Simultaneous Ad Auctions

Simultaneous Ad Auctions

Itai Ashlagi[a], Dov Monderer[b], Moshe Tennenholtz[b,c]

[a]Harvard Business School, Harvard University, USA

[b]Faculty of Industrial Engineering and Management, Technion, Haifa , Israel

[c]Microsoft Israel R&D Center, Herzeliya, Israel

Abstract

Different search engines conduct similar ad auctions simultaneously and advertisers have to choose in which search engine(s) to run their ad campaign. In this paper we discuss two models for a pair of simultaneous ad auctions, A and B: (i) single-campaign advertisers, which participate in a single ad auction, and (ii) multi-campaign advertisers, which participate in both auctions. We prove the existence and uniqueness of a symmetric equilibrium in the first model. Moreover, when the click-through rates in A are point-wise higher than those in B, we prove that the expected revenue in A is greater than the expected revenue in B in this equilibrium. In contrast, we show that higher click-rates do not necessarily imply higher revenues in the second model.

1 Introduction

Search engines and publishers conduct ad auctions for potentially every keyword. In an ad auction, advertisers compete over positions in the web page associated with the results of searching for the corresponding keyword. The advertisers submit bids, and the position of displayed ads on the web page is determined based on these bids. Moreover, an advertiser pays the search engine each time a user clicks on her ad where the charged price is also based

Email addresses: iashlagi@hbs.edu, dov@ie.technion.ac.il, moshet@ie.technion.ac.il

on the submitted bids. Due to their enormous impact on the revenues of publishers/search engines and advertisers and because of the important challenges they provide for auction designers and participants, ad auctions have become a central topic of study in economics, electronic commerce, and marketing.[1] However, the previous research has yet to account for the fact that similar ad auctions are held simultaneously by different search engines. That is, an advertiser has to choose not only how to bid, but where to bid. This paper initiates research on this question by examining two simultaneous ad auctions. We present two models. The first considers single-campaign advertisers, who choose a single search engine to run their campaign. In this case, the game has two stages. In the first stage each advertiser, after observing her type, chooses her probability for selecting each auction. We refer to this probability as the advertiser's participation strategy. After the realization of the participation strategy, the advertiser chooses her bid. No information is revealed after the first stage, and therefore, for technical convenience, the game can be viewed as having only one stage in which each advertiser chooses a participation strategy and two bids, one for each auction. Implicitly this model assumes that an advertiser has an infinite cost to run ad campaigns in both auctions. In the second model, we consider multi-campaign advertisers. These advertisers run their campaign and bid in both auctions.[2] A main issue in both settings is the effect of having more than one ad auction on the revenue of the different auctions; of particular interest is the relationship between the click-rate values (search engines' popularity) and the corresponding ad auctions' revenue. Do higher click-rates result in higher revenue? In addition, a full analysis of the first setting requires us to introduce equilibrium analysis

[1] See, e.g., (Varian, 2007; Lahaie, 2006; Edelman et al., 2007; Borgs et al., 2007; Mehta et al., 2007; Athey and Ellison, 2007).

[2] Our models of simultaneous ad auctions are not the only reasonable ones. One very interesting model is to consider a market in which advertiser chooses if to be single-campaign or multi-campaign, for example by incorporating costs for running ad campaigns (note that our models implicitly assume specific extreme costs). One can also consider at least two other potential models. In the first model, which is similar in spirit to our single-campaign model, the assumption that every bidder participates only in one auction is kept. However, the number of participants in each auction is revealed to all advertisers after the selection part is over. In the second model, each bidder has a limited budget, and her strategic decision is how to split this budget among the auctions.

for simultaneous ad auctions, a subject not tackled for such rich setting thus far. Note that, in the first setting, advertisers choose a single auction to participate in. As mentioned, advertisers often do tend to concentrate on only one search engine in certain keyword markets. One reason for this behavior is the burden that advertisers have in running and managing their campaigns in each search engine. Another reason arises from the lack of flexibility in copying advertising campaign data from some search engines to others (see e.g., (Edelman, 2008b), and (Edelman, 2008a)). We analyze simultaneous ad auctions with single-campaign advertisers within the standard symmetric independent private- value model. We prove our results for VCG ad auctions and explicitly extend them to regular ad auctions, where roughly speaking, a regular ad auction is one for which an appropriate revenue equivalence theorem makes it equivalent to a VCG ad auction. For VCG ad auctions, we assume that a bidder bids her true valuation, irrespective of the auction she selects. This assumption reduces the strategy set of every bidder to the set of participation strategies. We prove that the auction selection game has an essentially unique symmetric equilibrium, whose structure is analyzed. Particular cases are presented and discussed. Search engines use ad auctions as one of their main revenue sources. Intuitively, since an advertiser is charged each time a user clicks on her ad, one would expect that higher click rates will result in higher revenues. This implies that search firms should care about providing effective search engines, yielding high traffic to their sites. Indeed, in a setting with simultaneous ad auctions and single-campaign advertisers, we prove that when Auction A is stronger than Auction B, in the sense that the click rates in A are point-wise higher than those in B, the expected revenue of A is higher than the expected revenue of B in the essentially unique equilibrium. However, we show that this seemingly intuitive result is a consequence of the competition on the auction to be selected. For multi-campaign advertisers, in which advertisers participate in both auctions, we compare revenues in two monopolistic setups, where one of them has higher click-rates than the other. In this setting, we characterize cases in which a stronger auction may yield lower revenue than a weaker one.[3] We prove our results for a not necessarily symmetric model, and

[3] A relevant phenomenon was demonstrated for the VCG combinatorial auctions with complete information, where it was shown that higher valuations may reduce revenue. See, e.g., Rastegari et al. (2007) and the references there.

without any restriction on the distribution of types. Since the revenue equivalence principle is less useful for such general models, the theorems are proved only for VCG ad auctions. Nevertheless, the theorems are applied to next-price ad auctions[4] which are used in practice. This application builds on the pioneering papers of Varian (2007) and Edelman et al. (2007), where the relationships between next-price ad auctions and VCG ad auctions are explored using theoretical and empirical tools.

