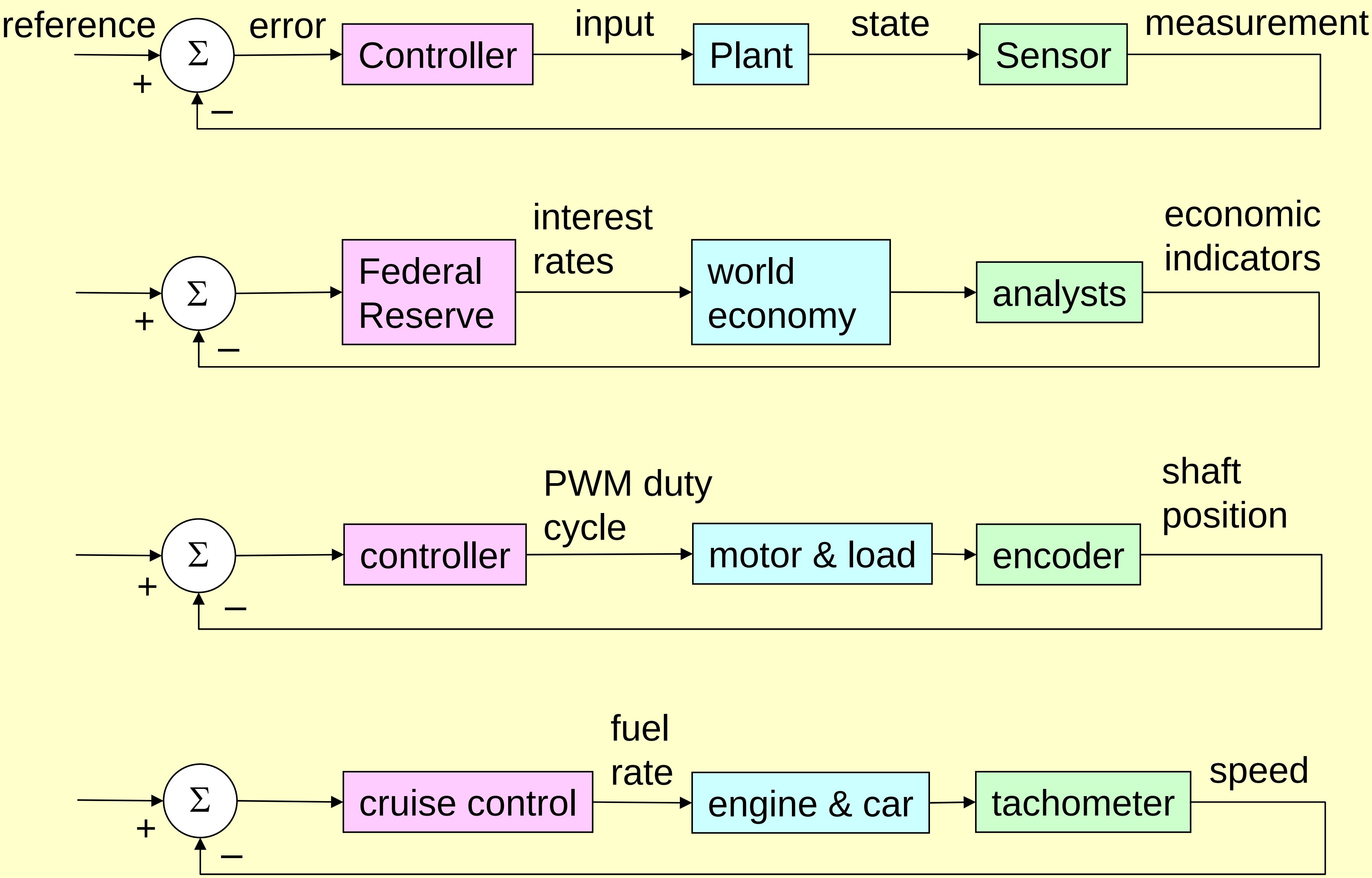
Feedback Control: Selected Topics

Feedback fundamentally creates a new dynamics!



