

22.05 Reactor Physics - Part Twelve

Application of One-Velocity Diffusion Equation to Multiplying Media

1. Multiplying Media:

A multiplying medium is one in which fission, either thermal or fast or both, does occur. Hence, the source term is retained in writing the diffusion equation. Thus, one has both a $\Sigma_a \phi$ term for absorption and a $\Sigma_f \phi$ term for fission. Both terms have the same mathematical form, a cross-section times a flux. So, what then is the difference between the solution for a non-multiplying medium and a multiplying one? The boundary conditions are different. For a non-multiplying medium, the source (or actually the current produced by the source) is a boundary condition. For a multiplying medium, one has conditions such as extrapolation distance or symmetry (e.g., zero current at the midsection of the multiplying medium.)

A second difference is that for non-multiplying media the solutions were in terms of exponential and/or hyperbolic functions. For multiplying medium, the solutions are in terms of sines and cosines. The latter repeat and hence one can have multiple solutions. As we will see, one of these modes corresponds to the desired critical condition.

2. Bare Slab Reactor:

The simplest multiplying medium example would be a bare slab that is of thickness, a , and is infinite in height and width. We start with the time-dependent diffusion equation:

$$\frac{1}{v} \frac{\partial \phi(r, t)}{\partial t} = v \bar{\Sigma}_F \phi(r, t) - \bar{\Sigma}_a \phi(r, t) + D \nabla^2 \phi(r, t)$$

For steady state conditions this becomes:

$$0 = v \Sigma_f \phi(x) - \Sigma_a \phi(x) + D \nabla^2 \phi(x)$$

The above equation is completely general provided that one obtains cross-sections that represent the complete range of neutron energies. This is unrealistic. Hence, it is common practice to limit the analysis to fast neutrons for which the cross-sections are relatively smooth and constant.

The quantity $v\Sigma_f\phi$ is the source term where v is the number of neutrons per fission and Σ_f is the macroscopic fission cross-section. Because we are limiting the analysis to fast reactors, it is common practice to express the source term in an alternative form.

The original motivation for this change of nomenclature was to express the source term in parameters that were measurable and readily computed. No such need now exists given present-day computing power. Nevertheless, familiarity with the traditional nomenclature is of benefit.

$$v\Sigma_f\phi = (v\Sigma_f\phi)\left(\frac{\Sigma_a^f}{\Sigma_a^f}\right)\left(\frac{\Sigma_a}{\Sigma_a}\right)$$

Where Σ_a^f is the macroscopic absorption cross-section of the fuel (i.e., absorption in the fuel that leads to either fission or radiative capture) and Σ_a is the total absorption cross-section of the core. That is,

$$\Sigma_a = \Sigma_a^f + \Sigma_a^M$$

Where Σ_a^M is the absorption cross-section of everything except the fuel.

To continue,

$$\begin{aligned} v\Sigma_f\phi &= \left(\frac{v\Sigma_f}{\Sigma_a^f}\right)\left(\frac{\Sigma_a^f}{\Sigma_a}\right)(\Sigma_a\phi) \\ &= (\eta)(f)(\Sigma_a)\phi \end{aligned}$$

Where f is called the “fuel utilization” because we are dealing with fast neutrons. (Note: The parameter for the neutron life cycle analysis that encompassed fast and thermal neutrons was called the “thermal utilization.”) The quantity $\eta f \Sigma_a \phi$ is the number of neutrons produced from fission. The quantity $\Sigma_a \phi$ is the number absorbed. So,

$$K_\infty = \frac{\text{Source}}{\text{Absorption}} = \frac{\eta f \Sigma_a \phi}{\Sigma_a \phi} = \eta f$$

Hence, we now write the source term as:

$$S = K_\infty \Sigma_a \phi$$

which also equals $\eta f \Sigma_a \phi$ and $v \Sigma_f \phi$.

