

## 22.05 Reactor Physics - Part Eighteen

### Group Theory and the Four Factor Formula<sup>1</sup>

#### 1. Background:

Group Theory can be used to provide rigorous definitions of the factors developed in life cycle analysis. Thus far, the energy spectrum of an infinite homogeneous medium has been identified. As part of that analysis, it was shown that, provided the reactor contains some fissionable material, a real positive value of  $\lambda$  exists which is the ratio of the neutron production rate to the neutron absorption rate. This number,  $\lambda$ , is the k-infinity of the medium. For a reactor finite size (i.e., leakage exists) it would be k-effective.

It is not possible to measure  $k_{\infty}$ . However,  $k_{\infty}$  can be factored as:

$$k_{\infty} = \epsilon p f \eta$$

And each of these factors can be measured through careful experiment. These types of measurements were first done in the late 1950s and early 1960s as a design tool for Generation I and II reactors including those used for Pu production (large D<sub>2</sub>O or graphite moderated reactors) and some of the early power production reactors. These were not homogeneous mediums of infinite dimension. Rather, they were repeating arrays of fuel rods embedded in a moderator. Such arrays are called a “lattice.” Two terms are defined for each repeating section of the lattices:

Cell: This is the fuel rod and its surrounding moderator.

Fuel: This is the fuel rod.

We previously defined  $\lambda$  (same as  $k_{\infty}$ ) as:

$$k_{\infty} = \frac{\sum_{g=1}^G v \Sigma_{fg} \Phi_g}{\sum_{g=1}^G \Sigma_{ag} \Phi_g}$$

for G energy groups. This can be rewritten as an integral:

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<sup>1</sup> Material in this section follows that of Henry, pp. 102-112. Portions that are verbatim are indicated by quotations.

$$k_{\infty} = \frac{\int_0^{\infty} v \Sigma_f(E) \Phi(E) dE}{\int_0^{\infty} \Sigma_a(E) \Phi(E) dE}$$

Now, we introduce spatial as well as energy dependence. So, the cross-section and flux become functions of position and energy. Thus, for a given cell:

$$k_{\infty} = \frac{\int_{\text{cell}} dV \int_0^{\infty} dE v \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{\infty} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

There is no fuel in the moderator. So, the numerator of the above expression could also be written as an integral over the fuel region alone. We do this in the expansion of  $k_{\infty}$  that is given below.

The above relation can be expanded to yield:

$$= \frac{\int_{\text{fuel}} dV \int_0^{\infty} dE v \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{fuel}} dV \int_0^{E_2} dE v \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)} \cdot \frac{\int_{\text{fuel}} dV \int_0^{E_2} dE v \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{fuel}} dV \int_0^{E_2} dE \Sigma_a^{\text{fuel}}(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

$$\frac{\int_{\text{fuel}} dV \int_0^{E_2} dE \Sigma_a^{\text{fuel}}(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{E_2} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)} \cdot \frac{\int_{\text{cell}} dV \int_0^{E_2} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{\infty} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

The energy  $E_2$  is the division, approximately 1.0 eV, that separates the thermal region from the epithermal one.

In words, the above equation approximates to:

$$\left( \frac{\text{Fast and Thermal Fission}}{\text{Thermal Fission}} \right) \left( \frac{\text{Thermal Fission}}{\text{Thermal Absorption in Fuel}} \right) \cdot \left( \frac{\text{Thermal Absorption in Fuel}}{\text{Thermal Absorption in Fuel and Moderator}} \right) \left( \frac{\text{Thermal Absorption in Fuel and Moderator}}{\text{Fast and Thermal Absorption in Fuel and Moderator}} \right)$$

## 2. Fast Fission Factor:

This factor is denoted as  $\varepsilon$  and is defined as:

$$\varepsilon \equiv \frac{\int_{\text{fuel}} dV \int_0^{\infty} dE \nu \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{fuel}} dV \int_0^{E_2} dE \nu \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

