22.05 Reactor Physics - Part Fourteen

Kinematics of Elastic Neutron Scattering

1. Multi-Group Theory:

The next method that we will study for reactor analysis and design is multi-group theory. This approach entails dividing the range of possible neutron energies into small regions, ΔE_i and then defining a group cross-section that is averaged over each energy group i. Neutrons enter each group as the result of either fission (recall the distribution function for fission neutrons) or scattering. Accordingly, we need a thorough understanding of the scattering process before proceeding.

2. Types of Scattering:

Fission neutrons are born at high energies (>1 MeV). However, the fission reaction is best sustained by neutrons that are at thermal energies with "thermal" defined as 0.025 eV. It is only at these low energies that the cross-section of U-235 is appreciable. So, a major challenge in reactor design is to moderate or slow down the fission neutrons. This can be achieved via either elastic or inelastic scattering:

a) Elastic Scatter: Kinetic energy is conserved. This mechanism works well for neutrons with energies 10 MeV or below. Light nuclei, ones with low mass number, are best because the lighter the nucleus, the larger the fraction of energy lost per collision.

b) Inelastic Scatter: The neutron forms an excited state with the target nucleus. This requires energy and the process is only effective for neutron energies above 0.1 MeV. Collisions with iron nuclei are the preferred approach.

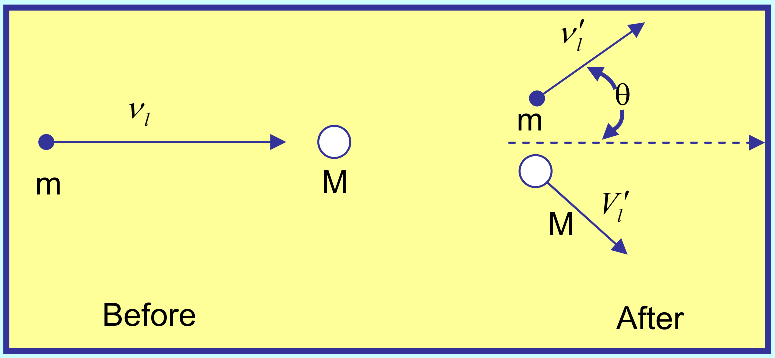
Neutron scattering is important in two aspects of nuclear engineering. One is the aforementioned neutron moderation. The other is shielding. In both cases, for reactors, one uses elastic collisions with light nuclei as the mechanism for energy loss. For accelerators, the situation is different because neutrons with energies in excess of 10 MeV are produced. For these, shields are made of iron so as to take advantage of the energy loss associated with inelastic scattering.

3. Methodology:

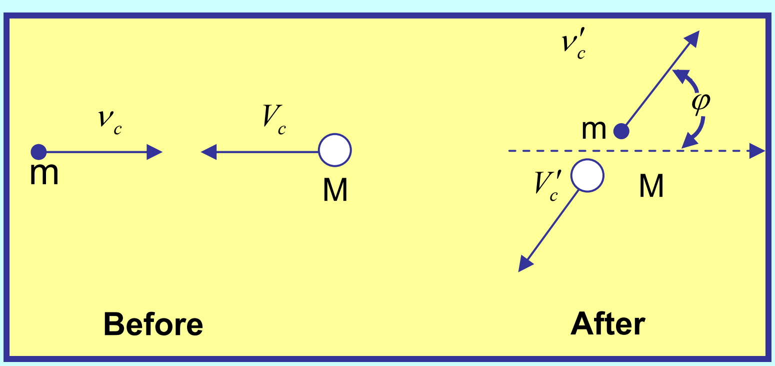
Neutron scattering is most easily analyzed in a center-of-mass frame of reference (COM). Collisions in a COM frame appear as rotations with the speeds the same both before and after the collision. However, the reactor is designed in a laboratory frame of reference (LAB). So, we will analyze the scattering collisions in a COM frame and then translate that result to a LAB frame. This is not difficult provided one keeps the notations clear.

4. Frames of Reference:

The material in this section follows that of Henry (p. 52).



Trajectories of a neutron and its target nucleus before and after a scattering event as seen from the laboratory reference system.



