# 22.05 Reactor Physics - Part Sixteen

## Solution of Group Equations

1. The multi-group equations in their most general form for an infinite homogeneous reactor are:

$$\begin{bmatrix} \Sigma_{t1}-\chi_1\nu\Sigma_{f1}-\Sigma_{11} & -\chi_1\nu\Sigma_{f2}-\Sigma_{12} & \cdots & -\chi_1\nu\Sigma_{fG}-\Sigma_{1G} \\ -\chi_2\nu\Sigma_{f1}-\Sigma_{21} & \Sigma_{t2}-\chi_2\nu\Sigma_{f2}-\Sigma_{22} & \cdots & -\chi_2\nu\Sigma_{fG}-\Sigma_{2G} \\ \vdots & \vdots & & \vdots \\ \chi_G\nu\Sigma_{f1}-\Sigma_{G1} & -\chi_G\nu\Sigma_{f2}-\Sigma_{G2} & \cdots & \Sigma_{tG}-\chi_G\nu\Sigma_{fG}-\Sigma_{GG} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_G \end{bmatrix} = 0$$

A non-trivial solution exists only if the determinant of the matrix is zero. The value of the determinant depends on the choice of materials that comprise the reactor. These include the fuel, coolant, moderator, reflector, control devices, and structural materials. It is unlikely (very, very unlikely) that any given initial choice of materials will result in a critical assembly. Accordingly, an interactive approach is needed where one selects a composition (perhaps guided by hand calculations using one-velocity theory), determines the proximity of the choice to criticality, readjusts a parameter (perhaps the fuel loading or its enrichment) and then redoes the calculations. The question therefore arises as how to assess proximity to criticality. We need some parameter that is a measure of the correctness of our design and in particular our choice of materials. We do this by pretending that the quantity $\nu$, which is the number of neutrons emitted per fission, can be arbitrarily varied. Thus, we replace $\nu$ by $\nu/\lambda$ where $\lambda$ is a real, positive number. The end result of the solution of the multigroup equations is the value of $\lambda$ that will make the determinant of the matrix go to zero. If too much fuel is present, $\lambda$ will be greater than 1. If too little, then $\lambda$ will be less than one. We then adjust some parameter (fuel loading, enrichment, more or less moderation, etc.) and redetermine the value of $\lambda$.

### 2. The Eigenvalue $\lambda$:

The role of the quantity $\lambda$ in the multi-group equations is most easily seen by rewriting those equations as the sum of two distinct matrices. Define two G x G matrices [A] and [M] whose $gg'$ elements are

$$A_{gg'} \equiv \Sigma_{tg}\delta_{gg'} - \Sigma_{gg'},$$

$$M_{gg'} \equiv \chi_g \nu \ \Sigma_{fg'}$$

where $\delta_{gg'}$ is the Kronecker delta. That is $\delta_{gg'} = 1$ for $g = g'$ and $\delta_{gg'} = 0$ for $g \neq g'$. The multi-group equations can then be written as:

$$[[A] - [M]/\lambda][\Phi] = 0$$

or

$$[A][\Phi] = [M][\Phi]/\lambda .$$

The [A] matrix combines the absorption plus scattering terms. The [M] matrix represents the source terms (fission). We quote directly from Henry (pp. 73-74) on the solution method.

"It is possible to prove that $[A]^{-1}$, the inverse of the matrix [A], exists for any physically real set of cross-sections and number densities (and, moreover, that every element of it is non-negative). Thus, we can obtain:

$$[A]^{-1}[M][\Phi] = \lambda[\Phi].$$

This result has the standard eigenvalue-equation form. The number $\lambda$ is thus an eigenvalue of the matrix $[A]^{-1}[M]$, and a nontrivial solution for $[\Phi]$ will exist if, and only if, the determinant of $[A]^{-1}[M] - \lambda[I]$ vanishes ($[I]$ being the G x G unit diagonal matrix).

In general, there will be G values of $\lambda$ for which there will be a nontrivial solution. However, in this case, it is possible (for any physically realizable set of multi-group parameters) to prove the following mathematical statements:

a) There is a unique real, positive eigenvalue of greater in magnitude than any other eigenvalue.

b) All the elements of the eigenvector corresponding to that eigenvalue are real and positive.

c) All other eigenvectors of either have some elements that are zero or have elements that differ in sign from each other.

