

22.05 Reactor Physics - Part Thirteen

Criticality Conditions

1. Objective:

Thus far, we have calculated the shape of the neutron flux in critical assemblies of certain geometries. However, we have not addressed the question of how to select a material composition and/or a geometrical configuration that will result in the critical condition. We do this by reviewing the definitions of multiplication factors and the parameters elucidated in the neutron life cycle and then relating these to the one-velocity, steady-state diffusion equation.

2. K-infinity and K-effective:

Definitions of both K-infinity and K-effective follow from our understanding of the neutron life cycle. We define K-infinity as “the ratio of the number of neutrons resulting from fission in the current generation to the number absorbed in the preceding generation in a system of infinite size.” For an infinite system, there is no loss from leakage. We define K-effective as “the ratio of the number of neutrons resulting from fission in the current generation to the total number lost by absorption and leakage in the preceding generation.”

The ratio of K-effective to K-infinity is therefore:

$$\begin{aligned} \frac{K_{\text{eff}}}{K_{\infty}} &= \frac{\frac{\text{Fission}}{\text{Absorption} + \text{Leakage}}}{\frac{\text{Fission}}{\text{Absorption}}} \\ &= \frac{\text{Neutrons Absorbed}}{\text{Neutrons Absorbed} + \text{Neutrons Leaking Out}} \end{aligned}$$

We define this ratio as P which is called the non-leakage probability for the system. Thus,

$$K_{\text{eff}} = K_{\infty} P$$

This relation is quite simple. Yet, it provides insight relative to the nature of criticality. An assembly of materials will be critical if its value of K-effective is exactly unity. We see that K-effective is the product of two factors: K-infinity

and the non-leakage probability. The former is in turn a function of the materials that form the assembly. The latter is a function of the arrangement (i.e., the geometry) of these materials. Thus, the achievement of criticality depends on both the choice of the fuel/moderator/coolant and on the size/shape and layout of each of these. Thus, when designing an actual reactor, two approaches often arise. One is to select geometry, hold it constant, and vary the material number densities. For example, vary the fuel loading or amount of moderator. A second approach is to select the materials, hold them constant, and vary the dimensions. For example, vary the reflector thickness.

3. **One-Group Critical Equation**: The steady-state, one-velocity (or one-group) diffusion equation is:

$$D\nabla^2\phi - \Sigma_a\phi + v\Sigma_f\phi = 0$$

Recall that this relation allows for neutron production from fission, neutron absorption, and leakage. Hence, this equation does apply to reactors of finite size. The above relation does not allow for neutron scattering. Hence, its use is limited to situations where all neutrons have approximately the same energy.

For an actual problem, the one-velocity equation would normally be solved numerically by first rewriting it as a set of difference equations and then programming it on a computer. A difference equation is one in which derivatives such as $d\phi/dx$ are approximations $((\phi_i - \phi_{i-1})/\Delta x)$. This approach has much to offer. It is, after all, a major computational tool. But, it doesn't provide physical understanding. For that we rewrite the one-velocity equation as follows:

$$D\nabla^2\phi - \Sigma_a\phi + S = 0$$

Where S is the source term $(v\Sigma_f\phi)$. For every neutron that is absorbed in the fuel, there are K_∞ fission neutrons produced. Thus,

$$S = K_\infty\Sigma_a\phi$$

And we have

$$D\nabla^2\phi - \Sigma_a\phi + K_\infty\Sigma_a\phi = 0$$

or

$$\nabla^2\phi + \left[\frac{(K_\infty - 1)\Sigma_a}{D} \right] \phi = 0$$

For one energy group, we have previously defined D/Σ_a as equaling L^2 , the square of the diffusion length. Thus,

$$\nabla^2 \phi + \frac{(K_\infty - 1)}{L^2} \phi = 0$$

or

$$\nabla^2 \phi + B^2 \phi = 0$$

Where B^2 is, also as previously named, the buckling and is equal to

$$B_c^2 = \frac{K_\infty - 1}{L^2}$$

The subscript c denotes that satisfaction of the above relation is essential for the achievement of criticality.

The above relation gives us considerable insight into the critical condition. The buckling, as was shown in part 12 of these notes, depends on the geometry and dimensions of the reactor. The quantity $(K_\infty - 1)/L^2$ depends on the materials used to construct the reactor.