Trajectories of a neutron and its target nucleus before and after a scattering event as seen from the center-of-mass system.

The notation used is as follows:

- Small letters m and v refer to the mass and speed of the neutron.
- Capital letters M and V refer to the mass and speed of the target nucleus
- A subscript l refers to the LAB system.
- A subscript c refers to the COM system.
- A prime on v or V indicates a measurement after the collision.
- V_{COM} is the speed of the center-of-mass in the LAB system.

Finally we note that the mass number of a neutron is 1 while that of the target nucleus is A.

5. <u>Analysis of a Scattering Collision</u>:

The material in this section follows that of Henry (pp. 53-55).

We begin the analysis in the COM system. Such a system is defined by the requirement that the total linear momentum of the neutron and the nucleus measured relative to that system vanish. The viewpoint of a COM frame of reference is that of an observer who is located at the center-of-mass. The neutron and target nucleus both approach with equal and opposite momentums. They collide and move off at some angle φ

Given the requirement that the total momentum of the system be zero, we have

$$v_c m - V_c M = 0 \quad \text{, Before} \tag{1}$$

$$v_c' m - V_c' M = 0 , \quad \text{After} \tag{2}$$

Because of the conservation of kinetic energy (elastic collision):

$$\frac{1}{2}\mathrm{mv}_c^2 + \frac{1}{2}\mathrm{MV}_c^2 = \frac{1}{2}\mathrm{mv}_c'^2 + \frac{1}{2}\mathrm{MV}_c'^2 \tag{3}$$

Using (1) and (2) to eliminate v_c and v_c', we get

$$\left[\frac{1}{2}m(M/m)^2 + \frac{1}{2}M\right]V_c^2 = \left[\frac{1}{2}m(M/m)^2 + \frac{1}{2}M\right]V_c'^2 \tag{4}$$

Therefore,

$$V_c = V_c' \tag{5}$$

And insertion of this result into (1) and (2) yields:

$$v_c = v_c', \tag{6}$$

So, the speeds of the neutron and target nucleus are the same, both before and after a collision when viewed in the COM frame.

Next we need to examine the LAB frame and obtain an expression for the speed of the center-of-mass in the LAB frame. The viewpoint of a LAB frame of reference is that of an external observer. The target nucleus is stationary. The neutron approaches with speed v. As a result of the collision, the neutron moves off at some angle relative to its original direction of travel. The target nucleus also moves off in a way so as to conserve momentum.

The speed of the center of mass in the LAB system is that of the neutron and the target nuclei weighted by their speeds. Thus,

$$\begin{aligned} V_{COM} &= \frac{mv_1 + MV_1}{m+M} \\ &= \frac{mv_1}{m+M} \\ &= \frac{v_1}{1+A} \end{aligned}$$

because the mass of the neutron is 1 and the initial velocity of the target nucleus is zero. The above relation is also obtained from a momentum balance. Namely, the momentum of the COM equals the sum of the momentums of the individual particles or

$$(m+M)V_{COM} = mv_1 + MV_1$$

Another way to obtain this relation is as follows: The target nucleus is at rest in the lab system and moving to the left with speed V_c in the COM system. Hence, the center-of-mass itself must be moving to the right in the LAB system with the speed V_c. Thus,

$$V_{COM} = V_c = V_c'$$

(Note: This says that the magnitudes are equal; the directions are opposite.) From this we obtain that the speed of the neutron in the COM is:

$$v_c = v_1 - V_{COM} = v_1 - V_c$$

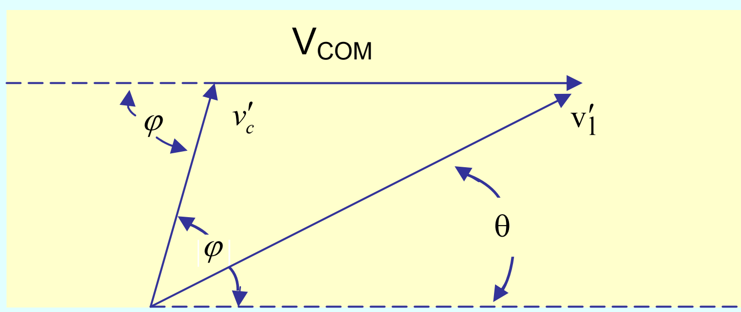
Using (1), we obtain:

