# 22.05 Reactor Physics - Part Eleven

## Application of One-Velocity Diffusion Equation to Non-Multiplying Media

### 1. Non-Multiplying Medium:

A non-multiplying medium is a material that does not contain any nuclide that can undergo fission. That is, no Th-233, U-235, or Pu-239 nor any nuclides that have a non-zero cross-section for fast fission. Many practical problems involve non-multiplying medium including:

- effectiveness of a shield for attenuating neutrons;
- transport of neutrons in the human body during a medical procedure; and
- effectiveness of neutrons as an imaging source in radiography.

### 2. Problem Solutions:

For each problem the equation is the same – the one-velocity neutron diffusion equation with the source term set to zero. Very different solutions result because of the boundary conditions.

### 3. Infinite Planar Source:

An infinite planar source might be a fission converter plate such as the one installed in the MIT Research Reactor for use in neutron capture therapy. For tumors of the brain, the idea is to inject a patient with a boronated drug that concentrates in tumor. Then, expose that tumor to neutrons. The boron splits in two:

$$ {}_0^1\mathrm{n} + {}_5^{10}\mathrm{B} \rightarrow {}_2^4\mathrm{He} + {}_3^7\mathrm{Li} + \mathrm{Q} $$

The He/Li nuclei are charged particles and travel less than a cell diameter. Hence, they destroy the cancer cell but do not travel to adjacent healthy cells. So, one has a selective means of targeting tumor provided, of course, that the drug is tumor-specific. In contrast, x-ray therapy targets all cells.

The fission converter works as follows: Neutrons from the reactor core leak out of the reactor and pass through a large region of graphite thereby becoming fully moderated. These neutrons then strike a fission plate, a linear array of fuel elements, that is of narrow thickness but of considerable width and height. That is:

$$\Delta x << y \text{ or } z \text{ dimensions}$$

Hence, it is in effect an infinite plate. These thermal neutrons cause fissions. The resulting fission spectrum neutrons pass through filters – materials that absorb unwanted energies such as fast neutrons that could travel large distances in a patient and give high dose. Also, thermals that would travel short distances and irradiate the skin but not provide any therapeutic benefit are filtered out. So, the purpose of the calculation might be to evaluate these filters for neutron therapy.

The starting point is the one-group diffusion equation.

$$\frac{1}{v}\frac{\partial\phi(r,t)}{\partial t} = \nu\overline{\Sigma}_F\phi(r,t) - \overline{\Sigma}_a\phi(r,t) + D\nabla^2\phi(r,t)$$

The neutron source is the infinite plate. We are interested in the movement of the neutrons in the materials adjacent to the plate. Those materials are non-multiplying, and we write the equation for those materials. So, the $\nu\Sigma_f\phi$ term is zero and the source condition (the plate) will be introduced as a boundary condition. Another simplification is that the problem is steady-state. So, the time-dependent term on the left is also zero. Finally, we are only interested in the x-direction because the geometry is taken as infinite in y and z and hence there is no variation with y and z. The equation becomes:

$$D\nabla^2\phi(x) - \Sigma_a\phi(x) = 0 \qquad ; x \neq 0$$

or

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = 0$$

Where $L^2=D/\Sigma_a$. The quantity $L^2$ is called the diffusion area and L is the diffusion length. A physical interpretation will be given later.

The general solution to the above equation is:

$$\phi(x) = Ae^{-x/L} + Ce^{x/L}$$

We need two boundary conditions. For the first, we have the requirement that the flux remain finite. Hence, the arbitrary constant, C, must be zero. Otherwise, the flux would become infinite as x increases. So, we are left with:

$$\phi(x) = Ae^{-x/L}$$

For the second boundary condition, we have

$$\lim_{x \to 0} \mathbf{J}(x) = S/2$$

Where S is the source strength. This arises because half the neutrons go in the +x and half in the –x direction. As, x approaches the surface of the source, the current (normal to the infinite plane) in the x-direction is S/2.