LaPlace Transform and Stability

- For linear systems, stability of a system refers to whether the impulse response has *exponentially growing components*.
- *No pre-determined input can stabilize an unstable system; no pre-determined input can destabilize a stable system.*
- Some examples you can work out:

$$\mathcal{L}(e^{-\alpha t}) = 1 / (s + \alpha)$$

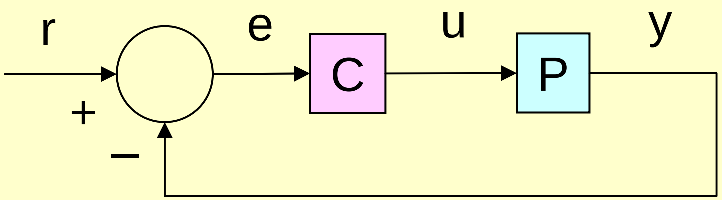
$$\mathcal{L}(t\, e^{-\alpha t}) = 1 / (s + \alpha)^2$$

$$\mathcal{L}[\, e^{-\alpha t} \sin(\omega t) \,] = \omega / (s^2 + 2\alpha s + \alpha^2 + \omega^2)$$

$$\mathcal{L}[\, \omega_d\, e^{-\zeta \omega n t} \sin(\omega_d t) / (1-\zeta^2) \,] = \omega_n^2 / (s^2 + 2\zeta\omega_n s + \omega_n^2)$$

Major observation: stable signal ←→ roots of $\mathcal{L}$ denominator have negative real parts: TRUE FOR ALL FIRST- AND SECOND-ORDER SYSTEMS

Basics in the Frequency Domain



e = r – y

u = Ce = C(r-y)

y = Pu = PCe = PC(r-y) → (PC + 1)y = PCr ➔ **y / r = PC / (PC + 1)**

Similarly, e = r – y = r – PCe → (PC+1)e = r ➔ **e / r = 1 / (PC + 1)**

u = C(r-Pu) → (PC + 1)u = Cr ➔ **u / r = C / (PC + 1)**

Why can we do this? Convolution in time domain = Multiplication in freq. domain!

P <u>must</u> roll off at high frequencies – because no physical plant can respond to input at arbitrarily high frequency. Same with controller C.

- Ideal case: e is a small fraction of r: e/r << 1, equivalent to y/r ~ 1
- This implies mag (PC + 1) >> 1 or mag (PC) >> 1.
- If plant P is given, then C has to be *designed* to make PC big.
- But mag (u / r) ~ mag(1 / P): HUGE when P gets small at high frequencies → excessive control action which will saturate or break actuators, excite unmodelled plant behavior, etc.. ← issues of *robustness*

Heuristic Tuning of PID loops

- Assuming a reasonably simple and stable plant, rule of thumb is:
  - Turn on the proportional gain and the derivative gain together until the system transient response is acceptable
  - Turn on the integral gain slowly so as to eliminate the steady-state error
- Why does it work?
  - Proportional gain is like a spring, the derivative gain is like damping. They are like *physical dissipative devices* and unlikely to destabilize your system (until you take the spring and damping too high)
  - Integral gain IS destabilizing → proceed cautiously!

1. Zeigler-Nichols Methods for Tuning of PID Controllers

- Ultimate cycle method
  - Increase proportional gain only until the system has sustained oscillations at a period T_u; this gain is K_u. (If no oscillations occur, don't use this method!)
  - For proportional-only control, use
    - $K_p = K_u / 2$
  - For proportional-integral control use
    - $K_p = 0.45 K_u$ and $K_i = 0.54K_u / T_u$
  - For full PID, use
    - $K_p = 0.6K_u$, $K_i = 1.2K_u / T_u$ and $K_d = 4.8K_u / T_u$

Assume the plant is of the form $P = k / (s^2 + 2\zeta\omega_n s + \omega_n^2)$

(no zeros, undamped natural frequency ω_n, damping ratio ζ)

With proportional-only control at K_u, the CL characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 + kK_u = 0$$

Because system has oscillations at frequency $2\pi/T_u$, we know that

$$\omega_n^2 + kK_u \sim [\, 2\pi/T_u \,]^2 \qquad \text{OR} \qquad kK_u = [\, 2\pi/T_u \,]^2 - \omega_n^2 = \mathbf{Q}$$

At this condition, the damping is not enough to counter the unmodelled dynamics that are causing the oscillation, so it is *ignored*.

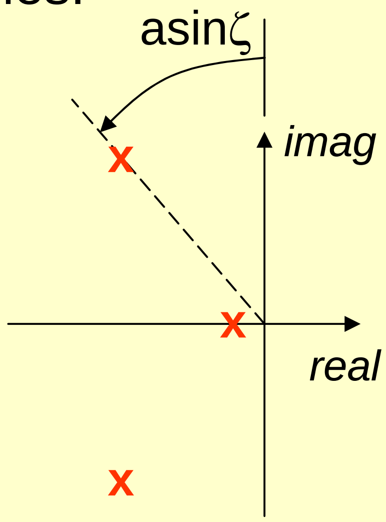
The characteristic equation with the Z-N PID gains becomes:

