22.05 Reactor Physics Part Seventeen

Energy Dependence of Flux[1]

1. Group Solution:

The fundamental assumption that allows one to move ahead with group theory is that the flux, $\phi(E)$, can be taken as a constant within each energy group. The result of the group calculation is then those group fluxes normalized to some power level. If the number of energy groups is large and the width of each group is small, this approximation gives an excellent result. The true energy-dependent flux $\phi(E)$, can then be approximated:

$$\Phi(E) = \frac{E_{g-1} - E}{E_{g-1} - E_g}\Phi(E_g) + \frac{E - E_g}{E_{g-1} - E_g}\Phi(E_{g-1}) \text{ for } E_g \le E \le E_{g-1}$$

While the computer approach is useful and practical, it provides no physical understanding of the reasons for the shape of the flux. We address that issue here by solving for $\phi(E)$ using various simplifications. The starting point is the neutron balance relation:

$$\Sigma_t(E)\phi(E) = \int_0^\infty dE'\left[\chi(E)\nu\Sigma_f(E') + \Sigma_s(E' \to E)\right]\phi(E')$$

The intent is to determine the shape of $\phi(E)$ in the three major energy ranges: fast, epithermal, and thermal. These correspond to the regions of neutron appearance from fission, neutron slowing down, and neutron absorption.

2. Fast Range:

This range is defined here as E > 1 MeV. For a given energy group, there are two possible neutron sources: fission and scatter. At these energies, fission dominates. Hence, the balance equation becomes:

$$\Phi(E) \approx \frac{\chi(E)}{\Sigma_t(E)}\int_0^\infty dE' \frac{1}{\lambda}\nu\Sigma_f(E')\Phi(E'), \text{ E in the fission} - \text{source range.}$$

[1] Material in this section follows that of Henry pp. 80-101. Portions that are verbatim are indicated by quotations.

The integral is a constant and $\Sigma_t(E)$ is fairly constant for E > 1 MeV. Hence, we conclude that $\phi(E)$ is proportional to $\chi(E)$, the fission spectrum. The two have the same shape.

3. **Epithermal Range**:

The epithermal range is defined here as extending from 50 keV to 1 eV. The upper limit, 50 keV, represents the energy at which $\chi(E)$ approaches zero Neutrons do not appear directly from fission at 50 keV and below. The lower limit 1 eV, is the upper end of the thermal range where neutron upscattering becomes significant. So, for this range we have:

$$\chi(E) = 0$$

$$\Sigma_s(E' \rightarrow E) = 0 \text{ for } E' < E \qquad \text{(no upscatter)}$$

The neutron balance equation therefore becomes:

$$\Sigma_t(E)\Phi(E) = \int_E^{\infty} \Sigma_s(E' \rightarrow E)\phi(E')dE' \qquad \text{(E in epithermal range)}$$

We earlier stated that the probability of a neutron of energy E' being scattered into an interval E to (E+dE) is:

$$P(E' \rightarrow E) = \frac{1}{E'(1+\alpha_j)} \text{ for } E'\alpha_j < E < E'$$

Hence,

$$\Sigma_t(E)\Phi(E) = \sum_j \int_E^{E/\alpha_j} \frac{\Sigma_s^j(E')\phi(E')dE'}{E'(1-\alpha_j)}$$

Where the subscript j denotes the isotope off which the neutron is scattered. The upper limit of integration, E/α_j, is the highest energy from which a neutron could be isotropically scattered in the COM and attain energy E.

The above relation has been explored for a number of different situations including moderation by hydrogen only, no neutron absorption, weak neutron absorption, absorption by narrow widely spread resonances, and moderation by heavy elements only. We will consider the first and fourth of these, moderation by hydrogen only and absorption by narrow resonances. We then give the more general result.

a) **Moderation by Hydrogen Only**: Recall that α is defined as $[(A-1)/(A+1)]^2$ where A is the mass number. For hydrogen, α is zero. For heavy nuclei, α approaches unity. We also showed earlier that the energy loss per collision was a function of α. For hydrogen, all of the colliding neutron's energy could be lost in a single collision. For heavy nuclei, very little energy is lost. We will use this information to separate the effects of hydrogen on the slowing down of the neutrons from those of other nuclides. The discussion that follows is taken from Henry (pp. 84-86).