Then, from Fick's Law,

$$\mathbf{J}(x) = -\frac{Dd\phi(x)}{dx} = \frac{DA}{L}e^{-x/L}$$

and as $x \to 0$

$$\mathbf{J}(x) = \frac{S}{2} = \frac{DA}{L}$$

i.e.,

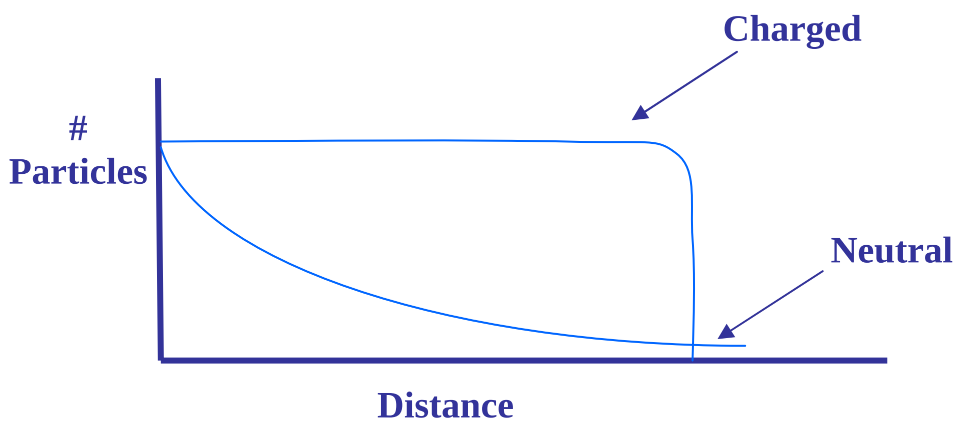
$$A = SL/2D$$

and

$$\phi(x) = \frac{SL}{2d}e^{-x/L}$$

This states that the neutrons drop off exponentially with distance. Is this a logical result? Remember that diffusion theory involves a number of assumptions. The only way to be certain of the validity of a solution is to compare it against measurement.

We know that neutral and charged particles behave very differently as they propagate through material. Neutral ones tend to undergo "all or nothing" collisions. That is, they either interact and are removed, or they don't interact and continue to migrate. Charged particles undergo many many "very small" collisions. That is, they lose small amounts of energy while undergoing many thousands of collisions. Thus,




Neutrons, being neutral, conform to the all or nothing mode. So, we expect exponential drop of. Will it be exactly as calculated? No, we have ignored scattering and variations of cross-section with energy. The former will cause some neutrons to undergo multiple (2 or 3) interactions; the latter will cause resonance effects. But, as a first estimate, the diffusion theory prediction is reasonable.

**4.** **Point Source**:

A point source that emits S neutrons per second isotropically might be a good approximation to a therapeutic device (Californium) that is embedded in a tumor. Neutrons are high LET radiation which means that they disrupt DNA directly as opposed to low LET radiation (photons) that damage indirectly. High LET radiation offers therapeutic advantages. But it is rarely used because of the difficulty in generating such sources. Low LET x-rays, on the other hand, are easily produced.

The geometry is spherical (r, $\theta$, $\phi$) and because a point source is specified, the flux depends only on r. So, we write the Laplacian $\left(\nabla^2\right)$term in spherical coordinates. Thus, the one-group diffusion equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} - \frac{1}{L^2}\phi = 0$$

Where we have set both the time-dependent and source terms to zero.