$$s^2 + 0 + \omega_n^2 + k * [\text{ PID controller }] = 0$$

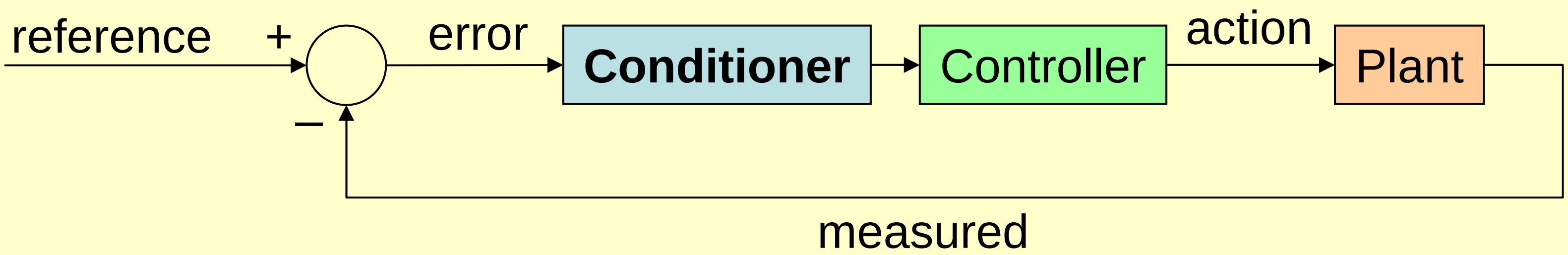
$$s^2 + 0 + \omega_n^2 + \mathbf{Q}\,[\, 0.6 + 1.2 / T_u / s + 4.8\, s / T_u \,] = 0$$

$$s^3 + [\, 4.8\, \mathbf{Q} / T_u \,]\, s^2 + [\, 4\pi^2 / T_u^2 - \mathbf{Q} + 0.6\, \mathbf{Q} \,]\, s + 1.2\, \mathbf{Q} / T_u = 0$$

For a wide range of **Q** and T_u, this will give ~20% overshoot (ζ~0.7) because the poles look like this:



2. The 2π Discontinuity in Heading Control



Objective of Conditioner is to make sure:

- Controller never gets an error signal that is discontinuous because of this effect
- Controller will always go for the shortest path – i.e., will turn 90 degrees left instead of 270 degrees right!

Simple logic:

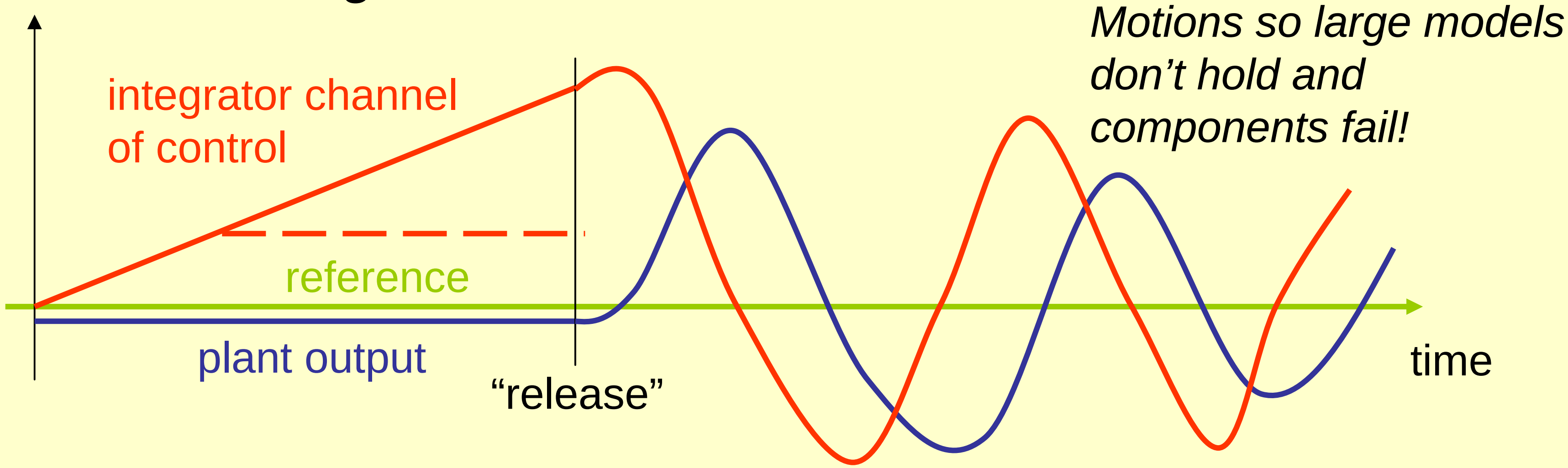
Subtract or add 2π to error to bring it into the range $[-\pi, \pi]$.