"If we take the mass of all isotopes j, except hydrogen, to be infinite, we see from the definition of α_j that for hydrogen $\alpha_H = 0$, while for all other isotopes $\alpha_j = 1$. Hence, for these other isotopes, the lowest energy $(\alpha_j E)$ attainable as the result of an elastic-scattering collision from initial energy E is E itself. That is, the scattering events lead to no energy loss. As a result, the range of integration tends to zero, and we can legitimately assume that, through this vanishingly small range,

$$\frac{\Sigma_s^j(E')\phi(E')}{E'} \rightarrow \frac{\Sigma_s^j(E)\phi(E)}{E}$$

Thus, our expression for the neutron balance in the epithermal range becomes

$$\Sigma_t(E)\phi(E) = \int_E^{E/\alpha_H} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'(1-\alpha_H)} + \sum_{j \neq H} \int_E^{E/\alpha_j} \frac{\Sigma_s^j(E)\phi(E)dE'}{E(1-\alpha_j)}$$

$$= \int_E^{\infty} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'} + \sum_{j \neq H} \frac{1}{\alpha_j}\Sigma_s^j(E)\phi(E)$$

Where $\Sigma_s^H(E')$ is the macroscopic scattering cross section for hydrogen. (Note: See next page if one has difficulty with the integration)

If we make use of the fact,

$\Sigma_t(E) = \Sigma_a(E) + \Sigma_s^H(E) + \sum_{j \neq H}\Sigma_s^j(E)$, then

$$\left[\Sigma_a(E) + \Sigma_s^H(E) + \sum_{j \neq H}\Sigma_s^j(E)\right]\phi(E) = \int_E^{\infty} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'} + \sum_{j \neq H} \frac{1}{\alpha_j}\Sigma_s^j(E)\phi(E)$$

Note on Integration

1. Consider the term:

$$\int_{E}^{E/\alpha_H} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'(1-\alpha_j)}$$

For hydrogen, α=0. Hence, E/ α is infinity and $E'(1-\alpha)$ is E'. Thus, one obtains:

$$\int_{E}^{\infty} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'}$$

2. Consider the term:

$$\sum_{j \neq H} \int_{E}^{E/\alpha_j} \frac{\Sigma_s^j(E)\phi(E)dE'}{E(1-\alpha_j)}$$

The integration is over dE'. So, $\Sigma_s(E)\phi(E)/E(1-\alpha)$ comes out of the integral. Thus,

$$\sum_{j \neq H} \frac{\Sigma_s(E)\phi(E)}{E(1-\alpha_j)} \int_{E}^{E/\alpha_j} dE'$$

$$= \sum_{j \neq H} \frac{\Sigma_s(E)\phi(E)}{E(1-\alpha_j)} \left[\frac{E}{\alpha_j} - E \right]$$

$$= \sum_{j \neq H} \frac{1}{\alpha_j} \Sigma_s^j(E)\phi(E)$$

and if we go to the limit of α_j=1 (valid for heavy elements) we get

$$\left[\Sigma_a(E)+\Sigma_s^H(E)\right]\Phi(E)=\int_E^{\infty}\frac{\Sigma_s^H(E')\phi(E')dE'}{E'}$$

As might be expected, the heavy element scattering, which does not degrade the neutron-energy, has disappeared from both sides of the equation.

This equation can be turned into a differential equation by the simple procedure of taking the derivative of both sides with respect to energy. The result is:

$$\frac{d}{dE}\left\{\left[\Sigma_a(E)+\Sigma_s^H(E)\right]\Phi(E)\right\}=-\frac{\Sigma_s^H(E)\Phi(E)}{E}$$