$$\frac{V_c M}{m} = v_1 - V_c = V_c'$$

or

$$V_c = \frac{v_1 m}{m + M} = \frac{v_1}{1 + A}$$

We now draw a vector diagram of the velocity of the neutron. The final velocity of the neutron in the LAB system is obtained by adding the vector that represents the movement of the center-of-mass in the LAB system to the vector that represents the velocity of the neutron after the collision in the COM system.



Vector relationship between the velocity of a neutron after a collision measured in the laboratory system and that measured in the COM system.

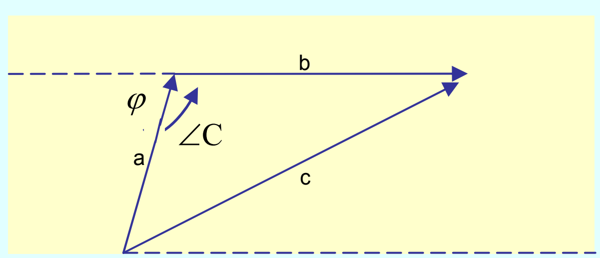


Illustration of law of cosines

The law of cosines states that $c^2 = a^2+b^2 - 2ab \cos C$ when $\angle C$ is opposite c. In the above diagram, $\angle C$ would be $(180^\circ - \varphi)$. Note that $\cos(180^\circ - \varphi) = -\cos\varphi$. Thus, application of the law of cosines to the above diagram yields:

$$(v_1')^2 = (v_c')^2 + (V_{COM})^2 + 2v_c' \, V_{COM} \cos\varphi$$

$$= \left[\left(\frac{M}{m}\right)^2 + 1 + 2\frac{M}{m}\cos\varphi\right]\left(\frac{v_1 m}{M+m}\right)^2$$

Next let M/m be defined as A. Also note that the ratio of the neutron's kinetic energies before and after the collision is:

$$E'/E = \frac{1}{2}mv_1'^2 / \frac{1}{2}mv_1^2 = v_1'^2 / v_1^2$$

Thus, we obtain:

$$\boxed{\frac{E'}{E} = \frac{A^2 + 1 + 2A\cos\varphi}{(1+A)^2}}$$

This equation gives the energy of the neutron before and after the collision in the laboratory frame of reference as a function of the scattering angle in the center-of-mass frame of reference.

A further simplification in the formula is possible. Define:

$$\alpha \equiv \left(\frac{A-1}{A+1}\right)^2$$

Then,

$$\boxed{\frac{E'}{E} = \left(\frac{1+\alpha}{2}\right) + \left(\frac{1-\alpha}{2}\right)\cos\varphi}$$

Finally, we note without derivation, the relation between the scattering angles in the LAB and COM systems. It is:

$$\text{Cos}\,\theta = \frac{1 + A\cos\varphi}{\left[A^2 + 2A\cos\varphi + 1\right]^{1/2}}$$

Or, for hydrogen,

$$\text{Cos}\ \theta = \left(\frac{1+\cos\varphi}{2}\right)^{1/2}$$

6. Insights into Neutron Scattering:

The material in this section follows that of Henry (p. 56).

The relation between E and E' as well as that relating θ to φ yields the following insights:

- For glancing or straight-ahead collisions, $\varphi = 0^\circ$ and $E' = E$. Thus, there is no energy loss in such collisions.
- For head-on collisions, $\varphi = 180^\circ$, in which the neutron reverses its direction of travel, E'/E takes on its minimum value which is α. Therefore,

  — Minimum possible energy of a neutron following an elastic collision is αE.

  — Maximum energy loss is $E(1-\alpha)$.

  Given that α is zero for hydrogen and near unity for large mass numbers, it is apparent that lighter elements are better moderators. That is, they result in greater energy losses per collision.