3. Integrator Windup

- A purely linear effect that has broken many systems and caused damage and injury!
- Basic issue: The integrator in the controller builds up a large control signal over time if the system is prevented from responding.

  PID: K_p*error + K_d*d(error)/dt + $K_i \int$ error dt

Solution: constrain this part of the control to be within a certain neighborhood of zero.



4. Sensor Noise & Outliers

- Most common model for sensor noise is Broadband, Gaussian:
  - Broadband means no particular frequency is favored – spectrum is flat; white noise.
  - Gaussian means samples fit the probability distribution function:

$$N(0,1) = 1 / \text{sqrt}(2\pi) * \exp [- X^2 / 2]$$

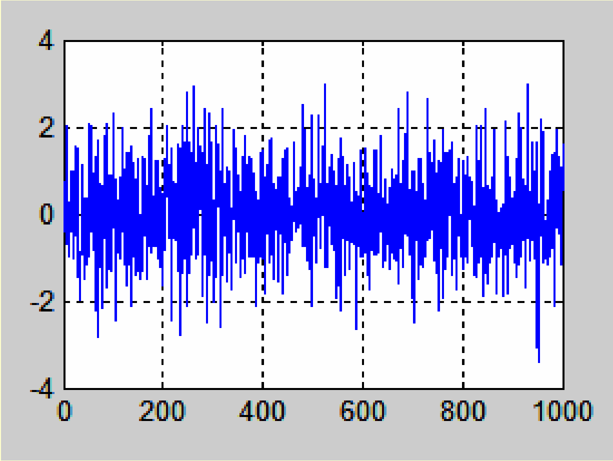
Such processes are defined completely by variance μ and mean value x_o:

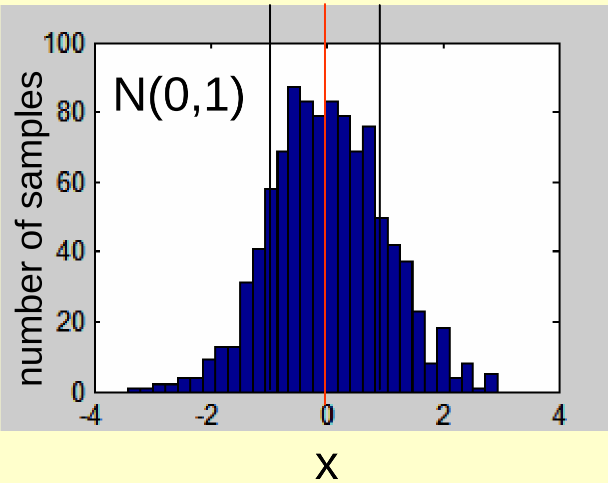
$$N(x_o,\mu) = x_o + \text{sqrt}(\mu)\ N(0,1)$$

Computing the variance from n samples:

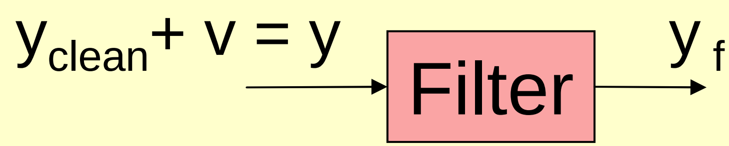
$$\mu = [(x_1-x_o)^2 + (x_2-x_o)^2 + \ldots + (x_n-x_o)^2] / (n-1)$$

1000 samples of a zero-mean, unit variance normal variable





Linear Filtering



Use good judgment!

filtering brings out trends, reduces noise BUT

filtering obscures dynamic response

<u>Causal filtering</u>: $y_f(t)$ depends only on *past measurements* – appropriate for real-time implementation

Example: $y_f(t) = (1-\varepsilon)\, y_f(t-\Delta t) + \varepsilon\, y(t)$ *("first-order lag")*

<u>Acausal filtering</u>: $y_f(t)$ depends on *all measurements*

– appropriate for post-processing

Example: $y_f(t) = [\, y(t+\Delta t) + y(t) + y(t-\Delta t)\,] / 3$ *("moving window")*

Convolution implies that the filter transfer function *F(s)* times the LaPlace transform of the input signal will give the LaPlace transform of the filter output:

$$Y_f(s) = F(s)\,[\, Y_{clean}(s) + V(s)\,]$$

Since a white noise process has uniform spectrum, the quantity $|F(j\omega)|$ determines what frequencies will get through → idea is to eliminate enough of the noise frequency band that the system dynamics can be seen.