Dividing both sides by $\left[\Sigma_a(E)+\Sigma_s^H(E)\right]\Phi(E)$ and integrating from some lower energy E in the slowing-down region to some higher energy E_1 then yields:

$$\ln\frac{[\Sigma_a(E_1)+\Sigma_s^H(E_1)]\Phi(E_1)}{[\Sigma_a(E)+\Sigma_s^H(E)]\Phi(E)}=-\int_E^{E_1}\frac{\Sigma_s^H(E')dE'}{[\Sigma_a(E')+\Sigma_s^H(E')]E'}$$

or

$$\left[\Sigma_a(E)+\Sigma_s^H(E)\right]\Phi(E)=[\Sigma_a(E_1)+\Sigma_s^H(E_1)]\Phi(E_1)\exp\left(\int_E^{E_1}\frac{\Sigma_s^H(E')dE'}{[\Sigma_a(E')+\Sigma_s^H(E')]E'}\right)$$

$$=[\Sigma_a(E_1)+\Sigma_s^H(E_1)]\Phi(E_1)\exp\left(\int_E^{E_1}\frac{dE'}{E'}-\int_E^{E_1}\frac{\Sigma_a(E')dE'}{[\Sigma_a(E')+\Sigma_s^H(E')]E'}\right)$$

(Note: If you have difficulty with the algebra, see the next page.)

Since $\exp(\ln(E_1/E))=E_1/E$, we obtain as the final result

$$\Phi(E)=\frac{[\Sigma_a(E_1)+\Sigma_s^H(E_1)]E_1\Phi(E_1)}{[\Sigma_a(E)+\Sigma_s^H(E)]E}\exp\left(-\int_E^{E_1}\frac{\Sigma_a(E')dE'}{[\Sigma_a(E')+\Sigma_s^H(E')]E'}\right)$$

Note on Algebra

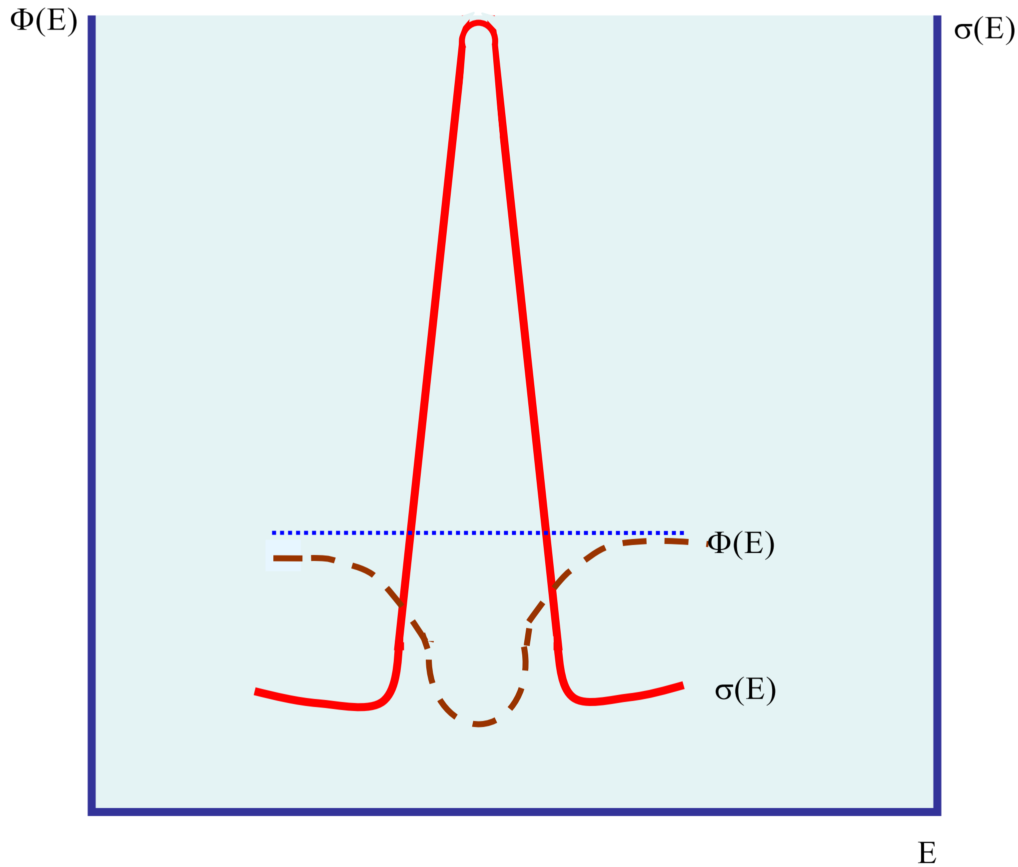
$$\frac{dE'}{E'} - \frac{\Sigma_a(E')dE'}{[\Sigma_a(E') + \Sigma_s^H(E')]E'}$$