- A neutron will lose all of its energy in a single head-on collision with hydrogen because hydrogen (a proton) is the same mass as a neutron.
- The scattering angle φ in the COM is always greater than (θ) in the LAB frame.
- For hydrogen, as φ goes from 0 to π, θ goes from 0 to $\pi/2$.

7. Maximum Energy Transfer:

The maximum energy transfer occurs for head-on collisions. That is for $\cos\varphi = -1$. It is:

$$E - E' = E - \left[\left(\frac{M}{m}\right)^2 + 1 - \frac{2M}{m}\right]\left(\frac{m}{m+M}\right)^2 E$$

$$= E - \left(\frac{M^2 + m^2 - 2Mm}{m^2}\right)\left(\frac{m^2}{(m+M)^2}\right)E$$

$$= \left(\frac{(m^2 + 2mM + M^2)E - (M^2 + m^2 - 2mM)E}{(m+M)^2}\right)$$

or

$$\boxed{\Delta E = \frac{4mM}{(m+M)^2} E}$$

This is the relation given earlier in the course without derivation. This relation is of fundamental importance in the understanding of how radiation causes damage to tissue. For large particles (proton or alpha) that slow down by collisions with atomic electrons, it shows that the energy loss per collision is small (~50 eV) and hence many thousands of collisions are needed. These particles travel in straight paths until stopping. For light particles (electrons) that also slow down by collisions with atomic electrons, complete energy loss is possible in one collision. These particles zig zag. For neutrons, the energy loss per collision is large and hence they would undergo only a few collisions before stopping in the human body. But, those collisions will be with water and will result in free protons. The protons then slow down through many collisions, each of which involves enough energy deposition to damage cellular structures.

8. **Consequences of Scattering**:

For neutron energies below 10 MeV, which is the upper limit for neutrons produced from fission, the scattering is isotropic in the COM. That is, quoting from Henry (p. 57), "If we imagine the scattering center to be surrounded by a sphere of unit radius so that the trajectories of the scattered neutrons lie along radii of that sphere, the probability that the scattered neutron will pass through any particular area on the surface of the unit sphere is just that area divided by 4π, the total area of the unit sphere. All areas of equal size on the surface of the sphere, no matter what their shape, have an equal probability that the scattered neutron will pass through them. In particular, if we consider a circular ring of area $2\pi \sin \varphi \, d\varphi$ on the surface of the sphere, the probability that the scattered neutron will pass through that ring is:

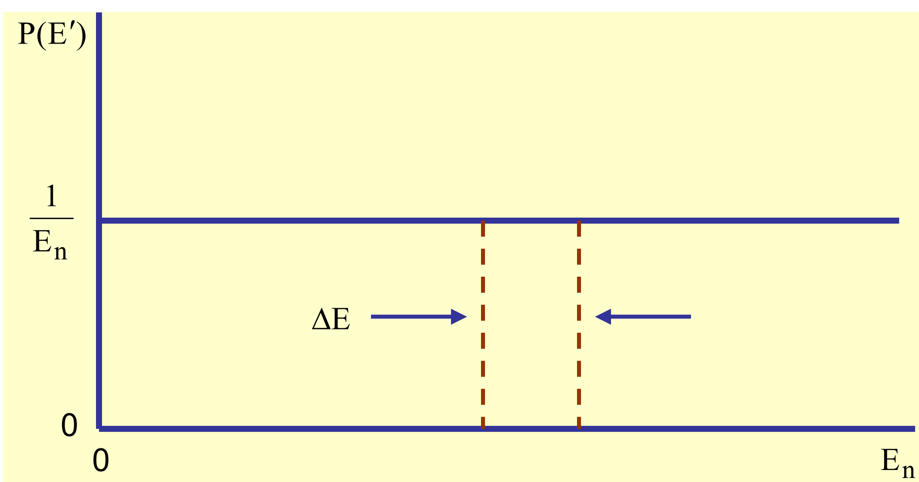
$$\frac{2\pi \sin\varphi \, d\varphi}{4\pi} = \frac{1}{2}\sin\varphi \, d\varphi = -\frac{1}{2}d(\cos\varphi) = \frac{-d\mu_c}{2},$$

where $d\mu_c$ is a negative number if $d\varphi$ is positive."