It is the ratio of the total (fast and thermal because the energy integral in the numerator is from 0 to  $\infty$ ) production rate of neutrons to the production rate from thermal fission alone. (Note: The quantity  $\nu$  could be canceled.) Measurements of  $\varepsilon$  are done by irradiating two foils of fissionable material (U-235, Pu-239, Th-233). For the first foil, irradiate it by itself so it responds to both fast and thermal fissions. For the second, cover the foil with cadmium. Cadmium has a huge (7500 barns) thermal absorption resonance that ends abruptly at 0.5 eV. So, the Cd-covered foil responds only to fast fission. Then,

$$\varepsilon \equiv \frac{B}{B - C}$$

Where B and C are the counts from the bare and covered foils respectively.

For natural uranium,  $\varepsilon$  is about 1.05. It decreases with enrichment (less U-238 which undergoes a fast fission).

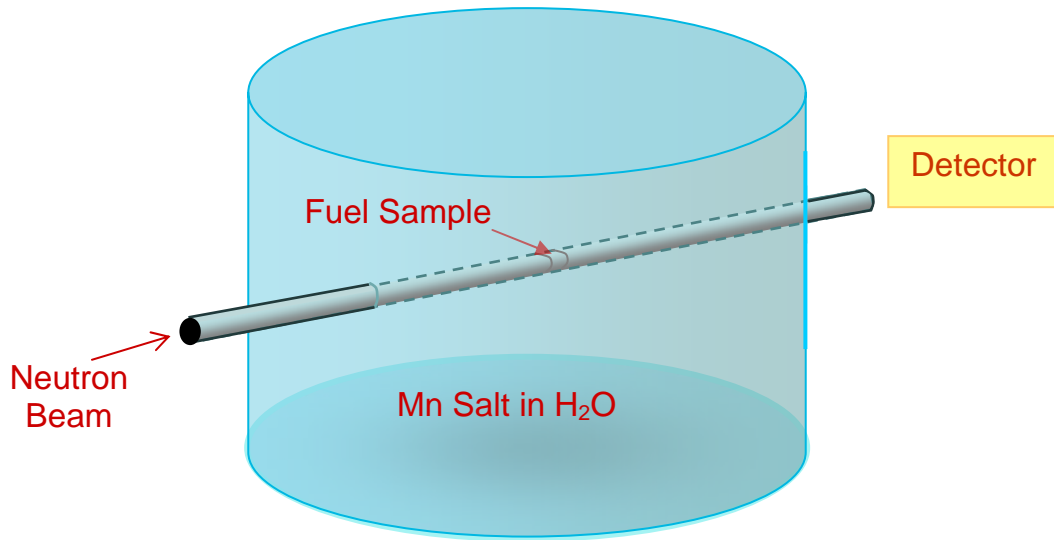
## 3. Thermal Reproduction Factor:

This factor is denoted by  $\eta$  and is defined as:

$$\eta \equiv \frac{\int_{\text{fuel}} dV \int_0^{E_2} dE \nu \Sigma_f(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{fuel}} dV \int_0^{E_2} dE \Sigma_a^{\text{fuel}}(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

It is the ratio of the rate of neutron production from thermal fission to the rate of thermal neutron absorption in the fuel.

Values of  $\eta$  can be determined experimentally using the following apparatus:



**The manganese-bath experiment for measuring  $\eta$**

A fuel sample is placed in the center of a tube which in turn is immersed in a bath of dissolved manganese salt. A detector is positioned to measure the neutrons emerging from the tube. The difference between the number of neutrons emerging from the tube and those entering the tube equals the number that were either: 1) absorbed in the fuel; or 2) scattered by the fuel. The latter is small and can often be ignored (or else a minor correction made). Thus, this measurement gives the denominator of the expression for  $\eta$ . To repeat, it is the number of neutrons absorbed in the fuel that result either in fission or U-236 production.

