Lecture XXXI

Linear Equation Systems

As we saw in the previous lecture, we can multiply $m \times n$ matrices by column n-vectors. Consider the rows of an $m \times n$ matrix A to be n-vectors:

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}, \quad \vec{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad then \qquad \vec{AC} = \begin{bmatrix} \vec{r}_1 \cdot \vec{C} \\ \vec{r}_2 \cdot \vec{C} \\ \vdots \\ \vec{r}_m \cdot \vec{C} \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}, \quad \vec{C} = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}, \qquad then \qquad \vec{AC} = \begin{bmatrix} 27 \\ 22 \end{bmatrix}$$

We can use multiplication by a column vector to solve equation systems. An $m \times n$ equation system has the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m$$

This system has m equations and n unknowns: $x_1, x_2, \cdots, x_n$. An example of a 2×3 system is

$$4x_1 + 3x_2 - x_3 = 1$$

$$x_1 - x_2 + x_3 = 6$$

This system has infinitely many solutions.

An example of a 3×2 system is

$$x_1 + x_2 = 4$$

$$2x_1 - x_2 = 6$$

$$x_1 - x_2 = 1$$

This system has no solutions.
An example of a 2×2 system is

$$2x_1 - x_2 = 4$$

$$x_1 + x_2 = 0$$

This system has an unique solution, namely $x_1 = \frac{4}{3}, x_2 = -\frac{4}{3}$.
A system that has all the coefficients on the right-hand side equal to zero, i.e. $d_1 = d_2 = \cdots = d_m = 0$, is called a *homogeneous system*.
An $m \times n$ system with the general form given above can by written as a matrix equality. Let D be an $m \times 1$ column-vector whose entries are $d_1, d_2, \cdots, d_m$ from top to bottom. Let $A = (a_{ij})$ be the $m \times n$ matrix those entries are the coefficients on the left-hand side of the system. Let X be an $n \times 1$ column-vector whose entries are $x_1, x_2, \cdots, x_n$ from top to bottom. Then the left-hand side of the system is given by the elements of AX, so the system can be written as $AX = D$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \textit{hence}$$

$$AX = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} = D$$

We define three elementary operations on the equations of a system that do not modify its solution-set:

(α) multiplying an equation by a non-zero scalar;

(β) adding to an equation some multiple of a different equation;

(γ) interchanging two equations.

By using these operations through the method of *row-reduction*, we can simplify the equation system without altering its solution-set. In the next lecture will define row-reduction, which is also known as *Gauss-Jordan reduction*.