A first-order filter transfer function in the freq. domain:

$$x_f(j\omega) / x(j\omega) = \lambda / (j\omega + \lambda)$$

At low ω, this is approximately one (λ/λ)

At high ω, this goes to zero magnitude, with 90 degrees phase lag ($\lambda/j\omega = -j\lambda/\omega$)

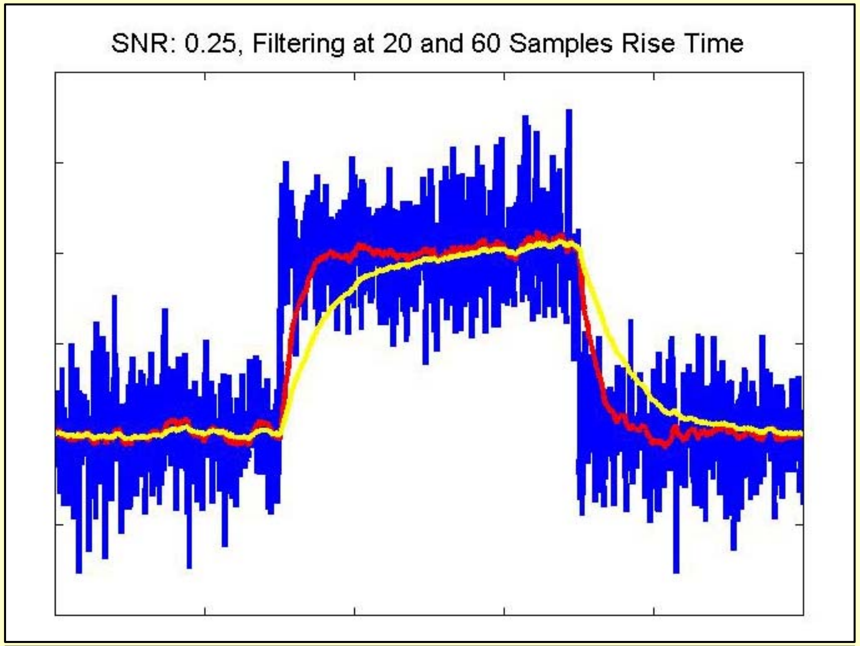
Time domain equivalent:

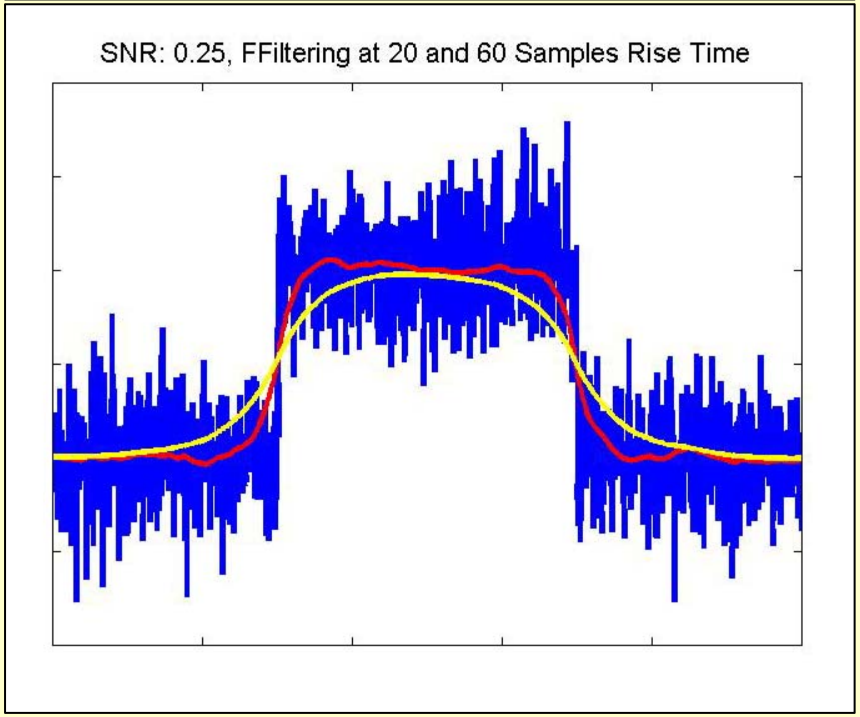
$$dx_f / dt = \lambda (x - x_f)$$

In discrete time, try

$$x_f(k) = (1-\lambda\Delta t)\, x_f(k-1) + \lambda\Delta t\ \ x(k)$$

(from Euler discretization!)