So, this equation is telling us that for a given set of materials, adjustment of the geometry could result in criticality. Conversely, for a given geometry, modification of the material composition could achieve criticality. An example (see #6.3, p. 283, Lamarsh) illustrates this. The simplest realistic geometry for a critical assembly is spherical because the only geometrical parameter that can be varied is the radius. So, from our previous analysis of a bare sphere, we have;

$$\begin{aligned} B^2 &= (\pi/R)^2 && \text{Geometry Requirement} \\ &= \frac{K_\infty - 1}{L^2} && \text{Materials Requirement} \end{aligned}$$

Hence,

$$R = \pi \left[\frac{L^2}{K_\infty - 1} \right]^{1/2}$$

For a given set of materials, this is readily solved for the critical radius, R . L^2 is the “diffusion area” and equals D/Σ_a . D is the diffusion length and is given by $1/3\Sigma_{Tr}$. Σ_a is the macroscopic absorption cross-section (fuel and other materials); Σ_{Tr} is the macroscopic transport cross-section. K_∞ , for a fast reactor, is given by:

$$K_\infty = \eta f$$

Tabulations of η , Σ_a , f , etc., are given in reactor handbooks.

4. Applicability of One-Velocity Analysis:

The analysis done so far has all been based on the one-velocity diffusion equation. This use of the equation is acceptable for an unreflected fast reactor.

Can this type of analysis be extended to unreflected thermal reactors? The answer depends on the extent to which neutron diffusion occurs. If diffusion is significant during the slowing down process, then the size of the reactor will be different from what our theory predicts. So, we seek some criterion that will let us determine the suitability of the approach for a thermal reactor. To do this, we will write two equations, one for the fast neutrons and one for the thermal ones. We will then determine under what conditions these two equations give the same result as the single one-velocity equation. The two equations are linked by their source terms. Thermal neutrons are created by the scattering of fast ones. Fast neutrons are created by the thermal fission of uranium-235. This analysis is called “modified one-group” or “modified one-velocity.”

5. Modified One-Velocity Theory:

The first issue is that of nomenclature. We use the subscript 1 to denote the fast group and 2 to denote the thermal group. The second issue is that of simplifying assumptions. These are:

- a) **Fast Removal:** It is assumed that there is no absorption of fast neutrons. The sole removal mechanism is scattering to thermal energies. The macroscopic scattering cross-section is Σ_{s1} .
- b) **Source Term for Fast Group:** The reality is that fast neutrons originate from both fast and thermal fission. So, we could write two fission terms ($\Sigma_{f1}\phi_1$ and $\Sigma_{f2}\phi_2$) because both fast and thermal fission produce neutrons in the fast energy range. Instead, we write the term for thermal fission and modify it by use of ϵ , the fast fission factor. Thus, the source term is:

$$\epsilon \eta f \Sigma_{a2} \phi_2$$

Where ϵ is the fast fission factor, η is the thermal reproductive factor, and f is the thermal utilization. The nomenclature and definitions are those of the neutron life cycle that was studied earlier. The fast source term is therefore:

$$S_1 = \epsilon \eta f \Sigma_{a2} \phi_2$$

$$= \frac{k_{\infty} \Sigma_{a2} \phi_2}{p}$$

Where p is the resonance escape probability.

- c) **Source Term for Thermal Group:** Neutrons enter the thermal range by scattering out of the fast range. The scattering rate is:

$$\text{Fast Scattering/cm}^3 \text{ s} = \Sigma_{s1} \phi_1$$

Of those that scatter out, some are absorbed by the U-238 resonances. Thus, the number attaining thermal energies is:

$$\text{Thermal Source Term} = p \Sigma_{s1} \phi_1$$

The two-group diffusion equations are:

Fast

$$D_1 \nabla^2 \phi_1 - \Sigma_{s1} \phi_1 + \frac{k_{\infty} \Sigma_{a2} \phi_2}{p} = 0$$

Thermal

$$D_2 \nabla^2 \phi_2 - \Sigma_{a2} \phi_2 + p \Sigma_{s1} \phi_1 = 0$$

The above equations apply to a bare (unreflected) reactor. Under such conditions, it can be shown that the two group fluxes (ϕ_1 and ϕ_2) have the same shape and that shape is the one given by the one-velocity equation. Thus,

$$\phi_1 = A_1 \phi$$

$$\phi_2 = A_2 \phi$$

Where A_1 and A_2 are constants and ϕ is given by an equation of the type:

$$\nabla^2 \phi + B^2 \phi = 0$$

The thermal and fast fluxes differ in magnitude with A_1 and A_2 indicating that difference.