In the last section we extend our results and conclusions to ad auctions with reserve prices. Our existence and uniqueness results in this setting generalize those in Burguet and Sakovics (1999), where the authors defined and analyzed a setting among two simultaneous identical second-price single item auctions with potentially distinct reserve prices. Our work could have had some implications for (or impact on) the Google-Yahoo Ad Deal, discussed by the U.S. House of Representatives Committee on the Judiciary Task Force on Competition Policy and Antitrust Laws in 2008 (see e.g., (Edelman, 2008a)) and by the European Commission(Cheng, 2008). Our results support the claim that this deal would harm advertisers and internet users in two ways. First, in the monopolistic setup, a search engine can, in certain circumstances, increase its revenue by decreasing click rates, which implies reducing quality of service to internet users. Second, the equilibrium structure we establish supports the claim that, when advertisers have the ability to select among alternatives auctions, those with low valuations participate in the auction with lower click rates. It is important to note that our paper does not discuss strategic organizers' competition in auction design, although it opens the road for such a model. In general, such a competition is modeled as a two stage game, where, in the first stage every auction organizer chooses the auction to conduct, and in the second stage each of the bidders decides which auction to attend and how to bid in this auction. Such an approach was taken in a restricted symmetric single-item auction setup, e.g., in (Burguet and Sakovics, 1999) and in (Monderer and Tennenholtz, 2004). However, as was already shown in (McAfee, 1993) the above two-stage game does not in general possess a sub-game perfect equilibrium, even in the simple single-item setup.[5] Therefore, most of

[4] Next-price ad auctions are also called generalized second-price ad auctions.

[5] In (Burguet and Sakovics, 1999) the authors proved the existence of a sellers' mixed strategy equilibrium in such setting.

the literature ((McAfee, 1993; Peters, 1997; Peters and Severinov, 1997)) on competition in auction design in the single-item setup has dealt with a model with many auctions and derived results about the limit (partially strategic) behavior of the market, when the number of auctions' organizers and buyers is approaching infinity. This approach does not seem the right one in the ad auction market, where only a few auctions' organizers (search engines) control the market.

The paper is organized as follows. In Section 2 we define the basic model of ad auctions. In Section 3 we define the model of single-campaign advertisers' competition in simultaneous ad auctions, and present our main results for the VCG setup: a theorem about existence and essential uniqueness of symmetric equilibrium, and a theorem about revenue dominance of the strong auction. The existence and uniqueness theorem is proved in Section 4, and examples are provided in Section 4.5. The theorem about revenue inequality is proved in Section 5. In Section 6 we define regular ad auctions and prove that our main theorems hold for them. Section 8 deals with multi-campaign advertisers, which participate in both auctions. We show that in a monopolistic setup, higher click-rates do not necessarily imply higher revenues. In Section 7 we extend our results to a setup with reserve prices.

2 Ad Auctions

There exist n advertisers, which we call *bidders*, $n \geq 2$; A generic bidder is denoted by i, $1 \leq i \leq n$. In an *ad auction* there is a *seller* who offers for sale k positions, $k \geq 1$; a generic position is denoted by j, $1 \leq j \leq k$. As the seller cannot sell more positions than the number of bidders, it is assumed that $n \geq k$. The positions are sold for a fixed period of time. For each position j there is a commonly-known number $\alpha_j > 0$, which is interpreted as the expected number of visitors at that position; α_j is called the *click rate* of position j. It is assumed that the positions have distinct click rates, and without loss of generality it is assumed that $\alpha_1 > \alpha_2 > \ldots > \alpha_k > 0$. For convenience, we add a dummy position, position $k+1$ with $\alpha_{k+1} = 0$. From now on, unless otherwise specified, the term"position" includes the dummy position. Let $K = \{1, \ldots, k\}$ be the set of non-dummy positions, and let $K_d = K \cup \{k+1\}$ denote the set of positions.

Given the above environment, an *ad auction* is defined by an allocation rule and a payment scheme. Each bidder i is requested to submit a bid $b_i \in B_i = [0,1]$. The set of bid profiles is denoted by $\mathbf{B} = B_1 \times \cdots \times B_n$. Based on the profile of bids, the allocation rule determines the allocation of positions to the bidders. However, in this paper we consider only ad auctions for which the allocation rule is a *welfare maximizer*; that is, whenever bids are distinct, the bidder with the highest bid receives the first position, the second highest bidder receives the second position etc. Each bidder that does not receive a position $j \in K$ is assigned to the dummy position, $k+1$. Distinct welfare maximizers allocation rules differ in the tie-breaking rule they use. In this paper tie problems will be successfully avoided. However, for completeness, it is assumed that ties are resolved by the following simple priority rule over bidders: $i < t$ implies that i has priority over t, whenever they make the same bid.[6] Hence, with this tie breaking rule there exists a unique allocation rule, which is a welfare maximizer. We denote by $s_i(\mathbf{b})$ bidder i's position with respect to the welfare maximizer allocation rule when the bid profile is $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbf{B}$.

