} \) at \( P \), and it is denoted by \( \text{div}\vec{F}|_P \).

Below are two important properties of divergence.

1. **Existence:** If \( \vec{F} \) is \( C^1 \) on \( D \), then the divergence of \( \vec{F} \) exists at every interior point of \( D \).

2. **Linearity:** If \( \vec{F} \) and \( \vec{G} \) are vector fields defined on \( D \), for any two scalar constants \( a \) and \( b \) the following equality holds:

\[
\text{div}(a\vec{F} + b\vec{G}) = a(\text{div}\vec{F}) + b(\text{div}\vec{G}).
\]

The following theorem helps us find a formula for divergence in Cartesian coordinates.
Theorem 1 (The parallel flow theorem in \( \mathbb{E}^3 \)) Let \( \vec{F} \) be a vector field on \( D \) in \( \mathbb{E}^3 \) such that there exists a scalar field on \( D \) and a constant unit vector \( \hat{w} \) for which \( \vec{F} = f \hat{w} \) on \( D \). Such a vector field \( \vec{F} \) is called a parallel flow. Suppose \( f \) is \( C^1 \). Then for any interior point \( P \) of \( D \), the following equality holds:

\[
\text{div} \vec{F}\big|_P = \frac{df}{ds}\big|_{\hat{w}, P}.
\]

Let \( \vec{F} = L \hat{i} + M \hat{j} + N \hat{k} \) be a \( C^1 \) vector field on a domain \( D \) in \( \mathbb{E}^3 \). By using the linearity of divergence and applying the parallel flow theorem to \( L \hat{i}, M \hat{j}, \) and \( N \hat{k} \), we get the following formula:

\[
\text{div} \vec{F}\big|_P = L_x(P) + M_y(P) + N_z(P) = \frac{\partial L}{\partial x}\bigg|_P + \frac{\partial M}{\partial y}\bigg|_P + \frac{\partial N}{\partial z}\bigg|_P.
\]

Through the following theorem, we can use divergence to compute surface integrals more easily.

Theorem 2 (The divergence theorem) Let \( \vec{F} \) be a \( C^1 \) vector field on \( D \) in \( \mathbb{E}^3 \). Let \( R \) be a regular region in \( D \), with outward directed outer boundary surface \( S \) and inward directed inner boundary surfaces \( S_1, \ldots, S_n \). Then

\[
\int \int \int_R \text{div} \vec{F} \, dV = \oint_S \vec{F} \cdot d\vec{\sigma} + \oint_{S_1} \vec{F} \cdot d\vec{\sigma} + \cdots + \oint_{S_n} \vec{F} \cdot d\vec{\sigma}.
\]

Let \( R \) be a regular region in \( D \), and let \( S \) be its boundary. Then, by the divergence theorem, we have that

\[
\int \int \int_R \text{div} \vec{F} \, dV = \oint_S \vec{F} \cdot d\vec{\sigma}.
\]