22.05 Reactor Physics Part Twenty-One

Extension of Group Theory to Reactors of Multiple Region Two Energy Groups[1]

1. "The basic approximation which leads to one-group theory is that $\Phi(\mathbf{r},E)$ has a separable energy dependence $\psi^k(E)$ throughout each given material composition k. This approximation is very questionable at interfaces between regions k and *l*, since the form of $\psi^k(E)$ will not in general match that of $\psi^l(E)$ at such boundary. We tried to minimize this problem by matching $R^k(\mathbf{r}_c)\psi^k(E)$ and $R^l(\mathbf{r}_c)\psi^l(E)$ in an energy-integral sense. But if $\psi^k(E)$ and $\psi^l(E)$ are quite different (as will be in the case if k is core material an *l* is reflector material), there are bound to be significant mismatches over particular subranges of the overall energy range 0 to ∞.

   For example, at an interface between core and reflector material, the net leakage of high-energy neutrons, which are originally created by fission in the core, is from the core to the reflector, whereas the net leakage of low-energy neutrons, created in abundance by the superior moderating power of the reflector material, is in the opposite direction. A one-group model cannot describe this process. Depending on the sign of $\mathbf{n}_k \cdot D^k \nabla R^k(\mathbf{r}) \left(= \mathbf{n}_k \cdot D^l \nabla R^l(\mathbf{r})\right)$, either a net number of neutrons having the energy distribution $\psi^k(E)$ leak from k to *l* per second or a net number having the distribution $\psi^l(E)$ leak from *l* to k.

   Two-group theory represents an attempt to improve the accuracy with which the flux can be described near such interfaces. The basic idea is to split the spectrum functions $\psi^k(E)$ into two parts, $\psi_1^k(E)$ (for energy group one, extending from a "cut-point" energy E_c to ∞) and $\psi_2^k(E)$ (for energy group two, extending from 0 to E_c) and to associate separate spatial functions $R_1^k(\mathbf{r})$ and $R_2^k(\mathbf{r})$ with neutrons belonging to each of these two groups. Continuity of flux and current across an interface is then required in an integral sense, individually, for the ranges 0 to E_c and E_c to ∞. Thus the two-group model permits a net leakage rate of group-one neutrons in one direction across an interface and a net leakage rate of group-two neutrons in the opposite direction. In the interior portions of a given region, where it is expected that the separable form $\Phi(\mathbf{r},E) = R^k(\mathbf{r})\psi^k(E)$ will be valid over the entire energy range, the two-group approximation will yield the result that $R_1^k(\mathbf{r})$ and $R_2^k(\mathbf{r})$ have the same spatial shape, so that $R_1^k(\mathbf{r})\psi_1^k(E)$ and $R_2^k(\mathbf{r})\psi_2^k(E)$ for the two energy ranges fit together to form a single function of the form $R^k(\mathbf{r})\psi^k(E)$.

[1] Material in this section follows that of Henry pp. 162-165. Portions that are verbatim are indicated by quotations.

Mathematically the assumptions of two-group theory may be summarized as follows:

The scalar flux function may be written, for **r** in region k, as:

$$\Phi(\mathbf{r},\mathrm{E}) = \begin{cases} \mathrm{R}_1^{\mathrm{k}}(\mathbf{r})\psi_1^{\mathrm{k}}(\mathrm{E}) \text{ for } \mathrm{E}_\mathrm{c} < \mathrm{E} < \infty \\ \mathrm{R}_2^{\mathrm{k}}(\mathbf{r})\psi_2^{\mathrm{k}}(\mathrm{E}) \text{ for } \mathrm{E} < \mathrm{E}_\mathrm{c} \end{cases}$$

Where $\psi_1^{\mathrm{k}}(\mathrm{E})$ and $\psi_2^{\mathrm{k}}(\mathrm{E})$ are the parts of $\psi^{\mathrm{k}}(\mathrm{E})$." (Henry, pp. 162-163)

2. Derivation of Two-Group Equations

a) Spectrum Equations:

The starting point is again the continuous energy diffusion equation and we follow our now standard procedure.