Hence, the steady-state diffusion equation becomes:

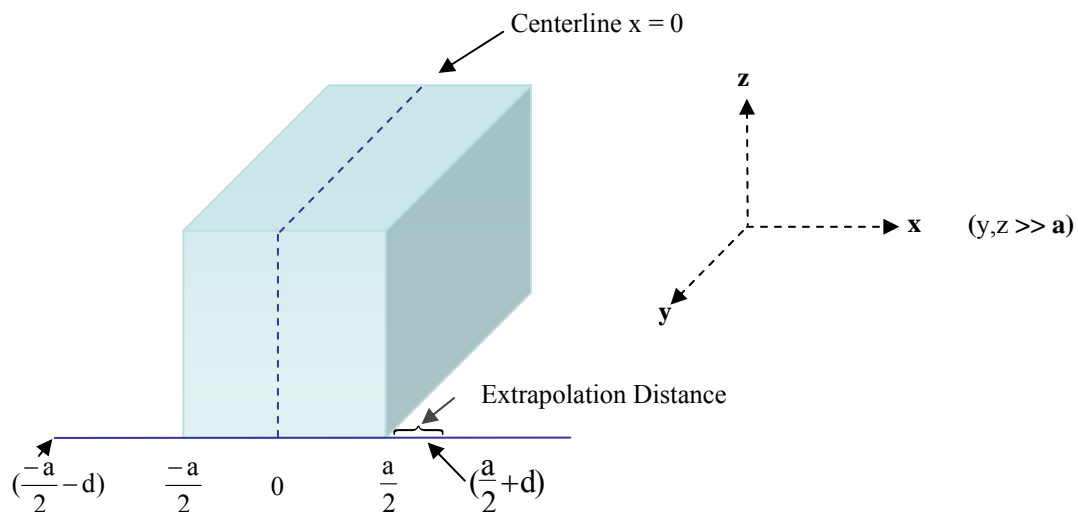
$$\begin{aligned} 0 &= K_\infty \Sigma_a \phi(x) - \Sigma_a \phi(x) + D \nabla^2 \phi(x) \\ &= \frac{(K_\infty - 1) \Sigma_a \phi(x)}{D} + \nabla^2 \phi(x) \\ &= \nabla^2 \phi(x) + B^2 \phi(x) \end{aligned}$$

Where:

$$\begin{aligned} B^2 &= \frac{(K_\infty - 1) \Sigma_a}{D} \\ &= (K_\infty - 1) / L^2 \end{aligned}$$

Where $L^2 = D / \Sigma_a$ is called the one-group diffusion area. (Note: The term one-group can be substituted for one-velocity. One group means one energy range.) It merits repeating that this change of nomenclature is not of significance in solving the diffusion equation. We could as well solve the equation as originally written by defining B^2 as $(v \Sigma_f - \Sigma_a) / D$ from the outset.

The physical meaning of the B^2 term will be explained later. For now, we wish to focus on solving the equation for the flux shape. The following figure shows the geometry of the problem.



The slab is of width a with the centerline at $x = 0$. So, the width of the slab goes from $\frac{-a}{2}$ to $\frac{a}{2}$. We take the extrapolation distance as d .

The boundary conditions are as follows:

- The flux must vanish at the extrapolation distances. To reiterate, this is a physically unreal requirement but its imposition results in the diffusion theory calculation of the flux being correct in the interior of the slab. Thus,

$$\phi\left(\frac{a}{2} + d\right) = \phi\left(\frac{-a}{2} - d\right) = 0$$

To simplify the notation, let $\left(\frac{a}{2} + d\right)$ be denoted as f . Thus,

$$\phi(f) = \phi(-f) = 0$$

- The problem is symmetric because of the geometry and the choice of $x = 0$ for the centerline. Hence,

$$d\phi/dx = 0 \text{ at } x = 0$$

Or in other words,

$$\phi(-x) = \phi(x) \text{ at } x = 0$$

The general solution of a differential equation of the form of

$$\frac{d^2\phi(x)}{dx^2} + B^2\phi(x) = 0$$

is

$$\phi(x) = A \cos Bx + C \sin Bx$$

Differentiation of this result and application of the second boundary condition gives $C=0$. So, the solution reduces to:

$$\phi(x) = A \cos Bx$$

To determine A, we use the extrapolation distance condition. The cosine is an even function, so we can use either $\phi(f) = 0$ or $\phi(-f) = 0$. Thus,

$$\phi(f) = 0 = A \cos(Bf)$$

This yields two solutions. The first is that $A=0$ and hence $\phi(x)$ is everywhere zero (the trivial solution). The second is:

$$\cos(Bf) = 0$$

$$\therefore B_n = \frac{n\pi}{2f}$$

Where n is an odd integer, 1, 3, 5, ...