- BUT linear filters will not handle <u>outliers</u> very well because outliers don't fit a typical Gaussian model.
- First defense against outliers: find out their origin and eliminate them at the beginning; quality metrics.
- <u>Detection</u>: Exceeding a known, fixed bound, or an impossible deviation from previous values. *Example: vehicle heading change per sample cannot be more than is physically possible from waves and control.*
- Second defense: set data to NaN (or equivalent), so it won't be used in calculations.
- Third defense: try to fill in.

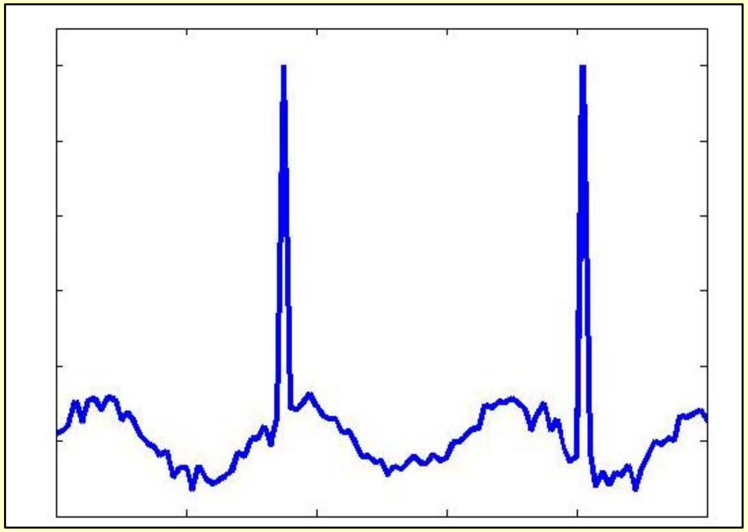
  Example:

  if abs(x(k) – x(k-1)) > MX,

  x(k) = x(k-1) ;

  end;

  → Limited usefulness!

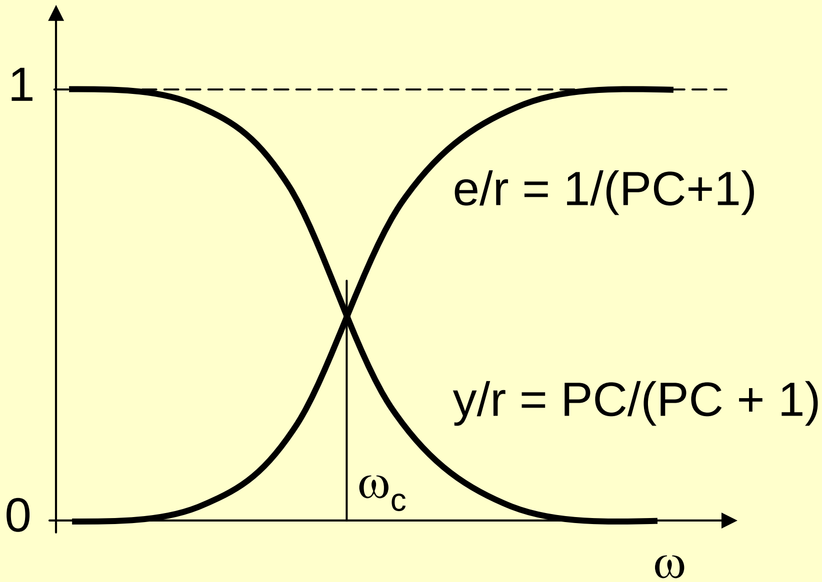


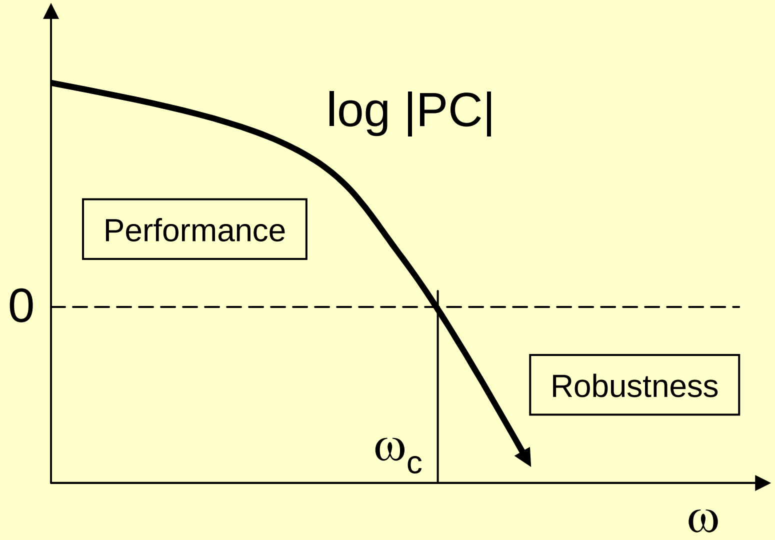
Feedback Control:
Supplementary Slides from 2.017