(i) Write out the continuous energy diffusion equation

$$-\nabla\cdot \mathrm{D}(\mathbf{r},E)\ \nabla\Phi(\mathbf{r},E) + \Sigma_t(\mathbf{r},E)\Phi(\mathbf{r},E)$$

$$= \int_0^\infty \left[\sum_j \chi^j(E)\nu^j\Sigma_f^j(\mathbf{r},E') + \Sigma_s(\mathbf{r},E'\rightarrow E) \right]\Phi(\mathbf{r},E')dE'$$

(ii) Modify this equation for a homogeneous medium (i.e., material parameters are not position dependent.)

$$-\mathrm{D}(E)\ \nabla^2\Phi(\mathbf{r},E) + \Sigma_t(E)\Phi(\mathbf{r},E)$$

$$= \int_0^\infty \left[\frac{1}{\lambda}\chi(E)\nu\Sigma_f(E') + \Sigma_s(E'\rightarrow E) \right]\Phi(\mathbf{r},E')dE'$$

(iii) Add superscripts k to denote the various regions of the reactor. We again chose k=2 as to describe two regions: Fuel/moderator and reflector. (Note: The regions are **NOT** the groups. The word "group" in group theory refers to the division of the neutron energies.)

$$-\nabla\cdot \mathrm{D}^{\mathrm{k}}(\mathrm{E})\ \nabla\Phi(\mathbf{r},\mathrm{E})+\Sigma_{\mathrm{t}}^{\mathrm{k}}(\mathrm{E})\Phi(\mathbf{r},\mathrm{E})$$

$$=\int_0^{\infty}\left[\frac{1}{\lambda}\chi^{\mathrm{k}}(\mathrm{E})\nu\Sigma_{\mathrm{f}}^{\mathrm{k}}(\mathrm{E}')+\Sigma_{\mathrm{s}}^{\mathrm{k}}(\mathrm{E}'\rightarrow\mathrm{E})\right]\Phi(\mathbf{r},\mathrm{E}')\mathrm{dE}'$$

$$(\mathrm{k}=1,\ 2,\ 3,\ ..,\ \mathrm{K}) \qquad (4.9.1)$$

(iv) Approximate the leakage term in each region as:

$$-\nabla\cdot \mathrm{D}^{\mathrm{k}}(E)\ \nabla\Phi(\mathbf{r},E)=\mathrm{D}^{\mathrm{k}}(E)\left(B_m^k\right)^2\Phi(\mathbf{r},E)$$

and rewrite the diffusion equation for each region as:

$$\left(\mathrm{D}^{\mathrm{k}}(\mathrm{E})\left(\mathrm{B}_{\mathrm{m}}^{\mathrm{k}}\right)^2+\Sigma_{\mathrm{t}}^{\mathrm{k}}(\mathrm{E})\right)\Phi(\mathbf{r},\mathrm{E})$$

$$=\int_0^{\infty}\left[\frac{1}{\lambda}\chi^{\mathrm{k}}(\mathrm{E})\nu\Sigma_{\mathrm{f}}^{\mathrm{k}}(\mathrm{E}')+\Sigma_{\mathrm{s}}^{\mathrm{k}}(\mathrm{E}'\rightarrow\mathrm{E})\right]\Phi(\mathbf{r},\mathrm{E}')\mathrm{dE}'$$

for **r** in the interior of region k; k = 1, 2.

(v) Substitute the two group definition of the scalar flux

$$\Phi(\mathbf{r},E)=\begin{cases}R_1^k(\mathbf{r})\psi_1^k(E)\ \ for\ E_c<E<\infty\\ R_2^k(\mathbf{r})\psi_2^k(E)\ \ for\ E<E_c\end{cases}$$

into the above relation and obtain two spectral equations for each region. For region 1, these would be

$$\left[\mathrm{D}^1(E)\left(B_m^1\right)^2+\Sigma_t^1(E)\right]R_1^1(r)\psi_1^1(E)$$

$$=\int_{E_c}^{\infty}\left[\chi'(E)\nu\Sigma_f^1(E')+\Sigma_s^1(E'\rightarrow E)\right]R_1^1(r)\psi_1^1(E')dE'$$

and

$$\left[\mathrm{D}^1(E)\left(B_m^1\right)^2+\Sigma_t^1(E)\right]R_2^1(r)\psi_2^1(E)$$

$$=\int_0^{E_c}\left[\chi'(E)\nu\Sigma_f^1(E')+\Sigma_s^1(E'\rightarrow E)\right]R_2^1(r)\psi_2^1(E')dE'$$

(<u>Note</u>: The difficult part of these equations is the notation. Superscripts denote regions; subscripts denote energy groups.)