The values of B_n , each a constant, are called the eigenvalues and the corresponding flux shapes, $\cos B_n x$, are called the eigenfunctions. The critical condition is the fundamental mode for which $n=1$. Thus,

$$\phi(x) = A \cos(\pi x / 2f)$$

or, for small values of the extrapolation distance,

$$\phi(x) = A \cos(\pi x / a)$$

(Note: Each eigenfunction contributes to the flux shape so that the flux would be a sum of the eigenfunctions (of which there are an infinite number) with each weighted by a time-dependent function. For the critical condition, all of these weighting functions go to zero except that for which $n=1$.)

So, one-group diffusion theory predicts that the flux in a bare slab of thickness a is:

$$\phi(x) = A \cos(\pi x / a)$$

3. **Power Level and Critical Condition:**

We have not yet evaluated the constant A in the above-equation for the flux. This is because the reactor can be critical at any power level. The value of A determines the magnitude of the flux which in turn determines the power level. One can not obtain a value of A from the geometry of the problem. Recall that:

$$\text{Power Level} = \left(\frac{200 \text{ MeV}}{\text{Fission}} \right) \left(\frac{1.6 \times 10^{-13} \text{ joules}}{\text{MeV}} \right) \Sigma_f \int_{-a/2}^{a/2} \phi(x) dx$$

$$P = (3.2 \times 10^{-11} \text{ joules}) \Sigma_f \int_{-a/2}^{a/2} A \cos(\pi x / a)$$

$$P = (3.2 \times 10^{-11} \text{ joules}) (A \Sigma_f) \left[\frac{a}{\pi} \sin \left(\frac{\pi x}{a} \right) \right]_{-a/2}^{a/2}$$

$$P = 3.2 \times 10^{-11} A \Sigma_f \left[\frac{a}{\pi} \sin \frac{\pi}{2} - \frac{a}{\pi} \sin \left(\frac{-\pi}{2} \right) \right]$$

$$P = 3.2 \times 10^{-11} A \Sigma_f (2a / \pi)$$

and hence

$$A = \pi P / (3.2 \times 10^{-11}) (2a \Sigma_f)$$

Thus, for a given power level (P) in a critical bare slab reactor, one group diffusion theory predicts that the flux is:

$$\phi(x) = \frac{\pi P}{3.2 \times 10^{-11} (2a \Sigma_f)} \cos \left(\frac{\pi x}{a} \right)$$

4. **Buckling:**

“Buckling” is a term that arose in the 1950s and 1960s as part of an effort to give physical meaning to the one-group equation. It is defined as the square of the lowest eigenvalue. Thus,

$$\text{Buckling} = B_1^2$$

and the diffusion equation would be:

$$\frac{d^2 \phi}{dx^2} + B_1^2 \phi = 0$$

Or, solving for B_1^2 ,

$$B_1^2 = \frac{1}{\phi} \frac{d^2 \phi}{dx^2}$$

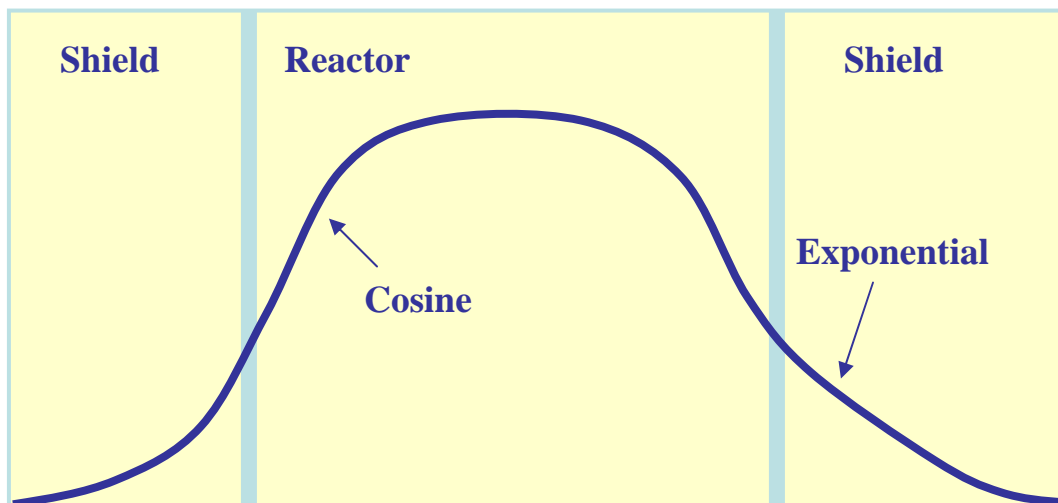
The buckling is therefore a measure of extent to which the flux curves or “buckles.” For a slab reactor $B_1^2 = (\pi/a)^2$ and the buckling goes to zero as “a” goes to infinity. This makes physical sense – there would be no buckling or curvature in a reactor of infinite width.