These properties guarantee that here always exists a solution to the multi-group equations corresponding to a real positive value of $\lambda$ and that the eigenvector $[\Phi]$ corresponding to that eigenvalue is the only physically acceptable (all-positive) solution. Moreover, the fact that the eigenvalue we seek is greater in magnitude than any other eigenvalue for the multi-group equations leads to a systematic

procedure for solving those equations that is guaranteed to yield the desired eigenvalue and eigenvector." (Henry, pp. 73-74)

*Reference should be made to Henry (p. 74) for the proof of these statements.*

## 3. Physical Meaning of λ and K-Effective:

It is evident from the relation $[A][\Phi] = [M][\Phi]/\lambda$ that λ can be interpreted as a ratio of neutron production to neutron loss. This concept becomes more precise if one writes out the multi-group equation for a single group and then sums over all groups. It can be shown that λ is:

$$\lambda = \frac{\sum_{g=1}^{G} \nu\Sigma_{fg}\Phi_g}{\sum_{g=1}^{G} \Sigma_{ag}\Phi_g}$$

Where $\Sigma_{ag}$ is the total macroscopic absorption cross-section for group g. If the reactor is critical, then λ is unity and it is the ratio of the total rate of neutron production to the total rate of neutron absorption. It is common practice to extend this interpretation to the case where λ is not unity. In that case, however, the reactor is artificially critical and λ is not a physically measurable quantity. Given this interpretation of λ and given that we are discussing an infinite homogeneous medium, λ is termed the infinite multiplication factor or K-infinity that was discussed when we considered neutron life cycle analysis.

The ideas developed above are not limited to an infinite homogeneous medium. For realistic reactors (heterogeneous with finite, complex geometries) it is still always possible to find some real, positive value of λ such that when it is divided into ν, the reactor assembly will be critical. Thus, we can generalize the interpretation of λ as the ratio of neutron production to that of the sum of both neutron absorption and leakage. In this case, λ is then the effective multiplication factor of K-effective.

Thus,

$$K_{\infty} = \frac{\text{Neutron Production}}{\text{Neutron Absorption}}$$

and applies to an infinite medium

$$K_{\text{effective}} = \frac{\text{Neutron Production}}{\text{Neutron Absorption} + \text{Neutron Leakage}}$$

and applies to reactors of finite size.

## 4. Reactivity:

Another very useful quantity, particularly in time-dependent reactor physics, is the reactivity. It is defined as:

$$\rho \equiv \frac{K_{effective} - 1}{K_{effective}}$$

or

$$\frac{\lambda - 1}{\lambda}$$

Reactivity is global property. That is, it is a property of the reactor as a whole. Nevertheless, it is common practice to speak of the reactivity associated with a control blade or a fuel element. But such statements are only valid for the reactor configuration in which the calculation (or measurement) of the reactivity was made. If the configuration is altered, then the reactivity numbers will be different. When we discuss reactor kinetics, we will see that failure to realize this has lead to some operational problems.

Data from the MIT Research Reactor provides strong evidence of the global nature of reactivity. Refer to Fig. 1 of the attached paper," Effect of Radial Power Distribution on MITR-II Fuel Element and Control Blade Worth." The reactor core consists of three rings (labeled A, B, and C) of fuel arranged in the shape of a hexagon. If fresh fuel is placed in the innermost (A) ring, power shifts to the core interior. The converse occurs if fresh fuel is placed in the C-ring. Fig. 1 shows this strong dependence of the reactivity worth of the shim blades, which are located exterior to the C-ring on the core perimeter. Why does this occur? The reactivity of a control device is a function of the number of neutrons that it absorbs which is in turn dependent on the neutron flux in which the device is immersed. So, when the core power shifts inward, the flux near the blades decreases and therefore the number of neutrons that strike the blades and hence are absorbed by them also decreases. Hence blade reactivity worth decreases. (Note: If one performs a thorough analysis of blade reactivity worth, one finds that it is approximately proportional to the square of the neutron flux. So, the effect is very pronounced.)

## 5. Sample Calculations:

Computer codes exist for the purpose of solving the multi-group equations. The design of those codes requires knowledge of both numerical methods and linear algebra, both of which are beyond the scope of this course. For accurate estimates of the energy spectrum, the number of groups may be as many as 3000. However,

for illustrative purposes, two or three group problems can be solved by hand. We summarize a three group example given by Henry (p. 79).

In this example, it is assumed that there is no upscattering (i.e., $\Sigma_{gg'} = 0$ for $g' > g$), that there is only one fissionable isotope (no need to average cross-sections over several isotopes) and that $\chi_3 = 0$ (no fission neutrons appear in group 3 which is the thermal group). Also, we make use of the notation for the total removal cross-section, $\Sigma_g$, which is defined as absorption plus scattering out. That is,

$$\Sigma_g \equiv \Sigma_{tg} - \Sigma_{gg} = \Sigma_{ag} + \Sigma_{sg} - \Sigma_{gg} = \Sigma_{ag} + \sum_{g' \neq g} \Sigma_{g'g}$$

The general form of the multi-group equations (first page of these notes) therefore reduces to this three-group example:

$$\begin{bmatrix} \Sigma_1 - \frac{1}{\lambda}\chi_1 \nu\Sigma_{f1} & -\frac{1}{\lambda}\chi_1 \nu\Sigma_{f2} & -\frac{1}{\lambda}\chi_1 \nu\Sigma_{f3} \\ -\frac{1}{\lambda}\chi_2 \nu\Sigma_{f1} - \Sigma_{21} & \Sigma_2 - \frac{1}{\lambda}\chi_2 \nu\Sigma_{f2} & -\frac{1}{\lambda}\chi_2 \nu\Sigma_{f3} \\ -\Sigma_{31} & -\Sigma_{32} & \Sigma_3 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = 0$$