A *payment scheme* is a tuple $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where for every i $p_i : \mathbf{B} \to \mathbb{R}_+$ is a non-negative bounded Borel measurable function, which is called bidder i's *payment function*; That is, $p_i(\mathbf{b})$ is bidder i's payment per click when the bid profile is $\mathbf{b}$. It is assumed that a bidder that bids 0 pays 0. Thus, the total payment of bidder i equals $\alpha_{s_i(\mathbf{b})} p_i(\mathbf{b})$. Therefore, although the payment per-click for a bidder that gets the dummy position can be positive, his total payment equals 0 since $\alpha_{k+1} = 0$. Given a fixed number of bidders, n, an ad auction with k non-dummy positions and a click rates vector $\alpha = (\alpha_1, \ldots, \alpha_k)$, which uses the welfare maximizer allocation rule and the payment scheme $\mathbf{p} = (p_1, \ldots, p_n)$ is denoted by $G(k, \alpha, \mathbf{p})$. In some discussions, some of the parameters in $G(k, \alpha, \mathbf{p})$ whose values are obvious are omitted.

If i holds a position, every visitor to this position gives i a revenue of $v_i \in [0,1]$, where v_i is called the *valuation* of i. Given the bidders' utility functions and their distribution over valuations are, an auction generates a Bayesian game. It is assumed that the bidders' utility functions are quasi-linear. That is, if bidder i is assigned to position j and pays p per click,

[6] Our results hold for every tie breaking rule, including randomized ones.

his utility is $\alpha_j(v_i - p)$.

We use the *independent-private-value model* to model the distribution of valuations; that is, each v_i is privately observed by i, and it is drawn from the interval $V_i = [0, 1]$ according to a random variable $\tilde{v}_i$, whose distribution function is F_i; The random variables, $\tilde{v}_i$ are independent. At this point, and unless otherwise specified F_i is not required to satisfy particular assumptions except for being a distribution function. That is, F_i is a non-decreasing and right-continuous function on $[0, 1]$, and $F_i(1) = 1$.

Let $\hat{F}$ denote the joint distribution of the bidders on $\mathbf{V} = V_1 \times V_2 \cdots \times V_n$. Throughout this paper we use the standard notation regarding the subscript $-i$. E.g., $\hat{F}_{-i}$ denotes the joint distribution of all bidders except i. The Bayesian game generated by the ad auction $G(k, \alpha, \mathbf{p})$ and by the vector of distribution functions $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is denoted by $G(k, \alpha, \mathbf{p}, \mathbf{F})$. With a slight abuse of notation $G(k, \alpha, \mathbf{p}, \mathbf{F})$ is also called an ad auction. Let $G = G(k, \alpha, \mathbf{p}, \mathbf{F})$ be an ad auction. A *strategy* for bidder i in G is a Borel measurable function $d_i : V_i \to B_i$ that assigns a bid, $d_i(v_i)$ to every possible value $v_i \in V_i$. We denote by $\Gamma_i = \Gamma_i(G)$ the set of possible strategies for bidder i.

For a profile of strategies $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ and a profile of valuations $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ we denote $\mathbf{d}(\mathbf{v}) = (d_1(v_1), d_2(v_2), \ldots, d_n(v_n))$. Let $U_i^G(v_i, \mathbf{d})$ be the expected utility of bidder i given that her valuation is v_i and every bidder t uses the strategy d_t. That is,

$$U_i^G(v_i, \mathbf{d}) = \int_{\mathbf{V}_{-i}} \alpha_{s_i(\mathbf{d}(\mathbf{v}))}(v_i - p_i(\mathbf{d}(\mathbf{v})))d\hat{F}_{-i}(\mathbf{v}_{-i}).$$

A strategy profile $\mathbf{d} = (d_1, \ldots, d_n) \in \Gamma_1 \times \cdots \times \Gamma_n$ is a *Bayesian equilibrium* in G if for every bidder i and every v_i,

$$U_i^G(v_i, \mathbf{d}) \geq U_i^G(v_i, d_i^{'}, \mathbf{d}_{-i}) \;\; \forall d_i^{'} \in \Gamma_i.$$

We are about to discuss special ad auctions. Before we do it, we need the following definition. For every vector of real numbers, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ we denote by $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ a permutation of the vector $\mathbf{x}$ for which $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$. For a bid profile, $\mathbf{b}$, whenever convenient, we let $b_{(j)} = 0$ for every $j > n$.

The leading search engines use variants of the next-price ad auctions:

Definition 2.1 (Next-price ad auction) The ad auction $G = G(k, \alpha, \mathbf{p})$ is called *the* *next-price ad auction* if its payment scheme $\mathbf{p}$ is defined for every $1 \leq i \leq n$ as follows:

$$p_i(\mathbf{b}) = b_{(s_i(\mathbf{b})+1)}. \tag{1}$$

In this paper we mainly deal with the standard VCG ad auction.

Definition 2.2 (Standard VCG ad auction) The ad auction $G = G(k, \alpha, \mathbf{p})$ is called *the standard VCG ad auction,* or in short, *the VCG ad auction,* if its payment scheme $\mathbf{p}$ is defined for every $1 \leq i \leq n$ as follows:

$$p_i(\mathbf{b}) = \begin{cases} \frac{1}{\alpha_{s_i(\mathbf{b})}} \sum_{j=s_i(\mathbf{b})+1}^{k+1} \mathbf{b}_{(j)}(\alpha_{j-1} - \alpha_j) & s_i(\mathbf{b}) \in K; \\ 0 & s_i(\mathbf{b}) = k+1. \end{cases} \tag{2}$$

In a non-standard VCG ad auction, every bidder may pay an additional amount depending only on the bids of the other bidders.