Buckling can be used to infer leakage. The greater the curvature, the more leakage would be expected.

5. **Flux Shape for Bare Slab Reactor:**

Does the predicted flux shape for a bare slab reactor make physical sense? The answer is yes given that Σ_a and Σ_f are uniform throughout the slab and net losses from leakage occur at the edges. We can think of the reactor as a series of mini slabs. Each produces neutrons, each absorbs some, and each results in diffusion to its neighbors. The largest sink is at $x = \pm a$ where leakage is greatest. Suppose the excess (production over absorption) is 10 neutrons per mini slab. So, the slab nearest the centerline has 10 neutrons to push out in order to remain at steady-state. The slab next to it has 20 (its own 10 plus the ones from its neighbor). The next mini slab has to move 30 and so on. Diffusion is proportional to the gradient of the flux. So, we expect the curvature of the flux shape to be greatest for mini slabs near the surface. And this is what the cosine function does show. It is flat near the centerline and most strongly curved as x approaches a .

6. **Bare Slab Reactor with a Shield:** The slab would be a multiplying medium while the shield would be a pure absorber. So, the flux shape would be:



7. Bare Spherical Reactor:

Earlier we analyzed a point source in a sphere of non-multiplying materials. The flux was only a function of the radial distance and was found to be a decaying exponential. Specifically, the one-group diffusion equation in spherical coordinates was:

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi(r)}{dr} - \frac{1}{L^2} \phi(r) = 0$$

To solve for ϕ , we let $w = r\phi$ and obtained:

$$\frac{d^2 w}{dr^2} - \frac{1}{L^2} w = 0$$

The solution was of the form:

$$w = Ae^{-r/L} + Ce^{r/L}$$

Note that if we substitute $Ae^{-r/L}$ into the differential equation for w , we obtain:

$$\left(\frac{-1}{L}\right)\left(\frac{-1}{L}\right)e^{-r/L} - \frac{1}{L^2}e^{-r/L} = 0$$

And conclude that $e^{-r/L}$ is indeed a solution. Similarly, $e^{r/L}$ can be shown to be a solution.

Now, what about a multiplying medium? The one-group diffusion equation for such a medium is:

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi(r)}{dr} + B^2 \phi(r) = 0$$

Where B^2 is equal to $(v\Sigma_f - \Sigma_a)/D$. We note immediately that an exponential cannot be the solution because the second term is now positive, not negative. Hence, the exponential and its second derivative will not cancel. We expect a solution that is a sine or cosine where the second derivative is the negative of the original function. Such functions are cyclic and that is a major difference between the multiplying and the non-multiplying medium. For the former, one gets sine, cosine or Bessel functions. For the latter one gets exponentials or a hyperbolic function.

To proceed, again let $w = r\phi$ and obtain:

$$\frac{d^2 w}{dr^2} + B^2 w = 0$$

The solution is:

$$\phi(r) = \frac{A \sin Br}{r} + \frac{C \cos Br}{r}$$

Next apply the boundary condition that the flux is finite at the center ($r=0$) of the sphere. Hence, $C=0$.

Thus,

$$\phi(r) = \frac{A \sin Br}{r}$$

If we assume that the extrapolation distance is small compared to the radius, R , of the sphere, then for a second boundary condition, we have that the flux at the sphere's surface must be zero.

$$\phi(R) = 0$$

From this, one obtains:

$$B_n = n\pi/R$$

where n is an integer. The first eigenvalue ($n=1$) corresponds to the critical condition. Thus,

$$B_1^2 = (\pi/R)^2$$

and the flux is:

$$\phi(r) = \frac{A \sin(\pi r/R)}{r}$$

The constant A is found by specifying the power level of the reactor (see Section 6.3 of Lamarsh). The final result is:

$$\phi(r) = \frac{P}{4E_r \Sigma_f R^2} \frac{\sin(\pi r / R)}{r}$$

Where P is the power level and E_r is the energy released per fission.