Let $w = r\phi$. Thus, the above equation becomes:

$$\frac{d^2w}{dr^2} - \frac{1}{L^2}w = 0$$

If you have difficulty in showing this, see next page.

$$\theta = w / r$$

$$d\theta / dr = (r\,dw / dr - w) / r^2$$

$$r^2\, d\theta / dr = (r dw / dr - w)$$

$$(1 / r^2)\frac{d}{dr}[r dw / dr - w] = (1 / r^2)[dw / dr + r d^2 w / dr^2 - dw / dr]$$

$$= (1 / r)[d^2 w / dr^2]$$

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$$(1 / r)[d^2 w / dr^2] - (1 / L^2)(w / r) = 0$$

or

$$\frac{d^2 w}{dr^2} - \frac{1}{L^2} w = 0$$

This relation is the same form as the infinite plane problem that was previously examined for the infinite planar source. The general solution is:

$$w = Ae^{-r/L} + Ce^{r/L}$$

or

$$\phi = \frac{Ae^{-r/L}}{r} + \frac{Ce^{r/L}}{r}$$

We again require C to be zero because the flux must be finite. The second boundary condition comes from a current condition. To derive it, consider a sphere of radius r that surrounds the point source. The number of neutrons per second that pass through its surface is $4\pi r^2 \mathbf{J}(r)$. Hence, in the limit as r goes to zero,

$$\lim_{r\to 0} \; r^2 \mathbf{J}(r) \;=\; S/4\pi$$

To apply this boundary condition, we apply Fick's Law.

$$\mathbf{J} = -Dd\phi/dr = DA\left[1/rL + 1/r^2\right]e^{-r/L}$$

And in the limit of $r \to 0$,

$$\begin{aligned} r^2\mathbf{J} &= DA[r/L + 1]e^{-r/L} \\ &= DA \end{aligned}$$

Hence $S/4\pi = DA$ and $A = S/4\pi D$

Therefore,

$$\phi = \frac{Se^{-r/L}}{4\pi Dr}$$

## 5. <u>**Multiple Point Sources**</u>:

Flux is a scalar. Hence, if there are multiple point sources, the total flux at any given location is found by simply adding the individual contributions. This is one of the challenges in medical therapy. How to position implants within the body so as to produce a uniform radiation field while at the same time allowing for the differing attenuations of soft tissue and bone and also minimizing patient discomfort.

## 6. **Bare Slab**:

Thus far, the boundary conditions have been in terms of currents. Another possible condition is that of the extrapolated flux. To illustrate its use, we consider a bare slab of thickness 2a with an infinite planar source that emits S neutrons/cm$^2$ s located at x = 0. We require the flux to go to zero at x = (a+d) and at x = (-a-d) where d is the extrapolation distance. Again, it is emphasized that the true flux does not go to zero at this distance. By imposing a condition on the diffusion theory calculation that the flux go to zero at the extrapolation distance, we force the theory to yield the correct result for the slab interior.

For x > 0, the general solution to the steady state zero source diffusion equation is, as before,

$$\phi(x) = Ae^{-x/L} + Ce^{x/L}$$

The first boundary condition is that

$$\phi(a+d) = Ae^{-(a+d)/L} + Ce^{(a+d)/L} = 0$$

Therefore,

$$C = -\,Ae^{-2(a+d)/L}$$

and

$$\phi = A\left[e^{-x/L} - e^{x/L-2(a+d)/L}\right]$$

The second boundary condition is obtained by requiring

$$\lim_{x\to 0} \mathbf{J}(x) = S/2$$

Application of Fick's Law yields:

$$\mathbf{J} = -Dd\phi/dx$$

$$= -DA\left[\frac{-e^{-x/L}}{L} - \frac{e^{-x/L-2(a+d)/L}}{L}\right]$$

and

$$\lim_{x\to 0} \mathbf{J}(x) = \frac{S}{2} = -DA\left[-\frac{1}{L} - \frac{e^{-2(a+d)/L}}{L}\right]$$

$$\therefore \qquad A = \frac{SL}{2D}\left[1 + e^{-2(a+d)/L}\right]^{-1}$$

And for x > 0

$$\phi(x) = \frac{SL}{2D}\left[\frac{e^{-x/L} - e^{x/L-2(a+d)/L}}{1 + e^{-2(a+d)/L}}\right]$$