The equations for region 2 are similar. These four equations are solved for the spectral functions to obtain $\psi_1(E)$ and $\psi_2(E)$.

b. <u>Spatial Equations</u>:

We now have the energy distribution of the scalar neutron flux and can obtain two group cross-sections.

Hence, we can proceed with the spatial analysis. We now quote again from Henry's Nuclear Reactor analysis (p. 163):

- "The scalar flux function may be written, for r in region k, as:

$$\Phi(\mathbf{r},E)=\begin{cases} R_1^k(\mathbf{r})\psi_1^k(E) \ for \ E_c<E<\infty \\ R_2^k(\mathbf{r})\psi_2^k(E) \ for \ E<E_c \end{cases} \qquad (4.11.1)$$

  Where $\psi_1^k(\mathrm{E})$ and $\psi_2^k(\mathrm{E})$ are parts of $\psi^k(\mathrm{E})$ found as described above.

- Boundary conditions, for k and *l* indicating any two adjacent compositions and $\mathbf{r}_c$ being a point on the interface separating them, require that

$$\int_{\mathrm{E_c}}^{\infty} \mathrm{dE}\, \mathrm{R}_1^{\mathrm{k}}(\mathbf{r}_c)\Psi_1^{\mathrm{k}}(\mathrm{E}) = \int_{\mathrm{E_c}}^{\infty} \mathrm{dE}\, \mathrm{R}_1^{l}(\mathbf{r}_c)\Psi_1^{l}(\mathrm{E})$$

$$\int_{0}^{\mathrm{E_c}} \mathrm{dE}\, \mathrm{R}_2^{\mathrm{k}}(\mathbf{r}_c)\Psi_2^{\mathrm{k}}(\mathrm{E}) = \int_{0}^{\mathrm{E_c}} \mathrm{dE}\, \mathrm{R}_2^{l}(\mathbf{r}_c)\Psi_2^{l}(\mathrm{E})$$

$$\int_{\mathrm{E_c}}^{\infty} \mathrm{dE}\ \mathbf{n}_{\mathrm{k}} \cdot \mathrm{D}^{\mathrm{k}}(\mathrm{E})\nabla\left[\mathrm{R}_1^{\mathrm{k}}(\mathbf{r}_c)\Psi_1^{\mathrm{k}}(\mathrm{E})\right] = \int_{\mathrm{E_c}}^{\infty} \mathrm{dE}\ \mathbf{n}_{\mathrm{k}} \cdot \mathrm{D}^{l}\ (\mathrm{E})\nabla\left[\mathrm{R}_1^{l}(\mathbf{r}_c)\Psi_1^{l}(\mathrm{E})\right]$$

$$\int_{0}^{\mathrm{E_c}} \mathrm{dE}\ \mathbf{n}_{\mathrm{k}} \cdot \mathrm{D}^{\mathrm{k}}(\mathrm{E})\nabla\left[\mathrm{R}_2^{\mathrm{k}}(\mathbf{r}_c)\Psi_2^{\mathrm{k}}(\mathrm{E})\right] = \int_{0}^{\mathrm{E_c}} \mathrm{dE}\ \mathbf{n}_{\mathrm{k}} \cdot \mathrm{D}^{l}(\mathrm{E})\nabla\left[\mathrm{R}_2^{l}(\mathbf{r}_c)\Psi_2^{l}(\mathrm{E})\right]$$