3 Simultaneous Ad Auctions with Single-Campaign Advertisers – Main Results

Consider two ad auctions, $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$, and recall the assumption that $n \geq k^A$ and $n \geq k^B$. In what follows, whenever necessary, it is assumed that $\alpha_j = 0$ for every $j > k^A$ and that $\beta_j = 0$ for every $j > k^B$. Auctions A and B form a game for the bidders $1, 2, \ldots, n$, denoted by $H(A, B, \mathbf{F})$, where $\mathbf{F}$ is the vector of distribution functions. In this game each bidder simultaneously chooses one auction to participate in, and how to bid at each auction. Bidders can use mixed strategies to select an auction. We assume that a bidder can not attend both auctions. This assumption captures single-campaign advertisers, who run their campaign with only a single search engine. As mentioned, single-campaign advertisement is popular in many keyword markets. Since bidders can always guarantee a non-negative utility by bidding zero there is no harm in assuming that bidders always choose to participate in some auction. A *strategy* for a bidder i in $H(A, B, \mathbf{F})$ is a tuple $\sigma_i = (\mathbf{q}_i, \mathbf{d}_i)$, where $\mathbf{q}_i = (q_i^A, q_i^B)$ is the *participation strategy* of i, and $\mathbf{d}_i = (d_i^A, d_i^B)$ is the *bidding strategy* of i.[7] More precisely, $q_i^L : [0,1] \to [0,1]$, a Borel measurable function, is the probability

[7]Note that a bidder's strategy includes her bid in an auction even if the probability of attending this auction is zero. This is consistent with the usual redundancy in the definition of strategies in game theory.

that i will attend the auction L, $L \in \{A, B\}$. In particular, $q_i^A(v_i) = 1 - q_i^B(v_i)$ for every valuation v_i; $d_i^L : [0,1] \to [0,1]$ is the bidding strategy in auction $L \in \{A, B\}$, which is also assumed to be measurable.

Let Σ_i be the set of strategies for bidder i. Denote by $U_i^{H(A,B,\mathbf{F})}(v_i, \sigma_1, \ldots, \sigma_n)$ the expected utility for bidder i given that her valuation is v_i and every bidder t uses the strategy σ_t. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_1 \times \cdots \times \Sigma_n$ is a *Bayesian equilibrium* in the game $H(A, B, \mathbf{F})$ if for every bidder i and every v_i,

$$U_i^{H(A,B,\mathbf{F})}(v_i, \sigma_1, \ldots, \sigma_n) \geq U_i^{H(A,B,\mathbf{F})}(v_i, \sigma_i', \sigma_{-i})$$

for every strategy $\sigma_i' \in \Sigma_i$.

When dealing with the above setting, we derive our results for a symmetric model. That is, the distribution functions of all bidders are identical. If the common distribution function is F, the ad auction $G(k, \alpha, \mathbf{p}, (F, F, \ldots, F))$ is denoted by $G(k, \alpha, \mathbf{p}, F)$ and the game $H(A, B, (F, F, \ldots, F))$ is denoted by $H(A, B, F)$. In a symmetric model it is natural to focus on symmetric strategies/equilibrium, and therefore we omit the bidder's index from strategies. Hence, we refer to a strategy of a bidder as a vector $(\mathbf{q},\mathbf{d}) = ((q^A, q^B), (d^A, d^B))$.

Let $(\mathbf{q},\mathbf{d})$ be a symmetric equilibrium strategy. Note that changing the participation function $\mathbf{q}$ in a set of valuations v with $F-$probability 0 for which $0 < q^A(v) < 1$, does not change the utilities of any of the bidders. This inspires the following definition:

Definition 3.1 (Essential Uniqueness) *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions, and let F be a distribution function. The game $H(A, B, F)$ possesses an essentially unique symmetric equilibrium if it possesses a symmetric equilibrium and for every two symmetric equilibria $\mathbf{q}, \hat{\mathbf{q}}$*

$$\mathbf{q}(v) = \hat{\mathbf{q}}(v) \ F-\textit{almost everywhere in } [0,1].$$

Moreover, we will frequently assume that F is standard, where

Definition 3.2 (Standard distribution functions) *The distribution function F defined on $[0,1]$ is standard if it is differentiable, its derivative is positive on $[0,1]$, and it has a density, that is $F(x) = \int_0^x F'(t)dt$ for every $x \in [0,1]$.*

We prove our main results for VCG ad auctions. In such case we naturally assume that *the bidding strategy of each bidder is the truth-telling strategy.* Hence, a strategy of a bidder in such case is determined by his participation strategy.

We prove:

Theorem 3.3 *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions and let F be a standard distribution function.*

The game $H(A, B, F)$ possesses an essentially unique symmetric equilibrium.

Moreover, If $\alpha_1 \geq \beta_1$ there exists a unique $0 \leq v^* \leq 1$ for which there exists a symmetric equilibrium, $\mathbf{q} = (q^A, q^B)$ with the following properties:

$$\begin{cases} 0 < q^B(v) < 1 & \text{for every } 0 < v < v^*; \\ q^B(v) = 0 & \text{for every } v^* < v \leq 1. \end{cases} \tag{3}$$

Furthermore, $v^* = 0$ if and only if $\alpha_n \geq \beta_1$, and $v^* = 1$ if and only if $\alpha_1 = \beta_1$.

The condition $\alpha_1 \geq \beta_1$ in Theorem (3.3) is without loss of generality; Otherwise exchange the names of the auctions.

In Section 4 we prove Theorem 3.3, and we provide tools for computing the cutting point v^* and the values of the probabilities below v^*. In Section 4.5 we apply the tools developed in the proof in order to explicitly find and discuss the equilibrium in special cases.

Before we state our main theorem regarding revenues we need the following definition:

Definition 3.4 ("Stronger than") *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be ad auctions. We say that A is stronger than B if $\alpha_j \geq \beta_j$ for every $1 \leq j \leq \max\{k^A, k^B\}$, and at least one inequality is strict.*

Theorem 3.5 *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions, and let F be a standard distribution function. If A is stronger than B, the expected revenue in A is greater than the expected revenue in B in the essentially unique symmetric equilibrium of the game $H(A, B, F)$.*

Theorem 3.5 is proved in Section 5.