The second measurement is made by sampling the Mn bath and counting the radioactivity of the activated Mn. This measurement is a function of the production of fission neutrons in the fuel because those neutrons are emitted into the bath where they are absorbed by the Mn nuclei, thereby producing radioactivity. Thus, we obtain the numerator of the expression for  $\eta$ . (Note: The thermal (0.25 eV) cross-section for Mn and H<sub>2</sub>O are 136b and 0.664b respectively. So, most of the fission neutrons do cause transmutation of the Mn.)

The value of  $\eta$  for natural uranium fuel is about 1.34.

#### **4. Thermal Utilization:**

This factor is denoted by  $f$  and is defined as:

$$f \equiv \frac{\int_{\text{fuel}} dV \int_0^{E_2} dE \Sigma_a^{\text{fuel}}(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{E_2} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

It is the ratio of the rate at which thermal neutrons are absorbed in the fuel to the total rate at which they are absorbed in the cell (fuel plus moderator). Flux shape is important in the measurement of  $f$  because, unlike  $\epsilon$  and  $\eta$ , the thermal utilization involves two different regions: the fuel and the moderator. Measurements are made by performing fuel irradiations throughout the lattice. The value of  $f$  in a natural uranium reactor is about 0.90.

## 5. Resonance Escape:

This factor is denoted by the symbol  $p$  and is defined as:

$$p \equiv \frac{\int_{\text{cell}} dV \int_0^{E_2} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{\infty} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

$$= 1 - \frac{\int_{\text{cell}} dV \int_{E_2}^{\infty} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}{\int_{\text{cell}} dV \int_0^{\infty} dE \Sigma_a(\mathbf{r}, E) \Phi(\mathbf{r}, E)}$$

“It is the ratio of the rate of thermal-neutron absorption throughout the cell to the rate of absorption at all energies throughout the cell. It is thus the fraction of all neutrons absorbed in the cell that are absorbed thermally. Hence it is the probability that a neutron will escape capture at energies above thermal. Finally, since, in natural or slightly enriched uranium lattices, most non-thermal capture is in the resonance part of the slowing-down region, it is the “resonance-escape probability.” (Henry, p. 110)

Measurements of  $p$  are challenging because (1) the entire lattice is involved and (2) the cross-sections vary sharply as a function of neutron energy. In general, measurements can only be made by assembling large mockups of the actual fuel cells that are proposed.

For a homogenized medium, the value of  $p$  for natural uranium is about 0.70. However, larger values exist for the heterogeneous case. This is because of a phenomenon called “spatial self-shielding.” The idea is to keep the fuel and moderator separate. Neutrons slow down in a stepwise manner. Some collisions will cause neutrons to “jump” over the energies that correspond to the U-238 resonances. Others will result in neutrons with energies that correspond to a resonance. What happens to these neutrons? Suppose a neutron has an energy close to that of a U-238 resonance. If a neutron scatters off U-238 it will lose only a small amount of energy. If it scatters off moderator (neutron mass equal to that of hydrogen) it will lose a lot of its energy. In the first case, the neutron is left near the resonance energy. In the second it is removed from that energy.

What then is the effect of homogeneous or heterogeneous fuel? If the fuel and moderator are separate and the neutron is in the moderator, chances are it will collide with another moderator atom and be scattered out of the resonance region. So, heterogeneous arrangements favor resonance escape. In contrast, if the fuel and moderator are infinitely mixed (homogeneous) or if the fuel rod/moderator volumes repeat on small scales with dimensions less than a diffusion length, the next collision may again be with fuel and the neutron will be absorbed. Heterogeneous arrays are sometimes called “lumped” meaning that the fuel and moderator are separate, widely spread entities.

It should be noted that if one separates the fuel and moderator (large amounts of moderator between fuel rods), one decreases the thermal utilization. For  $f$  to be large, one seeks to maximize absorption in the fuel and hence minimize spatial self-shielding which could lead to absorption in the moderator. Thus, there is a tradeoff between  $f$  and  $p$ . Both depend on the moderator-to-fuel volume ratio. Optimal designs have both  $f$  and  $p$  at about 0.9.