This is a homogenous equation and hence we will only be able to find the column vector $[\Phi]$ to within a multiplication constant. That is, if $[\Phi]$ is a solution, so is a constant times $[\Phi]$. We now quote directly from Henry:

"The critical condition is that the determinant be zero.

$$\begin{vmatrix} \Sigma_1 - \frac{1}{\lambda}\chi_1 \nu\Sigma_{f1} & -\frac{1}{\lambda}\chi_1 \nu\Sigma_{f2} & -\frac{1}{\lambda}\chi_1 \nu\Sigma_{f3} \\ -\frac{1}{\lambda}\chi_2 \nu\Sigma_{f1} - \Sigma_{21} & \Sigma_2 - \frac{1}{\lambda}\chi_2 \nu\Sigma_{f2} & -\frac{1}{\lambda}\chi_2 \nu\Sigma_{f3} \\ -\Sigma_{31} & -\Sigma_{32} & \Sigma_3 \end{vmatrix} = 0$$

If we multiply the second line of this determinant by $\chi_1$, subtract from the result the first line multiplied by $\chi_2$, then multiply the first line by $\lambda$ (assumed nonzero), we obtain

$$\begin{bmatrix} \lambda\Sigma_1 - \chi_1\nu\Sigma_{f1} & -\chi_1\nu\Sigma_{f2} & -\chi_1\nu\Sigma_{f3} \\ -\chi_{1}\Sigma_{21} - \chi_{2}\Sigma_{1} & \chi_{1}\Sigma_{2} & 0 \\ -\Sigma_{31} & -\Sigma_{32} & \Sigma_3 \end{bmatrix} = 0.$$

It is clear from this result that the equivalent algebraic equation will be one of first order in $\lambda$ and hence will yield only one value of that quantity. Expansion of the determinant shows that this value is:

$$\lambda = \frac{\left(\chi_1\Sigma_{21} + \chi_2\Sigma_1\right)\left(\nu\Sigma_{f3}\Sigma_{32} + \nu\Sigma_{f2}\Sigma_3\right) + \chi_1\Sigma_2\left(\nu\Sigma_{f3}\Sigma_{31} + \nu\Sigma_{f1}\Sigma_3\right)}{\Sigma_1\Sigma_2\Sigma_3}$$

Because the $\chi$'s and $\Sigma$'s in this equation are real positive numbers, we have thus demonstrated mathematically for this case that there always exists a real positive value of $\lambda$ that causes the determinant to vanish. Moreover, we have found that there is only one such value of $\lambda$.

The other two eigenvalues are both $\lambda$=0. The corresponding eigenvectors $\mathrm{Col}\{\Phi_1, \Phi_2, \Phi_3\}$ must be such that $\nu\Sigma_{f1\Phi_1} + \nu\Sigma_{f2\Phi_2} + \nu\Sigma_{f3\Phi_3} = 0$, and thus must have elements that differ in sign.

As noted above, we can find the column vector $[\Phi]$ only to within a multiplicative constant; or, to put it another way, we are free to set any one of the group fluxes $\Phi_1, \Phi_2,$ or $\Phi_3$ equal to an arbitrary constant and find the other two fluxes in terms of that constant. The constant would be determined by the power level of the reactor. Let us fix the element $\Phi_1$. The last two multi-group algebraic equations implied, then yield $(\text{for } \lambda \neq 0)$

$$\left(\Sigma_2 - \frac{1}{\lambda}\chi_2\nu\Sigma_{f2}\right)\Phi_2 - \frac{1}{\lambda}\chi_2\nu\Sigma_{f3}\Phi_3 = \left(\frac{1}{\lambda}\chi_2\nu\Sigma_{f1} + \Sigma_{21}\right)\Phi_1$$

$$-\Sigma_{32}\Phi_2 + \Sigma_3\Phi_3 = \Sigma_{31}\Phi_1$$

Inverting the matrix that multiplies $\mathrm{Col}\{\Phi_2, \Phi_3\}$ yields

$$\begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix} = \frac{\Phi_1}{\Sigma_3\left(\Sigma_2 - \lambda^{-1}\chi_2 \nu\Sigma_{f2}\right) - \lambda^{-1}\chi_2\Sigma_{32}\nu\Sigma_{f3}} \begin{bmatrix} \Sigma_3 & \frac{1}{\lambda}\chi_2\nu\Sigma_{f3} \\ \Sigma_{32} & \Sigma_2 - \frac{1}{\lambda}\chi_2\nu\Sigma_{f2} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda}\chi_2\nu\Sigma_{f1} + \Sigma_{21} \\ \Sigma_{31} \end{bmatrix}$$

This equation along with value of $\lambda$ constitutes the solution that is of physical interest."