Lecture XXXI
Linear Equation Systems

As we saw in the previous lecture, we can multiply $m \times n$ matrices by column $n$-vectors. Consider the rows of an $m \times n$ matrix $A$ to be $n$-vectors:

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{then} \quad A\vec{c} = \begin{bmatrix} \vec{r}_1 \cdot \vec{c} \\ \vec{r}_2 \cdot \vec{c} \\ \vdots \\ \vec{r}_m \cdot \vec{c} \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}, \quad \text{then} \quad A\vec{c} = \begin{bmatrix} 27 \\ 22 \end{bmatrix}$$

We can use multiplication by a column vector to solve equation systems. An $m \times n$ equation system has the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m$$

This system has $m$ equations and $n$ unknowns: $x_1, x_2, \cdots, x_n$. An example of a $2 \times 3$ system is

$$4x_1 + 3x_2 - x_3 = 1$$
$$x_1 - x_2 + x_3 = 6$$

This system has infinitely many solutions.

An example of a $3 \times 2$ system is

$$x_1 + x_2 = 4$$
\[2x_1 - x_2 = 6\]
\[x_1 - x_2 = 1\]

This system has no solutions.

An example of a 2 \times 2 system is
\[2x_1 - x_2 = 4\]
\[x_1 + x_2 = 0\]

This system has an unique solution, namely \(x_1 = \frac{4}{3}, x_2 = -\frac{4}{3}\).

A system that has all the coefficients on the right-hand side equal to zero, i.e. \(d_1 = d_2 = \cdots = d_m = 0\), is called a \textit{homogeneous system}.

An \(m \times n\) system with the general form given above can by written as a matrix equality. Let \(D\) be an \(m \times 1\) column-vector whose entries are \(d_1, d_2, \cdots, d_m\) from top to bottom. Let \(A = (a_{ij})\) be the \(m \times n\) matrix those entries are the coefficients on the left-hand side of the system. Let \(X\) be an \(n \times 1\) column-vector whose entries are \(x_1, x_2, \cdots, x_n\) from top to bottom. Then the left-hand side of the system is given by the elements of \(AX\), so the system can be written as \(AX = D\).

We define three elementary operations on the equations of a system that do not modify its solution-set:

(\(\alpha\)) multiplying an equation by a non-zero scalar;

(\(\beta\)) adding to an equation some multiple of a different equation;
(γ) interchanging two equations.

By using these operations through the method of row-reduction, we can simplify the equation system without altering its solution-set. In the next lecture will define row-reduction, which is also known as Gauss-Jordan reduction.