$$= \frac{\left[\Sigma_a(E') + \Sigma_s^H(E')\right]E'\ dE' - E'\ \Sigma_a(E')dE'}{E'\ [\Sigma_a(E') + \Sigma_s^H(E')]E'}$$

$$= \frac{\Sigma_s^H(E')dE'}{[\Sigma_a(E') + \Sigma_s^H(E')]E'}$$

Under the assumption that the masses of all isotopes other than hydrogen are infinite. Thus, with hydrogen the only moderator, the scalar flux density, assumed to have a value $\Phi(E_1)$ at some initial high energy, rises roughly as E_1/E as the energy decreases. The overall E_1/E behavior is modified by the presence of $[\Sigma_a(E) + \Sigma_s^H(E)]$ in the denominator and by the exponential term that accounts for the decrease in flux at lower energies due to neutron absorption in the range E to E_1.

The $\left(\Sigma_a + \Sigma_s^H\right)$ term leads to an interesting effect when the system contains materials having large narrow resonances as shown in the figure below. Under such circumstances $\Sigma_a(E)$ will be very large for E in the range of a resonance, and the term $[\Sigma_a(E) + \Sigma_s^H(E)]$ will cause the flux to dip at energies corresponding to the resonance. (<u>Note</u>: Remember, the numerator is merely some number because everything is evaluated at E_1.)



Behavior of the neutron flux density in the energy range above a large absorption resonance (in an infinite homogeneous medium, with hydrogen the only moderator). (Source: Henry, p. 86)

The denominator is a function of E and as E decreases, $\Sigma_a(E)$ rises (1/v behavior).) At energies just below the resonance where $\Sigma_a(E)$ once more becomes small, the flux $\Phi(E)$ will again rise. Physically such behavior is due to the fact that, in collisions with hydrogen, neutrons of initial energy E can emerge with any energy in the range 0 to E; hence, there is little likelihood that they will emerge in the energy range corresponding to a given narrow resonance. Thus the neutron population at energies just below an absorption resonance will build up because of the appearance of neutrons scattered from energies above the range of the resonance." Scattering is a stepwise, not a continuous process.

"Of course, since a resonance removes neutrons from the system, there will be some lasting effect on $\Phi(E)$. Mathematically this is accounted for through the exponential term

$$\exp\left(-\int_{E}^{E_1} \frac{\Sigma_a(E')dE'}{[\Sigma_a(E') + \Sigma_s^H(E')]E'}\right)$$

Even here, however, the magnitude of the effect is kept moderately small by the fact that $\Sigma_a(E')$ appears in the denominator as well as in the numerator of the integrand."

The exponential term shown above illustrates a phenomenon called "energy self-shielding." If, at a given resonance energy, $\Sigma_a(E) >> \Sigma_s(E)$ then the integral reduces to:

$$\exp\left[-\int_{E}^{E+\Delta E} \frac{dE'}{E'}\right]$$

$$= \exp\left[-\ln\left(\frac{E+\Delta E}{E}\right)\right]$$

$$= \left[\frac{E}{E+\Delta E}\right]$$

Hence, the magnitude of the absorption is a function of ΔE, which is the width of the resonance. The first of the six large U-238 resonances is at 6.67 eV and it has a width of 0.027 eV at low temperature. So, E/(E+ΔE) is 6.67/6.695 or 0.996. Hence, only 0.4% of the neutrons would be absorbed despite the magnitude of the resonance.

To summarize, the above analysis (moderation by hydrogen only) is useful for the physical insights that it provides:

- $\Phi(E)$ in the epithermal range varies inversely with E.
- $\Phi(E)$ dips at energies corresponding to a resonance but it then recovers for lower energies.