4 Proof of Theorem 3.3

Consider the game $H(A, B, F)$, where A and B are VCG auctions, and F is a standard distribution function, whose density is denoted by f. Without loss of generality assume that $\alpha_1 \geq \beta_1$.

If $\alpha_n \geq \beta_1$ then in particular $\alpha_n > 0$ and therefore $n = k^A$. In such case, for an arbitrary bidder i, independently of all other bidders' strategies, the maximal utility in B, $\beta_1 v_i$ does not exceed his minimal utility in A, $\alpha_n v_i$, and we say that the competition is *degenerate*. We cover this trivial case in some more detail in Section 4.4. In the following proof of Theorem 3.3 we assume non-degenerate competition. That is, $\alpha_n < \beta_1$.

4.1 Preparations

Let $\mathbf{q}$ be a symmetric strategy. For an arbitrary bidder i let

$$\varphi(v, \mathbf{q}) = Prob_F(\tilde{v}_i \geq v,\ i \text{ chooses to participate in } B).$$

That is,

$$\varphi(v, \mathbf{q}) = \int_v^1 q^B(x)dF(x). \tag{4}$$

When all other bidders but t use the strategy $\mathbf{q}$, bidder t should compare his utilities in A and in B. When he computes his utility in A, he faces a random number of participants. Equivalently, bidder t can consider lack of participation in A as participation in A of a bidder with valuation 0. Hence, bidder i can assume that there exist exactly additional $n-1$ bidders in A such that the distribution function of each of them is F_q^A, where

$$F_q^A(v) = \varphi(v, \mathbf{q}) + F(v). \tag{5}$$

Similarly, for auction B, let $\psi(v, \mathbf{q}) = \int_v^1 q^A(x)dF(x)$, and let $F_q^B(v) = \psi(v, \mathbf{q}) + F(v)$. Since $q^A(v) = 1 - q^B(v)$ for all v, $\psi(v, \mathbf{q}) = 1 - F(v) - \varphi(v, \mathbf{q})$, and therefore

$$F_q^B(v) = 1 - \varphi(v, \mathbf{q}). \tag{6}$$

Denote by $P^A(v, \mathbf{q})$ ($P^B(v, \mathbf{q})$) the expected total payment in A (B) experienced by a bidder with valuation v given that each of the other bidders uses the strategy $\mathbf{q}$. Similarly, denote by $Q^A(v,\mathbf{q})$ ($Q^B(v,\mathbf{q})$) the expected click rate in A (B), and denote by $U^A(v,\mathbf{q})$ ($U^B(v,\mathbf{q})$) the expected utility in A (B). Obviously

$$U^L(v,\mathbf{q}) = vQ^L(v,\mathbf{q}) - P^L(v,\mathbf{q}) \quad L \in \{A,B\},\, v \in [0,1]. \tag{7}$$

Note that a bidder with valuation v obtains position j in auction A if there are exactly $n-j$ other bidders each of whom has a lower valuation than v in A and there are exactly $j-1$ bidders each of whom has a higher valuation than v in A. Since ties have probability zero, the probability that the bidder obtains j and the above condition is not satisfied equals 0. Therefore,

$$Q^A(v,\mathbf{q}) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (F_q^A(v))^{n-j}(1-F_q^A(v))^{j-1}. \tag{8}$$

Similarly,

$$Q^B(v,\mathbf{q}) = \sum_{j=1}^{k^B} \alpha_j \binom{n-1}{j-1} (F_q^B(v))^{n-j}(1-F_q^B(v))^{j-1}. \tag{9}$$

We need the following proposition whose proof is standard in mechanism design theory.

Proposition 4.1 *For every $L \in \{A,B\}$ and every symmetric strategy $\mathbf{q}$:*

1. *$Q^L(\cdot,\mathbf{q})$ is non-decreasing and continuous in $[0,1]$.*
2. *For every symmetric strategy $\mathbf{q}$ and for every $v \in [0,1]$*

$$U^L(v,\mathbf{q}) = \int_0^v Q^L(x,\mathbf{q})dx. \tag{10}$$

Consequently, since Q^L is continuous, the derivative of $U^L(\cdot,\mathbf{q})$ equals $Q^L(\cdot,\mathbf{q})$ everywhere in $[0,1]$.

Proof: We prove the proposition for $L = A$. 1. Let $v,w \in [0,1]$. Because truth-telling is a dominant strategy for a bidder, bidding v when his valuation equals v yields at least as bidding w, that is, $vQ^A(v,\mathbf{q}) - P^A(v,q) \geq vQ^A(w,\mathbf{q}) - P^A(w,\mathbf{q})$. Similarly, $wQ^A(w,\mathbf{q}) - P^A(w,\mathbf{q}) \geq wQ^A(v,\mathbf{q}) - P^A(v,\mathbf{q})$. Combining these inequalities yields

$$(v-w)(Q^A(v,\mathbf{q}) - Q^A(w,\mathbf{q})) \geq 0 \text{ for every } v,w \in [0,1], \tag{11}$$

which implies that Q^A is non-decreasing. Since F_q^A is continuous in $[0,1]$, Q^A is continuous as well by (8).