Thus far, we solved the problem only for the positive x direction. But, because of symmetry, the solution is the same for the negative x direction. So, substitute $|x|$ for x to obtain the complete solution. Finally, if the numerator and denominator are both multiplied by exp (a+d)/L, one obtains

$$\phi = \frac{SL}{2D}\frac{e^{(a+d-|x|)/L} - e^{-(a+d-|x|)/L}}{e^{(a+d)/L} + e^{-(a+d)/L}}$$

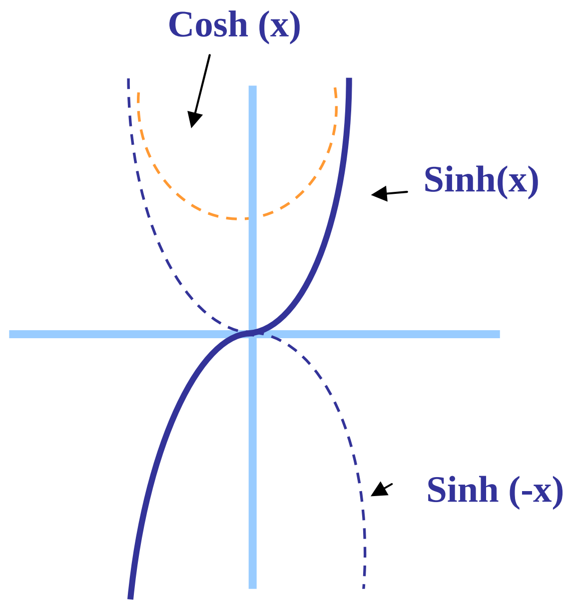
$$= \frac{SL}{2D}\frac{\sinh\left[\left((a+d) - |x|\right)/L\right]}{\cosh\left[(a+d)/L\right]}$$

Where sinh and cosh are hyperbolic functions defined as:

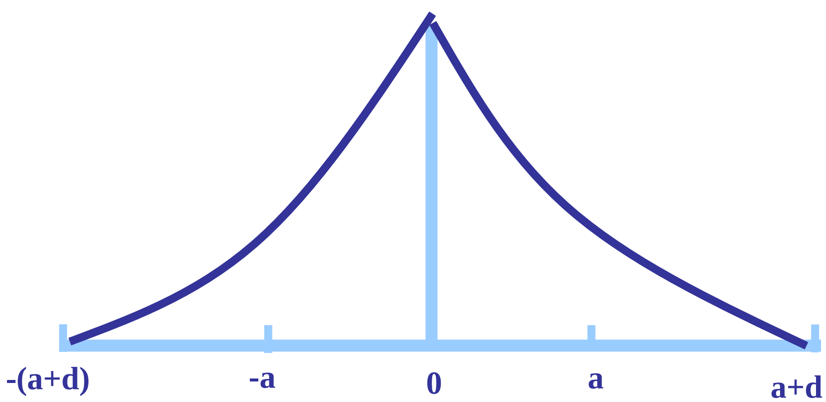
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

What does this look like and is the calculation logical?

First, note that hyperbolic functions are very different than trigonometric ones. The sinh and cosh functions are unbounded, although the cosh is never less than unity. The following figure is a plot of these functions.




For our slab problem, the solution has the following shape:




So, the neutron flux drops off with an exponential shape in both the +x and –x directions. This is to be expected for the same reasons as given for the infinite planar source.

7. **<u>Diffusion Length</u>**:

Earlier we defined the parameter $L^2$ as $D/\Sigma_a$. $L^2$ is called the diffusion area and has units of $cm^2$. Its square root, L, has units of length and is called the diffusion length.

$L^2$ has a physical interpretation. It can be shown that:

$$L^2 = (1/6)\overline{r^2}$$

Where r is the distance of a neutron in spherical coordinates from its point source. $\overline{r^2}$ is the average of the square of the "crow-flight" distance that a neutron travels between emission and absorption. (Note: This discussion pertains to fast neutrons and we are considering only fission (production) and absorption (removal)).