Henry goes on to show that the probability that a neutron having an initial energy E will, as a result of an elastic scatter, emerge with energy dE' is:

$$P(E \to E')dE' = \frac{dE'}{E(1-\alpha)}$$

This is a crucial result because, again quoting Henry (p. 58), it shows that, "for the isotropic-elastic-scattering case, the probability of scattering into any energy interval dE' in the accessible range of energies $E(1-\alpha_j)$ is uniform, i.e., does not depend on E' and is, hence, equally likely for all equal intervals dE' in the range." In other words, the energy loss spectrum for neutrons scattering off hydrogen is flat. Consider a mono-energetic neutron source with energy E_n. The probability of losing an amount of energy ΔE is simply $\Delta E/E_n$ and is independent of where ΔE is located.



This is a very practical result for multi-group theory because it shows that for a hydrogen moderator, the number of neutrons that scatter into a given group is a function of the group's width and not the absolute value of the energy of that group. Thus, the probability that a 10 MeV neutron scatters off hydrogen to a group defined as 1.1 to 1.2 MeV is the same as for a group defined as 0.4 to 0.5 MeV because both have the same width (0.1 MeV). For moderators other than hydrogen, a similar result holds. Specifically, quoting Henry (p. 58),

"for the isotropic-eleastic-scattering case, the probability of scattering into an any energy interval dE' in the accessible range of energies $E(1-\alpha_j)$ is a uniform, i.e., does not depend on E' and is, hence, equally likely for all equal intervals dE' in the range."

9. Average of the Cosine of the Laboratory Scattering Angle:

It will be recalled from the discussion of Fick's Law that the diffusion constant D was defined in terms of a transport mean free path. Specifically,

$$D = \lambda_{tr}/3$$

and

$$\lambda_{tr} = 1/\Sigma_{tr} = 1/[\Sigma_t - \bar{\mu}_0 \Sigma_s]$$

Σ_t refers to the total rate of which neutrons are removed by all processes from a defined volume and energy region. Σ_s refers to the scattering rate. $\bar{\mu}_0$ is the average of the cosine of the laboratory scattering angle. If absorption is small compared to Σ_s or at least weak, then Σ_t may be replaced by Σ_s and we obtain:

$$\lambda_{tr} = \frac{1}{\Sigma_s(1-\bar{\mu}_0)} = \frac{\lambda_s}{(1-\bar{\mu}_0)}$$

and

$$D = \frac{1}{3\Sigma_s(1-\bar{\mu}_0)}$$

We are now in a position to compute $\bar{\mu}_0$

The scattering of neutrons with energies less than 10 MeV (i.e., fission neutrons) is isotropic in the center-of-mass frame of reference. So, all angles from 0 to π (0 to 180°) are equally likely. The average is $\pi/2$ (90°) and the average of cos φ is 0 where φ is the COM scattering angle.

The scattering angle in the laboratory system is always less than that of the COM system. This means that scattering in the LAB system is in the forward direction. For hydrogen, which is the most common scatterer, the LAB angle of scatter ranges from 0 to 90°.

To obtain the average value of the cosine of the scattering angle, multiply each value of μ_0 by the fraction of neutrons that are scattered between μ_0 and (μ_0 plus $d\mu_0$) and then integrate over μ_0. The result (see Henry p. 58) is:

$$\bar{\mu}_0 = \frac{2}{3A}$$

For a mixture of isotopes one would then average again, this time weighting each $\bar{\mu}_0$ by the percent of the isotope present.

What does the above mean? If scattering in the LAB system were isotropic (instead of forward directed), $\bar{\mu}_0$ would be zero and the transport mean free path would equal the scattering mean free path. Hence, the quantity $1/(1-\bar{\mu}_0)$ serves as a correction factor for anistropic scattering. Note that $\bar{\mu}_0$ decreases as the mass number increases. Hence, scattering is essentially isotropic for neutrons that strike heavy nuclei.