8. Bare Infinite Cylinder:

This geometry is of interest because fuel elements in a PWR or BWR consist of cylindrical pellets that are stacked to form long cylinders. In this case, the flux is again only dependent on the radial distance. The one-group diffusion equation in cylindrical coordinates is:

$$\frac{1}{r} \frac{d}{dr} r \frac{d\phi(r)}{dr} + B^2 \phi(r) = 0$$

or

$$\frac{d^2 \phi(r)}{dr^2} + \frac{1}{r} \frac{d\phi(r)}{dr} + B^2 \phi(r) = 0$$

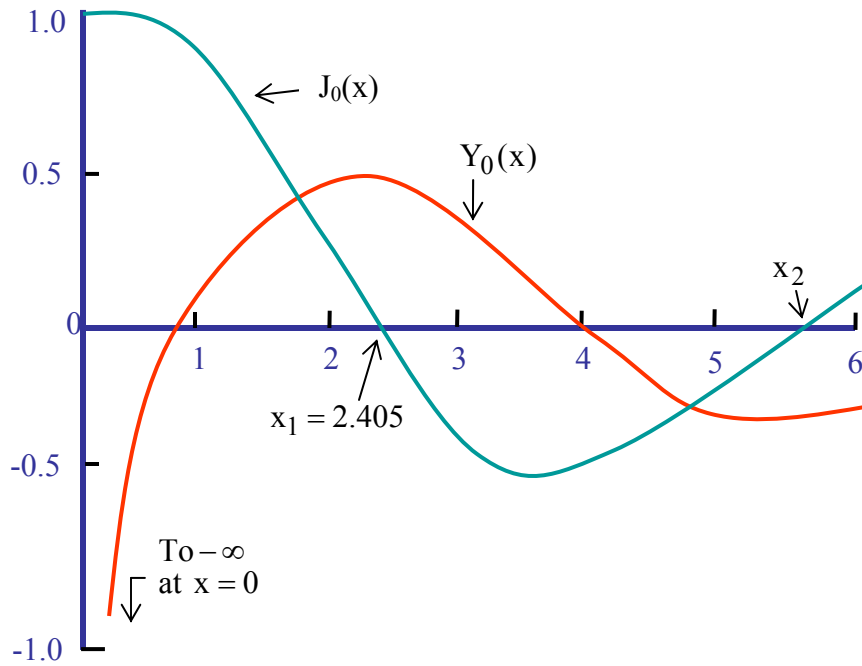
This is a form of Bessel's equation which is of the type:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \left(B^2 - \frac{m^2}{r^2} \right) \phi = 0$$

Here, for our case, $m=0$. The solutions are $J_m(Br)$ and $Y_m(Br)$ which are called ordinary Bessel functions of the first and second kind. For our case, $m=0$ and the general solution is:

$$\phi(r) = AJ_0(Br) + CY_0(Br)$$

The attached figure shows both J_0 and Y_0 . First note that they are cyclic as is to be expected for a multiplying medium. Second note that Y_0 goes to $-\infty$ as $x \rightarrow 0$.



The Bessel Functions $J_0(x)$ and $Y_0(x)$

Hence, the constant $C=0$. Thus, we have:

$$\phi(r) = AJ_0(Br)$$

Again, assume that the extrapolation distance is small compared to the cylinder's radius. So,

$$\phi(R) = 0$$

Examination of the figure for the Bessel functions shows that J_0 equals zero for various values of x . Denote these as x_n . Hence, we require:

$$B_n = x_n / R$$

The critical condition corresponds to $n=1$. So,

$$B_1^2 = (x_1 / R)^2 = (2.405 / R)^2$$

and

$$\phi(r) = AJ_0(2.405 r / R)$$

The constant A is again determined from the power level (see Section 6.3 of Lamarsh). The final result is:

$$\phi(r) = \frac{0.738P}{E_r \Sigma_f R^2} J_0(2.405r/R)$$