2. Let $v, w \in [0,1]$. Recall that $U^A(v, \mathbf{q}) = vQ^A(v, \mathbf{q}) - P^A(v, \mathbf{q})$. Therefore, by the two inequalities we derived in part 1 of this proof, and by (7),

$$U^A(v, \mathbf{q}) - U^A(w, \mathbf{q}) \geq Q^A(w, \mathbf{q})(v - w).$$

By (Rockafellar, 1970) this easily implies that $U^A(\cdot, \mathbf{q})$ is a convex function, whose derivative equals Q^A almost everywhere in $[0,1]$, and since $U^A(0, \mathbf{q}) = 0$, the required integral equality follows. $\square$

The following functions are extensively used in our proofs. For every $0 \leq x \leq 1$ and every $0 \leq y \leq 1$ let

$$\tilde{Q}^A(x, y) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (x+y)^{n-j} (1-x-y)^{j-1}, \tag{12}$$

$$\tilde{Q}^B(x) = \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} (1-x)^{n-j} x^{j-1}, \tag{13}$$

and let

$$Q(x, y) = \tilde{Q}^A(x, F(y)) - \tilde{Q}^B(x). \tag{14}$$

Note that by (8), (9), (5), and (6), for every $0 \leq v \leq 1$,

$$(i) \quad Q^A(v, q) = \tilde{Q}^A(\varphi(v, \mathbf{q}), F(v)); \quad (ii) \quad Q^B(v, q) = \tilde{Q}^B(\varphi(v, \mathbf{q})), \tag{15}$$

and therefore

$$Q(\varphi(v, \mathbf{q}), v) = Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}). \tag{16}$$

For a function $\phi(x, y)$ we denote by ϕ_x, ϕ_y the derivatives with respect to the first and second variable respectively. Similarly, if $\phi(x)$ is a function of one variable, ϕ_x denotes the derivative of ϕ. The following technical lemma will be useful for us.

Lemma 4.2

(i) $\tilde{Q}^A_y(x, y) = \tilde{Q}^A_x(x, y)$ *for every* x, y.

(ii) $\tilde{Q}^A_x(x, y) > 0$ *for every* x, y *for which* $0 < x + y < 1$.

(iii) $\tilde{Q}^B_x(x) < 0$ *for every* $0 < x < 1$.

Consequently,

(iv) $Q_x(x, y) > 0$ *for every* x, y *for which* $0 < x+F(y) < 1$ *or* $[x+F(y) = 1$ *and* $0 < x, y < 1]$.

(v) $Q_y(x, y) > 0$ *for every* x, y *for which* $0 < x + F(y) < 1$.

Proof: The equality (i) is obvious. (ii) Recall the standard convention that for nonnegative integers $a < b$, $\binom{a}{b} = 0$. Note that

$$\tilde{Q}^A_x(x, y) =$$

$$\sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1}(n-j)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1}(j-1)(x+y)^{n-j}(1-x-y)^{j-2} =$$

$$\sum_{j=1}^{k^A} \alpha_j \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=2}^{k^A} \alpha_j \binom{n-2}{j-2}(n-1)(x+y)^{n-j}(1-x-y)^{j-2},$$

where the last equality follows since $\binom{n-1}{j-1}(n-j) = \binom{n-2}{j-1}(n-1)$ and $\binom{n-1}{j-1}(j-1) = \binom{n-2}{j-2}(n-1)$. Therefore,

$$\tilde{Q}^A_x(x, y) =$$

$$\sum_{j=1}^{k^A} \alpha_j \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=1}^{k^A-1} \alpha_{j+1} \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1} =$$

$$\alpha_{k^A} \binom{n-2}{k^A-1}(n-1)(x+y)^{n-k^A-1}(1-x-y)^{k^A-1} + \sum_{j=1}^{k^A-1} (\alpha_j - \alpha_{j+1}) \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1}. \quad (17)$$

Assume $0 < x + y < 1$, which implies, in particular, that both summands in $RHS(17)$ are nonnegative. If $n > k^A$, the first summand in $RHS(17)$ is positive. If $n = k^A$, $k^A \geq 2$, and since $\alpha_j - \alpha_{j+1} > 0$ for every $1 \leq j \leq k^A - 1$, the second summand in $RHS(17)$ is positive. Therefore, $RHS(17) > 0$, which completes the proof of (ii). Note that $\tilde{Q}^B(x) = \hat{Q}(1-x, 0)$, where $\hat{Q}$ is defined as $\tilde{Q}^A$, except that we replace α_j with β_j for every relevant j and k^A with k^B. Therefore, (iii) follows from (ii). □

Before we present our next proposition we present a variant of the Implicit Function Theorem needed in its proof:

Theorem 4.3 (Implicit Function Theorem) *Let* $F : E \times J \to \mathbb{R}$ *be a continuously differentiable function, where* E *and* J *are open intervals. and let* $(x_0, y_0) \in E \times J$ *be a point for* *which $F(x_0, y_0) = 0$ and $F_x(x_0, y_0) \neq 0$. Then, there exists an open interval $I = (y_0-\delta, y_0+\delta)$, $\delta > 0$, and a unique function $g : I \to J$ such that $g(y_0) = x_0$ and $F(g(y), y) = 0$ for all $y \in I$. Moreover g is continuously differentiable on I and for every $y \in I$*

$$g'(y) = -\frac{F_y(g(y), y)}{F_x(g(y), y)}. \tag{18}$$

Proposition 4.4

1. *There exist a unique valuation in $[0, 1]$ denoted by v^* for which $Q(0, v^*) = 0$. Moreover, $v^* > 0$, and $v^* = 1$ if and only if $\alpha_1 = \beta_1$.*

2. *There exists a unique function $h : (0, v^*) \to [0, 1]$ such that the following two conditions hold for every $0 < v < v^*$:*

$$0 \leq h(v) \leq 1 - F(v).$$

$$Q(h(v), v) = 0.$$

   Moreover, this unique function denoted by h satisfies $0 < h(v) < 1 - F(v)$ for every $0 < v < v^*$.