Components of Engineered Feedback Systems

- **Plant**: the system whose behavior is to be controlled.
  Examples: vehicle attitude, temperature, chemical process, business accounting, team and personal relationships, global climate
- **Actuator**: systems which alter the behavior of the plant.
  Examples: motor, heater, valve, law enforcement (!), pump, FET, hydraulic ram, generator, US Mint
- **Sensor**: system which measures certain states of the plant.
  Examples: thermometer, voltmeter, Geiger counter, opinion poll, balance sheet, financial analyst
- **Controller**: translates sensor output into actuator input.
  Examples: computer, analog device, human interface, committee

- Extreme variability in time scales:
  - active noise cancellation requires ~100 *kiloHertz* sensing and actuation
  - Social Security is assessed and corrected at ~3 *nanoHertz* (10 years)





Good tracking only possible at low frequencies → leads to a "formula" for design:

Make |PC| *large at low frequencies*, e/r ~ 0, y/r ~ 1;
Good regulation and tracking at low frequencies

Make |PC| *small at high frequencies*, e/r ~ 1, y/r ~ 0, u/r ~ C
Poor tracking at high frequencies, but reasonable control action

The frequency where |PC| = 1 is the crossover frequency ω_c ;
Above this point, closed loop t.f. y/r = PC/(PC+1) drops off to zero.
So ω_c is about the *bandwidth* of the closed-loop t.f.

LaPlace vs. Fourier XFM

Fourier Transform integrates **$x(t)\ e^{-j\omega t}$** over the time range from
<u>negative infinity</u> to positive infinity

Laplace Transform integrates **$x(t)\ e^{-st}$** over the time range from
<u>zero</u> to positive infinity

Result: $X(j\omega)$ can describe *acausal* systems, $X(s)$ describes only *causal* ones!

Many important results of Fourier Transform carry over to LaPlace Transform:

$\mathcal{L}(x(t)) = X(s)$ (notation)

$\mathcal{L}(ax(t)) = a\,X(s)$ (linearity)

$\mathcal{L}(x(t) * y(t)) = X(s)Y(s)$ (convolution)

$\mathcal{L}(x_t(t)) \leftarrow\rightarrow sX(s)$ (first time derivative)

$\mathcal{L}(x_{tt}(t)) \leftarrow\rightarrow s^2X(s)$ (second and higher time derivatives)

$\mathcal{L}(\int x(t)dt) \leftarrow\rightarrow X(s) / s$ (time integral)

$\mathcal{L}(\delta(t)) = 1$ (unit impulse)

$\mathcal{L}(1(t)) = 1/s$ (unit step)

Decoding the transfer function

Numerator polynomials are a snap:

$(s + 2)/(s^2+s+5) = s/(s^2 + s + 5) + 2/(s^2+s+5)$

"input derivative plus two times the input, divided by the denominator"

For higher-order polynomials in the denominator: use partial fractions, e.g.,

$(s+1)/(s+2)(s+3)(s+4) = -0.5/(s+2) + 2/(s+3) -1.5/(s+4)$ *(all real poles)*

$(s+1)/s(s^2+s+1) = -s/(s^2+s+1) + 1/s$ *(some complex poles)*

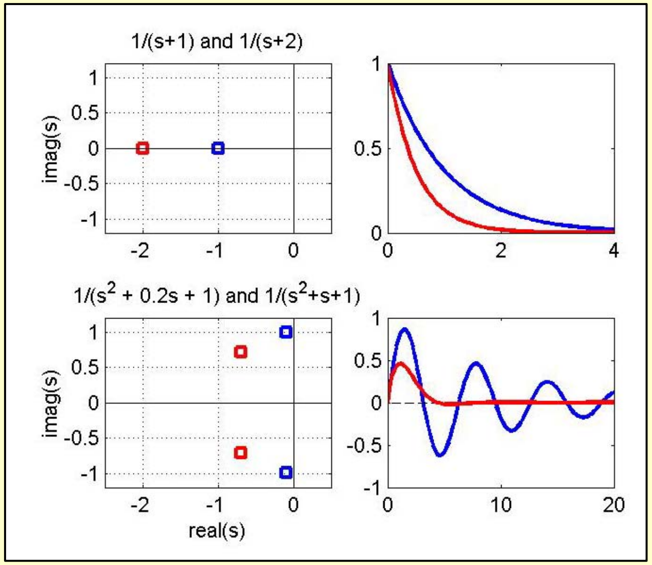
Any high-order transfer function can always be broken down into a sum of transfer functions with factored first- and second-order polynomials in the denominator.

stability ←→ the roots of the characteristic equation have negative real part.