As with the one group case, it simplifies the algebra if, after having found the shape in energy of $\psi_1^k(\mathrm{E})$ and $\psi_2^k(\mathrm{E})$ for the material of region k, we renormalize these two segments $\psi^k(\mathrm{E})$ so that

$$\int_{E_c}^{\infty} \Psi_1^k(E)\,dE = 1 \quad (k=1,2,....,K),$$

$$\int_{0}^{E_c} \Psi_2^k\ (E)\,dE = 1 \quad (k=1,2,....,K).$$

Then the continuity equations show that $R_1^k(\mathbf{r}_c) = R_1^l(\mathbf{r}_c)$ and $R_2^k(\mathbf{r}_c) = R_2^l(\mathbf{r}_c)$ for points $\mathbf{r}_c$ on the interface between regions k and *l*. Thus, as in the one-group case, the superscript k on the functions R are superfluous, and we shall change notation by replacing the various $R_1^k(\mathbf{r})$ by the single, everywhere-continuous function $\Phi_1(\mathbf{r})$ and similarly replacing $R_2^k(\mathbf{r})$ by $\Phi_2(\mathbf{r})$. The functions $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ are called the two-group fluxes. Note that they are not fluxes per unit energy (in the way that $\Phi(\mathbf{r},E) = v(E)n(\mathbf{r},E)$ is). Physically $\Phi_1(\mathbf{r})$ is the two-group approximation to the number of neutrons per cc having energies in the range E_c to ∞ $\left(\int_{E_c}^{\infty} n(\mathbf{r},E)dE\right)$ multiplied by the average "fast" speed.

$$v_1 = \frac{\int_{E_c}^{\infty} v(E)n(\mathbf{r},E)dE}{\int_{E_c}^{\infty} n(\mathbf{r},E)dE}$$

And $\Phi_2(\mathbf{r})$ is the two-group approximation to the number in the range 0 to E_c multiplied by the average thermal speed.

$$v_2 = \frac{\int_0^{E_c} v(E)n(\mathbf{r},E)dE}{\int_0^{E_c} n(\mathbf{r},E)dE}$$

(These interpretations are somewhat ambiguous since (4.11.1), on which they are based, is an approximation which cannot be rigorously correct.)

To find differential equations for the two-group fluxes $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ we substitute the approximation (4.11.1) into (4.9.1) which is the homogenized diffusion equation for two regions. This will give us different results for $E > E_c$ and $E < E_c$.

$$\begin{aligned} &-D^k(E)\psi_1^k(E)\nabla^2\Phi_1(\mathbf{r}) + \Sigma_t^k(E)\psi_1^k(E)\Phi_1(\mathbf{r}) \\ &= \int_{Ec}^{\infty}\left[\frac{1}{\lambda}\chi^k(E)v\Sigma_f^k(E')\psi_1^k(E') + \Sigma_s^k(E'\to E)\psi_1^k(E')\right]dE'\Phi_1(\mathbf{r}) \\ &\quad + \int_0^{E_c}\left[\frac{1}{\lambda}\chi^k(E)v\Sigma_f^k(E')\psi_2^k(E')\right]dE'\Phi_2(\mathbf{r}) \qquad \text{for } E > E_c \end{aligned} \tag{4.11.4}$$

and

$$- D^k(E)\psi_2^k(E)\nabla^2\Phi_2(\mathbf{r}) + \Sigma_t^k(E)\psi_2^k(E)\Phi_2(\mathbf{r})$$

$$= \int_{E_c}^{\infty}\left[\frac{1}{\lambda}\chi^k(E)\nu\Sigma_f^k(E')\psi_1^k(E') + \Sigma_s^k(E'\to E)\psi_1^k(E')\right]dE'\Phi_1(\mathbf{r})$$

$$+ \int_0^{E_c}\left[\frac{1}{\lambda}\chi^k(E)\nu\Sigma_f^k(E')\psi_2^k(E') + \Sigma_s^k(E'\to E)\psi_2^k(E')\right]dE'\Phi_2(\mathbf{r}) \qquad for\ E \le E_c \tag{4.11.5}$$

Where on physical grounds we have omitted from (4.11.4) the scattering from $E' < E_c$ to $E > E_c$ (For thermal reactors the cut point E_c will always be such that $\chi^k(E)$ is zero for $E < E_c$; hence no fission terms will appear in (4.11.5). We retain them for possible application to fast reactors, for which E_c will be much higher.