3. *h is continuously differentiable and $0 < -\frac{h'(v)}{f(v)} < 1$ for every $0 < v < v^*$, where f is the density function, that is $f = F'$.*

4. *The function h can be continuously extended to the closed interval, $[0, v^*]$. Denote this extension also by h; h satisfies $h(v^*) = 0$.*

Proof:

1. By part (v) of Lemma 4.2, $Q(0, v)$ is increasing in $v \in [0, 1]$. Since $Q(0, 0) = \alpha_n - \beta_1 < 0$ and $Q(0, 1) = \alpha_1 - \beta_1 \geq 0$, the requested results follow.

2. Let $v \in (0, v^*)$. Since $Q(0, y)$ is increasing in $y \in [0, 1]$ and $Q(0, v^*) = 0$, $Q(0, v) < 0$. By part (iv) in Lemma 4.2, $Q(x, v)$ is increasing in $x \in [0, 1 - F(v)]$, and therefore the proof is completed if we show that $Q(1 - F(v), v) > 0$. Indeed, $Q(1 - F(v), v) = \tilde{Q}^A(1-F(v), F(v)) - \tilde{Q}^B(1-F(v))$. However, $\tilde{Q}^A(1-F(v), F(v)) = \alpha_1$ and by Part (iii)

of Lemma 4.2, $\tilde{Q}^B(1 - F(v)) < \tilde{Q}^B(0) = \beta_1$. Therefore, $Q(1 - F(v), v) > \alpha_1 - \beta_1 \geq 0$, which completes the proof of this part.

3. Let $0 < v < v^*$. Since $Q(h(v), v) = 0$ and by Part (iv) in Lemma 4.2, $Q_x(h(v), v) > 0$, the Implicit Function Theorem, Theorem 4.3, implies that there exists an interval $(v - \delta, v + \delta)$ around v and a unique real-valued function, g defined on this interval such that $g(v) = h(v)$ and $Q(g(y), y) = 0$ for every y in this interval. More over g is continuously differentiable in this interval. For sufficiently small $\delta > 0$, $g(y)$ is sufficiently close to $g(v) = h(v)$, and since $0 < h(v) < 1 - F(v)$, $g(y) \in [0, 1 - F(v)]$. Therefore, by what we proved in the previous part of this theorem, $g(y) = h(y)$ for every y in this smaller neighborhood of v. This implies that h is continuously differentiable in this smaller neighborhood of v, and in particular it is differentiable in v. By the Implicit Function Theorem,
$$h'(v) = -\frac{Q_y(h(v), v)}{Q_x(h(v), v)}.$$
By parts (iv) and (v) of Lemma 4.2, $-h'(v) > 0$. It remains to prove that $-\frac{h'(v)}{f(v)} < 1$. That is, we have to prove that
$$\frac{Q_y(h(v), v)}{Q_x(h(v), v) f(v)} < 1. \tag{19}$$
Note that $Q_y(h(v), v) = \tilde{Q}^A_y(h(v), F(v)) f(v)$, and by part (i) of Lemma 4.2, $\tilde{Q}^A_y(h(v), F(v)) f(v) = \tilde{Q}^A_x(h(v), F(v)) f(v)$. Also, $Q_x(h(v), v) = \tilde{Q}^A_x(h(v), F(v)) - \tilde{Q}^B_x(h(v)) > \tilde{Q}^A_x(h(v), F(v))$ by part (iii) of Lemma 4.2. Hence, (19) holds.

4. We first prove that $\lim_{v \to v^*} h(v) = 0$. Indeed, Since $h'(v) < 0$ for $0 < v < v^*$, h is decreasing in $(0, v^*)$, and since in addition $h(v)$ is bounded below by 0, there exists $c \geq 0$ such that $\lim_{v \to v^*} h(v) = c$. We proceed to prove that $c = 0$. Since $Q(h(v), v) = 0$ for every $0 < v < v^*$, $Q(c, v^*) = 0$. Moreover, $0 \leq c \leq 1 - F(v^*)$. Hence, $c = 0$ if $v^* = 1$. Consider the case $v^* < 1$: Since by part (iv) of Lemma 4.2, $Q(x, v^*)$ is increasing in $x \in [0, c]$, and $Q(0, v^*) = 0$, it must be that $c = 0$.

   Similarly, because h is decreasing and bounded from above in $(0, v^*)$ the limit, $\lim_{v \to 0} h(v)$ exists, and is denoted by $h(0)$.

$\square$

We end this subsection with the following useful lemma:

Lemma 4.5

1. *For every* $v \in [0, v^*)$, $Q(0, v) < 0$*; For every* $v \in (v^*, 1]$, $Q(0, v) > 0$.

2. *For every* $v \in (v^*, 1]$ *and for every* $x \in [0, 1 - F(v)]$, $Q(x, v) > 0$.

3. *For every* $v > v^*$ *and for every strategy* $\mathbf{q}$, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) > 0$.

Proof:

1. By part (v) in Lemma 4.2, $Q(0, y)$ is increasing in $y \in [0, 1]$. Since $Q(0, v^*) = 0$ the result follows.

2. By part (ii) of Lemma 4.2, $Q(x, v)$ is increasing in $x \in [0, 1 - F(v)]$, and therefore $Q(x, v) \geq Q(0, v) > 0$ for every $x \in [0, 1 - F(v)]$.

3. By (16), $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = Q(\varphi(v, \mathbf{q}), v) > 0$ by the previous part because $\varphi(v, \mathbf{q}) \leq 1 - F(v)$.

