Basic Physics of Underwater Acoustics

Definitions

$p$: pressure, measured *relative to hydrostatic*, Pa

$\rho$: density, measured *relative to hydrostatic*, kg/m³

$E$: bulk modulus of the fluid, Pa,

$$\delta p = E \left[ \frac{\delta \rho}{\rho} \right]$$

$[u,v,w]$: deflections in [x,y,z]-directions, relative to the hydrostatic condition, m

Then in one dimension (*pipe*)

$$p = E \left[ -\frac{\delta u}{\delta x} \right]$$
One-dimensional Case cont.

Newton’s Law:
\[ \delta p = - \rho \ u_{tt} \ \delta x \quad \text{OR} \]
\[ p_x = - \rho \ u_{tt} \]

Constitutive Law:
\[ p = - E \ \delta u / \delta x \quad \text{OR} \]
\[ p = - E \ u_x \]

\[ p_{xx} = \left[ \rho / E \right] p_{tt} \]

a wave equation!
Let $p(x,t) = P_0 \sin(\omega t - kx)$

Insert this in the wave equation:

$$-P_0 k^2 \sin(\ ) = - \left[ \frac{\rho}{E} \right] P_0 \omega^2 \sin(\ ) \Rightarrow$$

$$\left[ \frac{\omega}{k} \right]^2 = \frac{E}{\rho} \Rightarrow$$

Wave speed $c = \frac{\omega}{k} = \left[ \frac{E}{\rho} \right]^{1/2}$

This is sound speed in water, independent of pressure, or frequency.

$\rho \sim 1000 \text{ kg/m}^3$, $E \sim 2.3\text{e}9 \text{ N/m}^2 \Rightarrow c \sim 1500 \text{ m/s}$

Wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = c/f$; $1\text{kHz} : 1.5\text{m}$
In Three Dimensions: A CUBE

Newton’s Law:

\[ p_x = -\rho u_{tt} \rightarrow p_{xx} = -\rho u_{ttx} \]
\[ p_y = -\rho v_{tt} \rightarrow p_{yy} = -\rho v_{tty} \]
\[ p_z = -\rho w_{tt} \rightarrow p_{zz} = -\rho w_{ttz} \]

Constitutive Law:

\[ -E u_x = p / 3 \rightarrow -E u_{ttx} = p_{tt} / 3 \]
\[ -E v_y = p / 3 \rightarrow -E v_{tty} = p_{tt} / 3 \]
\[ -E w_z = p / 3 \rightarrow -E w_{ttz} = p_{tt} / 3 \]

All directions deform uniformly

Lead to Helmholtz Equation:

\[ p_{xx} + p_{yy} + p_{zz} = p_{tt} / c^2 \]

or

\[ \Delta p = p_{tt} / c^2 \]

where \( \Delta \) is the LaPlacian operator.
Particle Velocity

Consider one dimension again:
\[ p_x = -\rho \ u_{tt} \rightarrow p_x = -\rho \ (u_t)_t \]

If \( p(x,t) = P_0 \sin(\omega t - kx) \) and \( u_t(x,t) = U_{to} \sin(\omega t - kx) \) \( \rightarrow \)

\[ -kP_0 \cos(\ ) = -\rho \ \omega \ U_{to} \cos(\ ) \rightarrow U_{to} = P_0 / \rho \ c \]

*Note velocity is in phase with pressure!*

[\( \rho \ c \): *characteristic impedance*;

- water: \( \rho c \sim 1.5e6 \) Rayleighs “hard”
- air: \( \rho c \sim 500 \) Rayleighs “soft”

In three dimensions:
\[ \nabla p = -\rho \ V_t \]

where
\[ \nabla p = p_x \ i + p_y \ j + p_z \ k \quad \text{and} \]
\[ \underline{V} = u_t \ i + v_t \ j + w_t \ k \]
Note equivalence of the following:
\[ \lambda = \frac{c}{f} \quad \text{and} \quad \frac{\omega}{k} = c \]
There is no dispersion relation here; this is the only relationship between \( \omega \) and \( k \)!

Consider Average Power through a 1D surface:
\[
P(x) = \left[ \frac{1}{T} \right] \int_{0}^{T} p(\tau, x) u_i(\tau, x) \, d\tau
= \left[ \frac{1}{T} \right] \int_{0}^{T} P_0 U_0 \sin^2(\omega \tau - kx) \, d\tau
= P_0 U_0 \frac{2}{\rho c}
= \frac{P_0^2}{2} \rho c = U_0^2 \frac{\rho c}{2}
\]

*Acoustic Intensity in W/m^2*

If impedance \( \rho c \) is high, then it takes little power to create a given pressure level; but it takes a lot of power to create a given velocity level.
Spreading in Three-Space