Hence, the two-group equations become:

$$\begin{aligned} -\left(D_1 B^2 + \Sigma_{s1}\right) A_1 + \frac{k_{\infty} \Sigma_{a2} A_2}{\rho} &= 0 \\ \rho \Sigma_{s1} A_1 - \left(D_2 B^2 + \Sigma_{a2}\right) A_2 &= 0 \end{aligned}$$

The above relations constitute a set of linear homogeneous equations. These will have a nontrivial solution if the determinant of the coefficients of A_1 and A_2 is zero. Thus,

$$\begin{vmatrix} -\left(D_1 B^2 + \Sigma_{s1}\right) & \frac{k_{\infty} \Sigma_{a2}}{\rho} \\ \rho \Sigma_{s1} & -\left(D_2 B^2 + \Sigma_{a2}\right) \end{vmatrix} = 0$$

or

$$k_{\infty} \Sigma_{a2} \Sigma_{s1} - \left(D_1 B^2 + \Sigma_{s1}\right) \left(D_2 B^2 + \Sigma_{a2}\right) = 0$$

or

$$\left(\frac{K_{\infty} \Sigma_{a2} \Sigma_{s1}}{\left(D_1 B^2 + \Sigma_{s1}\right) \left(D_2 B^2 + \Sigma_{a2}\right)}\right) = 1$$

or

$$\left(\frac{K_{\infty}}{\left(1 + B^2 L_T^2\right) \left(1 + B^2 \tau_T\right)}\right) = 1$$

Where we have divided numerator and denominator by $\Sigma_{a2}\Sigma_{s1}$ and where we define

$$L_T^2 = D_2 / \Sigma_{a2} \quad \text{and}$$

$$\tau_T = D_1 / \Sigma_{s1}$$

The quantity L_T^2 is the “thermal diffusion area” and τ_T is termed the “neutron age.” Both have units of area.

To continue, the quantities in the denominator of the above relation are the non-leakage probabilities. Thus, the probability that a thermal neutron will not leak out is:

$$p_T = 1 / (1 + B^2 L_T^2)$$

And the probability that a fast neutron will not do so is:

$$p_F = 1 / (1 + B^2 \tau_T)$$

Here, we can write

$$K\text{-effective} = K_\infty P_T P_F$$

If the reactor is designed so that neutron leakage is small then the product of the leakage terms $(B^4 L_T^2 \tau_T)$ will be very small and can be ignored. Thus, we obtain:

$$\frac{K_\infty}{1 + B^2 (L_T^2 + \tau_T)} = 1$$

or

$$\boxed{\frac{K_\infty}{1 + B^2 M_T^2} = 1}$$

Where

$$M_T^2 = L_T^2 + \tau_T$$

M_T^2 is termed the “thermal migration area.”

We are now in a position to compare our derived two-group analysis with the previously analyzed one-group analysis. The one velocity (or one-group) critical equation for an unreflected reactor is:

$$\frac{K_\infty}{1 + B^2 L^2} = 1$$

The two-group result, or as it is more correctly called, the modified one-group equation, is:

$$\frac{K_\infty}{1 + B^2 M_T^2} = 1$$

The two equations are of the same form and if τ_T is much less than L_T^2 , the two equations are the same. So a comparison of τ_T (neutron age) and L_T^2 (thermal diffusion area) provides a means of determining if one-velocity theory will be sufficient for estimating critical size and/or composition:

Comparison of τ_T and L_T^2		
Moderator	τ_T	L_T^2
Light Water	27	8.1
Heavy Water	131	9600
Be	102	480
Graphite	368	3500

This shows that the one-velocity approach does suffice for moderators of heavy water and graphite, that it is marginal for moderators of Be, and it fails entirely for moderators of light water. Quantities such as the “neutron age” and the “thermal diffusion area” are no longer widely used in reactor analysis. *What is important to recognize in the above discussion is that one-velocity theory is only applicable if neutron scattering is not significant.*

Examples of the use of modified one-velocity theory for the determination of composition are given in Section 6.5 of Lamarsh.

6. **Relevance of One-Velocity and Modified One-Velocity Theory:**

These two approaches combine neutron life cycle analysis with diffusion theory to provide a quantitative method for calculating flux shape given a geometry and for estimating both critical dimensions and compositions. This methodology was the principal design tool for Generation I and some of the smaller Generation II reactors. It allows the engineer to obtain a hands-on feel for the calculations. However, it is not a viable approach for the design of reactors that are moderated by light-water (which are to date the vast majority) or of modern large reactors. For that, we require a thorough understanding of neutron scattering and the development of either multi-group methods (cross-sections averaged over small energy ranges) or Monte Carlo methods (exact cross-sections).