There is no solution $\Phi_1(\mathbf{r})$, $\Phi_2(\mathbf{r})$ that will satisfy (4.11.4) and (4.11.5) at all energies since the form (4.11.1) is not sufficiently general. However, we can force equality of the right- and left-hand sides in an integral sense and in that way find equations which, when solved, will give us $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$. Accordingly, we shall integrate (4.11.4) from E_c to ∞ and (4.11.5) from 0 to E_c and require that the resultant equations be valid at all **r**. To simplify the result, we first define a set of "two-group constants":

$$D_1^k \equiv \int_{E_c}^{\infty} D(E)\Psi_1^k(E)\,dE, \qquad D_2^k \equiv \int_0^{E_c} D(E)\Psi_2^k(E)\,dE,$$

$$\Sigma_{t1}^k \equiv \int_{E_c}^{\infty} \Sigma_t^k(E)\Psi_1^k(E)\,dE, \qquad \Sigma_{t2}^k \equiv \int_0^{E_c} \Sigma_t^k(E)\Psi_2^k(E)\,dE$$

$$\chi_1^k \equiv \int_{E_c}^{\infty} \chi^k(E)\,dE, \qquad \chi_2^k \equiv \int_0^{E_c} \chi^k(E)\,dE$$

$$\nu\Sigma_{f1}^k \equiv \int_{E_c}^{\infty} \nu\Sigma_f^k(E)\Psi_1^k(E)\,dE, \qquad \nu\Sigma_{f2}^k \equiv \int_0^{E_c} \nu\Sigma_f^k(E)\Psi_2^k(E)\,dE,$$

$$\Sigma_{11}^k \equiv \int_{Ec}^{\infty} dE, \int_{E_c}^{\infty} dE'\Sigma_S^k(E'\to E)\psi_1^k(E'), \qquad \Sigma_{22}^k \equiv \int_0^{E_c} dE, \int_0^{E_c} dE'\Sigma_S^k(E'\to E)\psi_2^k(E')$$

$$\Sigma_{21}^k \equiv \int_0^{E_c} dE, \int_{E_c}^{\infty} dE'\Sigma_S^k(E'\to E)\psi_1^k(E') \qquad \Sigma_1^k \equiv \Sigma_{t1}^k - \Sigma_{11}^k, \quad \Sigma_2^k \equiv \Sigma_{t2}^k - \Sigma_{22}^k$$

Using these definitions, integrating (4.11.4) from E_c to ∞. and integrating (4.11.5) from 0 to E_c, we get

$$-D_1^k\nabla^2\Phi_1(r)+\Sigma_1^k\Phi_1(r)=\frac{1}{\lambda}\chi_1^k\left[\nu\Sigma_{f1}^k\Phi_1(r)+\nu\Sigma_{f2}^k\Phi_2(r)\right]$$

$$-D_2^k\nabla^2\Phi_2(r)+\Sigma_2^k\Phi_2(r)=\frac{1}{\lambda}\chi_2^k\left[\nu\Sigma_{f1}^k\Phi_1(r)+\nu\Sigma_{f2}^k\Phi_2(r)\right]+\Sigma_{21}^k\Phi_1(r)$$

(4.11.7)

The boundary conditions become:

1. $\Phi_1(\mathrm{r})$, $\Phi_2(\mathrm{r})$ must be continuous everywhere,

2. $\mathrm{n}\cdot D_1\nabla\Phi_1(\mathrm{r})$ and $\mathrm{n}\cdot D_2\nabla\Phi_2(\mathrm{r})$ must be continuous across interfaces separating different material compositions,

3. $\Phi_1(\mathrm{r})=\Phi_2(\mathrm{r})$ on the outer boundary of the reactor.

The above equations (4.11.7) are the "two-group diffusion equations." They are the standard workhorses of thermal-reactor design. It can be proved that a unique, positive solution corresponding to a most-positive real eigenvalue λ always exists for the two-group equations. Thus, a physically acceptable solution for the group fluxes Φ_1 and Φ_2 can always be found, and from this the two-group approximation $\left(\Phi_1(\mathrm{r})\psi_1^k(\mathrm{E})\right), \left(\Phi_2(\mathrm{r})\psi_2^k(\mathrm{E})\right)$ for the scalar flux $\Phi(\mathbf{r},\mathrm{E})$ can be constructed throughout each material composition k." (Henry, pp. 163-165)