The second of these results has general validity. The first, however, does not because its derivation has ignored the effects of the heavier elements.

b) Narrow, Widely Spread Absorption Resonances: This situation gives a result that has greater general validity than the previous derivation which allowed for hydrogen moderation only. The general result is:

$$\Phi(E) = \frac{[\Sigma_a(E_1) + \Sigma_s(E_1)]E_1\xi(E_1)\Phi(E_1)}{[\Sigma_a(E) + \Sigma_s(E)]E\xi(E)} \exp\left(-\int_E^{E_1} \frac{\Sigma_a(E')dE'}{E'\xi(E')\left[\Sigma_a(E') + \Sigma_s(E')\right]} \right)$$

The principal differences between this relation and the one derived above for hydrogen-moderation only are: 1) the presence of the lethargy $\xi(E)$ in the denominator of both the main term and the exponential, and 2) scattering is not restricted to hydrogen.

For small absorption cross-sections, the exponential is close to unity and the 1/E dependency of $\Phi(E)$ that is predicted for hydrogen-moderation only will be seen. But, this is normally not the case. If $\Sigma_a(E)$ is small relative to Σ_s and especially if $\xi(E)$ is small, the exponential will be small and $\Phi(E)$ will be attenuated as the neutrons slow down. This suppresses the 1/E behavior with the result that the flux decreases as neutron energy decreases. This decrease is exacerbated if heavy elements are present. Specifically $\xi(E)$ is the average loss in the logarithm of the neutron energy per collision and an absence of light elements results in a small $\xi(E)$ and greater attenuation of the flux.

4. Thermal Range:

The thermal range is defined here as covering neutron energies $E \leq 1\text{eV}$. No neutrons are produced directly from fission (i.e., $\chi(E) = 0$ for this range). So that is a simplification. A complication is that neutron upscattering may occur because neutron energies are the same as the thermal energies of the nuclei with which they collide. While this complication renders a hand calculation of $\Phi(E)$ for thermal energies too difficult, it provides arguments for the observed result. We again quote Henry (p. 96):

"If there were no absorption of neutrons in the infinite homogeneous medium under consideration and if there were no production of neutrons due to fission, any neutron population present would neither decay nor increase. Instead the neutrons would come into thermal equilibrium with the material of the medium. They would behave like a very rare gas and would thus have a Maxwellian distribution of velocities given by:

$$n(v)\,dv \propto v^2 \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right)dv$$

Where m and v are the mass and speed of the neutron, n(v) dv is the number of neutrons per unit volume having speeds between v and v + dv, T is the absolute temperature of the medium, and k is the Bolzmann constant.

The quantity n(v) dv is also equal to the number of neutrons per cc having energies between E and E+dE, where $E = \frac{1}{2}mv^2$ is the kinetic energy corresponding to speed v. Thus, if we define n(E), a number density per unit energy such that n(E) dE = n(v)dv, we get:

$$n(E)\,dE \propto v^2 \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right)dv$$

(Note that we are again using the physics convention that n( ) stands for a physical quantity and not a mathematical function; thus n(E) and n(v) have different units.) Since the scalar flux density is $\Phi(E) = v(E)n(E)$ and $E \propto v^2$, we then find that

$$\Phi(E)dE \propto E\exp\left(\frac{-E}{kT}\right)dE \text{ if } \Sigma_a(E) = 0\text{"}$$

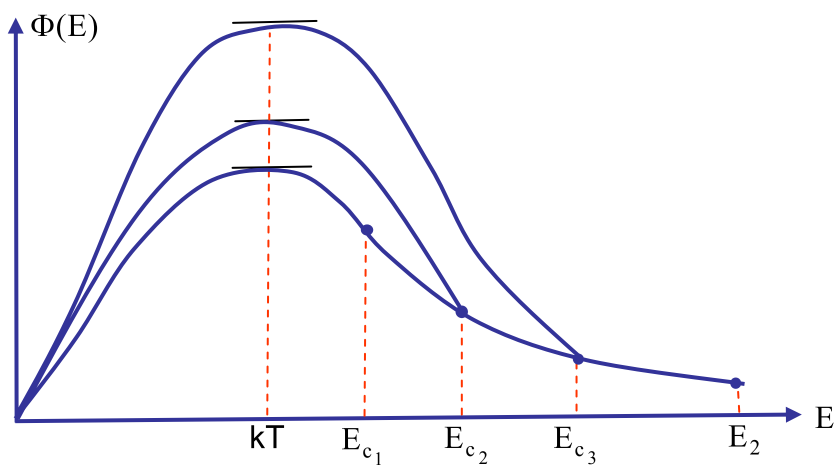
When absorption and fission are present, the rationale for assuming a Maxwellian distribution is not valid. Comparison with the multi-group calculation shows that the largest differences occur at the higher-energies (0.1 – 1 eV). In this range, the Maxwellian distribution goes to zero while the $\Phi(E)$ falls off as 1/E. This is to be expected from our analysis of the slowing down process which showed that, in the absence of upscatter and at energies for which $\chi(E) = 0$ (no fission source), $\Phi(E)$ exhibits 1/E behavior.