More details:

real negative root $-\alpha$: the mode decays with time constant $1/\alpha$

complex roots at $-\omega_n\zeta$ +/- $j\omega_d$: the mode decays with frequency ω_d and exponential envelope having time constant $1/\zeta\omega_n$



Example with a double integrator: e.g., a motor or dynamic positioning

System is $mx_{tt}(t) = u(t)$ where:

m is mass

$x_{tt}(t)$ is double time derivative of position

$u(t)$ is control action; thrust

Let a *Control law* be: **$u = -k_p x$ (Proportional Control: P)**

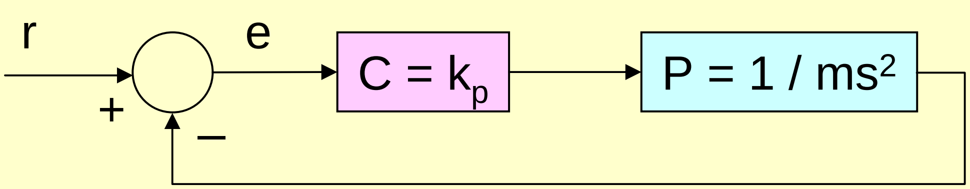
Closed-loop system dynamics become $mx_{tt} + k_p x = 0$

Response to an initial condition is undamped oscillations at frequency $\omega_n = \text{sqrt}(k_p/m)$

$P = 1/ms^2$

$C = k_p$

$PC = k_p/ms^2 \rightarrow$



$e/r = 1/(PC + 1)$

$= ms^2 / (ms^2 + k_p)$

Tracking error is small when s is small; large when s is large, as desired.

BUT characteristic equation $ms^2 + k_p = 0$ has two imaginary poles – undamped!

Try the control law $u = -k_p x - k_d x_t$ **(Proportional + Derivative: PD)**
Closed-loop system dynamics become $m x_{tt} + k_d x_t + k_p x = 0$
Recall for a second-order underdamped oscillator:

$0 < k_d < 2\, \mathrm{sqrt}(k_p/m)$

$\omega_n = \mathrm{sqrt}(k_p/m)$ (undamped natural frequency)

$\zeta = k_d / 2\, \mathrm{sqrt}(k_p m)$ (damping ratio)

$\omega_d = \omega_n\, \mathrm{sqrt}(1-\zeta^2)$ (damped natural frequency)

Response to an initial condition is either:

- *Damped oscillations at frequency* $\omega_d = \mathrm{sqrt}(1-\zeta^2)\omega_n$, *inside an exponential envelope with time constant* $1/\zeta\omega_n$ OR
- *Sum of two decaying exponentials* (overdamped case)

Consider a constant disturbance: $m x_{tt} + k_d x_t + k_p x = F$;
System will settle at $x = F/k_p$; this is a steady-state error!
But k_p cannot be increased arbitrarily – natural frequency will be too high and too much control action

Try the control law $u = -k_p x - k_d x_t - k_i \int x\, dt$ **(Proportional + Derivative + Integral: PID)**
Closed-loop system dynamics become $m x_{tt} + k_d x_t + k_p x + ki \int x\, dt = F$
If the system is stable ($ms^3 + k_d s^2 + k_p s + k_i = 0$ has roots with negative real part), then differentiate:
$m x_{ttt} + k_d x_{tt} + k_p x_t + k_i x = 0$ ➔ settles to $x = 0$!

The PID

$C = k_p + k_d s + k_i/s$
$= (k_p s + k_d s^2 + k_i) / s$

High-frequency response is $\sim k_d s$; increases with frequency and disobeys our prior rule about infinite power. High frequency errors will lead to very large control action!

Sensor noise solutions:

- use a very clean and high-res. sensor for x, which can be easily differentiated numerically, *e.g., motor encoder*
- use a sensor that measures xt directly, *e.g., tachometer*
- filter the measurement. For a low-pass, we would get

$$C_f = [(k_p s + k_d s^2 + k_i) / s] \times [\lambda / (s+\lambda)]$$
$$= \lambda (k_p s + k_d s^2 + k_i) / s (s+\lambda)$$

But combine with a double integrator plant $P = m/s^2$

$PC = m(k_p s + k_d s^2 + k_i) / s^3$, which *does* go to zero at high frequencies, as desired → the system does have a real bandwidth, which can be tuned.