10. Average Loss of the Logarithm of Energy:

Another useful quantity in the study of the slowing down of neutrons is the average value of the decrease in the natural logarithm of the neutron energy per collision. This quantity is denoted by the Greek symbol ζ. If E_1 and E_2 are the neutron energies before and after the collision, then ζ is the average for all collisions of the quantity ($\ln E_1 - \ln E_2$). The probability that a neutron with energy E_1 scatters into an interval defined by (E_2 to $E_2 + dE$) is:

$$P(E_1 \to E_2)dE_2 = \frac{dE_2}{E_1(1-\alpha)}$$

Note that this probability depends on the width (dE_2) and on the initial energy (E_1) and it does NOT depend on the final energy (E_2).

Therefore,

$$\zeta = \int_{\alpha E_1}^{E_1} \left(\ln E_1 - \ln E_2\right) \frac{dE_2}{E_1(1-\alpha)}$$

$$\cong \frac{2}{A + 2/3}$$

(Note: See Henry p. 59 for the integration.)

The above relation is very accurate for $A > 10$. For $A = 2$, it is accurate to 3 %. Of importance is that the value of ζ is independent of the neutron's energy. This allows us, as shown in the next section, to use it to estimate the effectiveness of moderators.

11. Lethargy:

The material in this section follows that of Henry (p. 60).
Lethargy, which is also called the logarithmic energy decrement, is denoted by the symbol u. It is defined as:

$$u \equiv \ln(E_0 / E)$$

Where E_0 is the highest energy of a fission neutron, typically 10 MeV. Hence, for $E=E_0$, $u=0$. For $E=0$, $u=\infty$.

Consider a neutron that scatters from E to E'. We can write:

$$\ln E - \ln E' = [\ln E_0 - \ln E' - (\ln E_0 - \ln E)]$$
$$= \ln \frac{E_0}{E'} - \ln \frac{E_0}{E}$$
$$= u' - u$$

We previously defined ζ as the average value of $\ln (E/E')$. So, ζ is also the average change in lethargy of a neutron per collision.

A useful application of the lethargy concept is for the calculation of the number of collisions needed to thermalize a neutron. The most probable energy of a fission neutron is 0.85 MeV. Thermal energy is defined as 0.025 eV. So the needed change in lethargy to slow down from 0.85 MeV to 0.25 eV is ln (.85 x 10^6/.025) or 17.34.

From the relation 2/(A+2/3), we obtain ζ for various elements. For carbon, it is 0.158. So, (17.34/0.158) = 110 collisions. The table below (Source: Glasstone, p. 133) lists some data for several moderators.

Scattering Properties of Nuclei

Element	Mass No.	α	ζ	Collisions to Thermalize
Hydrogen	1	0	1.000	18
Deuterium	2	0.111	0.725	25
Helium	4	0.360	0.425	43
Beryllium	9	0.640	0.206	86
Carbon	12	0.716	0.158	114
Uranium	238	0.983	0.000838	2172

The time required for a neutron to attain thermal energy is a function of the number of collisions that it must undergo to attain 0.025 eV. For moderators made of Be or graphite, that time will be significantly greater than for those made of light water. Thus, leakage will be greater in reactors that do NOT use light water as a moderator. That is, if it takes longer to slow down, the likelihood of neutron leakage out of the core goes up. So, reactors that use Be or graphite moderators will have to be larger than those that use light water.

12. Moderating Ratio:

The moderating ratio is a figure of merit for the evaluation of moderators.

$$MR = \frac{\zeta \Sigma_s}{\Sigma_a}$$

The desired properties are a high energy loss per collision, a high scattering cross-section, and a low absorption cross-section. Some values (Glasstone, p. 139) are:

Moderator	MR
Light Water	58
Heavy Water	21000
Helium	45
Beryllium	130
Graphite	200

MR values should be taken as guidelines only. For example, heavy water has the highest moderating ratio. But, that is because of its very, very low absorption cross-section. Light water is a better scatterer and hence would allow a more compact core. So, light water is the best choice for a moderator and heavy water the best for a reflector where the objective is not to absorb the already thermalized neutrons. Also, while helium has a good MR value, it is gaseous and hence its use would also make for a large reactor.