At time $t_1$, perturbation is at radius $r_1$; at time $t_2$, radius $r_2$ →

\[ P(r_1) = \frac{P_o^2(r_1)}{2 \rho c} \]
\[ P(r_2) = \frac{P_o^2(r_2)}{2 \rho c} \]

Assuming no losses in water; then

\[ P(r_2) = P(r_1) \frac{r_1^2}{r_2^2} = \frac{P_o^2(r_1)}{2 \rho c r_2^2} \]

and

\[ P_o(r_2) = P_o(r_1) \frac{r_1}{r_2} \]

Let $r_1 = 1$ meter (standard!) →

\[ P(r) = \frac{P_o^2(1m)}{2 \rho c r^2} \]
\[ P_o(r) = \frac{P_o(1m)}{r} \]
\[ U_{to}(r) = \frac{P_o(1m)}{\rho c r} \]

Pressure level and particle velocity decrease linearly with range.
Decibels (dB)

10 * \( \log_{10} \) (ratio of two positive scalars):

Example:
\( x_1 = 31.6 \); \( x_2 = 1 \) \( \rightarrow \) 1.5 orders of magnitude difference
\[ 10 \log_{10}(x_1/x_2) = 15 \text{dB} \]
\[ 10 \log_{10}(x_2/x_1) = -15 \text{dB} \]

RECALL
\[ \log(x_1^2/x_2^2) = \log(x_1/x_2) + \log(x_1/x_2) = 2 \log(x_1/x_2) \]

In acoustics, acoustic intensity (power) is referenced to 1 W/m\(^2\);
pressure is referenced to 1 \( \mu \text{Pa} \)

\[ 10 \log_{10}[ P(r) / 1 \text{ W/m}^2 ] = 10 \log_{10}[ [ P_0^2(r) / 2 \rho c] / 1 \text{ W/m}^2 ] \]
\[ = 20 \log_{10}[ P_0(r) ] - 10 \log_{10}(2\rho c) \]
\[ = 20 \log_{10}[ P_0(r) / 1\mu \text{Pa} ] - 120 - 65 \]
Spreading Losses with Range

Pressure level in dB is
\[ 20 \log_{10} \left( \frac{P_0(r)}{1 \mu Pa} \right) - 185 \]
\[ = 20 \log_{10} \left( \frac{P_0(1m)}{r} / 1 \mu Pa \right) - 185 \]
\[ = 20 \log_{10} \left( \frac{P_0(1m)}{1 \mu Pa} \right) - 20 \log_{10} [r] - 185 \]

Example: At 100m range, we have lost
40dB or four orders of magnitude in sound intensity
40dB or two orders of magnitude in pressure
(and particle velocity)
Attenuation Losses with Range

Acoustic power does have losses with transmission distance – primarily related to relaxation of boric acid and magnesium sulfate molecules in seawater. Also bubbles, etc.

At 100 Hz, ~1dB/1000km: OK for thousands of km, ocean-scale seismics and communications

At 10kHz, ~1dB/km: OK for ~1-10km, long-baseline acoustics

At 1MHz, 3dB/10m: OK for ~10-100m, imaging sonars, Doppler velocity loggers

$$TL = 20\log_{10} r + \alpha r$$

(pressure transmission loss)

Linear approximation!
Francois & Garrison (1982) model
The Piezo-Electric Actuator

strain = constant X electric field

\[ \varepsilon = d \times E \]

or

\[ \Delta t / t = d \times (V / t) \]

where \( d = 40-750 \times 10^{-12} \text{ m} / \text{V} \)  

Drive at 100V, we get only 4-75 nm thickness change!

Series connection amplifies displacement

**still capable of MHz performance**
The Piezo-Electric Sensor

electric field = constant X stress

\[ E = g \times \sigma \quad \text{or} \quad V = t \cdot g \cdot \sigma \]

where \( g = 15-30 \times 10^{-3} \text{ V/mN} \)

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**Ideal Actuator:** Assume the water does not impede the driven motion of the material

**Ideal Sensor:** Assume the sensor does not deform in response to the water pressure waves
Typical Transducer:

120 to 150 dB re 1\(\mu\)Pa, 1m, 1V

means

\(10^6 - 10^{7.5}\) \(\mu\)Pa at 1m for each Volt applied or

1-30 Pa at 1m for each Volt applied

Typical Hydrophone:

-220 to -190 dB re 1\(\mu\)Pa, 1V

means

\(10^{-11} - 10^{-9.5}\) V for each \(\mu\)Pa incident or

\(10^{-5} - 10^{-3.5}\) V for each Pa incident

So considering a transducer with 16Pa at 1m per Volt, and a hydrophone with \(10^{-4}\) V per Pa:

If \(V = 200\) V, we generate 3200Pa at 1m, or 3.2Pa at 1km, assuming spreading losses only;

The hydrophone signal at this pressure level will be 0.00032V or 320\(\mu\)V!