"This qualitative discussion suggests that a rough approximation to $\Phi(E)$ in the thermal range can be obtained by assuming it to have a Maxwellian shape KE exp(-E/kT), where K is a constant, up to some energy cut point $E_c \approx 0.1$ eV and to be proportional to 1/E throughout the remainder of the thermal range. To connect these two segments together and to connect the thermal range to the slowing-down range, we simply require that $\Phi(E)$ be a continuous function of energy and that neutron balance be maintained.

The approximate value of $\Phi(E)$ that results from this procedure is then

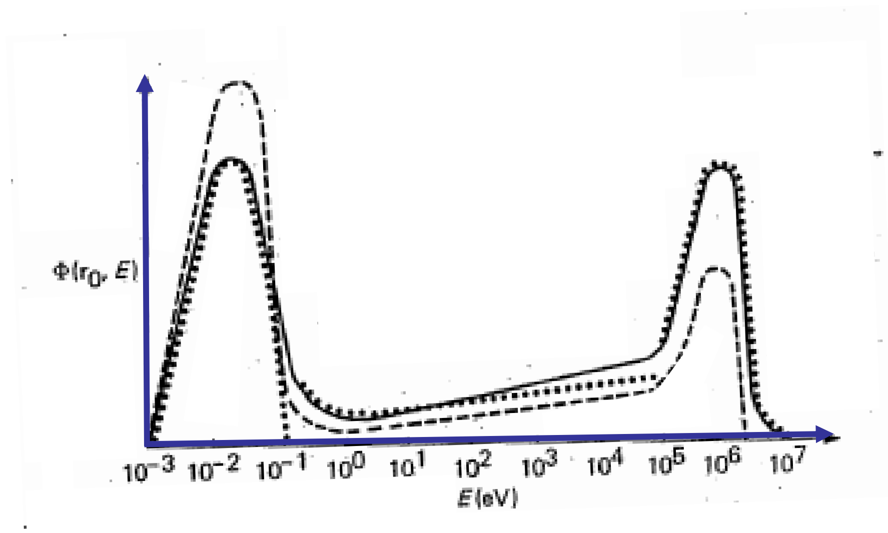
$$\Phi(E) = \begin{cases} \dfrac{E_2 E \Phi(E_2)}{E_c^2} \exp\left(\dfrac{E_c - E}{kT}\right) & \text{for } 0 \le E \le E_c, \\ \dfrac{E_2 \Phi(E_2)}{E} & \text{for } E_c < E \le E_2 \approx 1\,\text{eV}. \quad " \end{cases}$$

Where $E_2 \equiv 1$ eV (the upper end of the thermal range) and E_c is the "energy cut point" which depends on temperature and the cross-sections. The following figure shows the thermal flux approximated by a Maxwellian distribution joined to a 1/E tail at several possible cut points.



The thermal flux approximated by a Maxwellian distribution joined to a 1/E tail at various cuts point E_c. (Source: Henry, p. 101)

The final figure is calculation of $\Phi(r, E)$ for three energy groups. The energy dependence is given by the dashed line.



A three-group approximation to $\Phi(\mathbf{r}_0, E)$. The solid line is the true flux at $\mathbf{r}_0$; the dashed line is the asymptotic spectrum $\psi(E)$; and the dotted line is the three-group approximation to $\Phi(\mathbf{r}_0, E)$ using pieces of $\psi(E)$ separately normalized over ranges 0 -10^{-1} eV; 10^{-1} – 10^5 eV; and 10^5-10^7 eV. (Source: Henry, p. 189)