$\square$

4.2 Existence

We define $\tilde{\mathbf{q}} = (\tilde{q}^A, \tilde{q}^B)$ by defining $\tilde{q}^B$ as follows: The values, $\tilde{q}^B(0), \tilde{q}^B(v^*)$ are left unspecified, and for other $0 < v \leq 1$

$$\tilde{q}^B(v) = \begin{cases} -\frac{h'(v)}{f(v)} & 0 < v < v^*; \\ 0 & v^* < v \leq 1. \end{cases} \tag{20}$$

In order to prove that $\tilde{\mathbf{q}}$ is a symmetric equilibrium it suffices to prove the following two claims:

(a) $U^A(v, \tilde{\mathbf{q}}) - U^B(v, \tilde{\mathbf{q}}) = 0$ for every $0 \leq v \leq v^*$;

(b) $U^A(v,\tilde{\mathbf{q}}) - U^B(v,\tilde{\mathbf{q}}) > 0$ for every $v > v^*$.

We first compute the function $\varphi(v,\tilde{\mathbf{q}})$. Obviously, for every $v > v^*$, $\varphi(v,\tilde{\mathbf{q}}) = \int_v^1 \tilde{q}^B(x)f(x)dx = 0$. We will show that:

$$\varphi(v,\tilde{\mathbf{q}}) = h(v) \quad \text{for every } 0 \le v \le v^*. \tag{21}$$

Indeed, if $v = v^*$ the proof is obvious since $\varphi(v^*,\tilde{\mathbf{q}}) = 0 = h(v^*)$. Let then $0 \le v < v^*$. For sufficiently small $\epsilon > 0$, $\varphi(v,\tilde{\mathbf{q}}) = I(\epsilon) + \int_{v+\epsilon}^{v^*-\epsilon} \tilde{q}^B(x)f(x)dx + J(\epsilon)$, where $I(\epsilon)$ equals the integral over the interval $[v, v+\epsilon]$ and $J(\epsilon)$ equal the integral over the interval $[v^*-\epsilon, v^*]$. Therefore, $\varphi(v,\tilde{\mathbf{q}}) = I(\epsilon) + h(v+\epsilon) - h(v^*-\epsilon) + J(\epsilon)$. Because h is continuous in $[0, v^*]$ and $I(\epsilon), J(\epsilon)$ converge to zero when $\epsilon \to 0$, $\varphi(v,\tilde{\mathbf{q}}) = h(v) - h(v^*)$, and since by part 4 in Proposition 4.4 $h(v^*) = 0$, (21) holds. Since $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) = Q(\varphi(v,\tilde{\mathbf{q}}), v)$, by (21), $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) = Q(h(v), v)$ for every $0 \le v \le v^*$. Therefore, since by Proposition 4.4 $Q(h(v), v) = 0$ for every $0 \le v \le v^*$, $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) = 0$ for every $0 \le v \le v^*$. Therefore, $U^A(v,\mathbf{q}) - U^B(v,\mathbf{q}) = \int_0^v (Q^A(x,\mathbf{q}) - Q^B(x,\mathbf{q}))dx = 0$ for every $0 \le v \le v^*$, which proves (a). By Lemma 4.5, $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) > 0$ for every $v > v^*$. Hence, for every $v > v^*$, $U^A(v,\mathbf{q}) - U^B(v,\mathbf{q}) = \int_{v^*}^v (Q^A(x,\mathbf{q}) - Q^B(x,\mathbf{q}))dx > 0$ by Lemma 4.1, which proves (b).

4.3 Uniqueness

Let $\mathbf{q}$ be a symmetric equilibrium. We first prove that:

$$\begin{cases} U^A(v,\mathbf{q}) - U^B(v,\mathbf{q}) = 0 & \text{for every } v \in [0, v^*]; \\ U^A(v,\tilde{\mathbf{q}}) - U^B(v,\mathbf{q}) > 0 & \text{for every } v > v^*. \end{cases} \tag{22}$$

Establishing (22) proves the essential uniqueness as follows: In the interval $(v^*, 1)$ a bidder in equilibrium must choose A, and therefore $q^B(x) = 0 = \tilde{q}^B(x)$ for every $x > v^*$. Considering the other interval, the derivative of $U^A(v,\mathbf{q}) - U^B(v,\mathbf{q})$ equals zero almost everywhere in $(0, v^*)$. Therefore, $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) = 0$ almost everywhere in this interval. Since $Q^A(v,\mathbf{q})$ and $Q^B(v,\mathbf{q})$ are continuous in v, $Q^A(v,\mathbf{q}) - Q^B(v,\mathbf{q}) = 0$ for every $v \in [0, v^*]$. Therefore $Q(\varphi(v,\mathbf{q}), v) = 0$ for every $v \in [0, v^*]$, and since $\varphi(v,\mathbf{q}) = h(v)$ for every $v \in [0, v^*]$, it must be that $\varphi(v, \mathbf{q}) = \varphi(v, \tilde{\mathbf{q}})$ in this interval. This finally implies that $q^B(x) = \tilde{q}^B(x)$ almost every where in $[0, v^*]$.

In order to prove (22) we need the following technical lemma, the proof of which is standard and hence omitted.

Lemma 4.6 *Let $\phi : [a, b] \rightarrow R$ be a continuous function with $\phi(a) = \phi(b) = 0$, which satisfies the following property: for every $a \leq c < d \leq b$ for which $\phi(c) = \phi(d) = 0$ there exists $c < z < d$ such that $\phi(z) = 0$. Then, $\phi(x) = 0$ for every $a \leq x \leq b$.*

Lemma 4.7 *Let $0 \leq c < d \leq 1$ be two valuations for which $U^A(c, \mathbf{q}) - U^B(c, \mathbf{q}) = 0$ and $U^A(d, \mathbf{q}) - U^B(d, \mathbf{q}) = 0$. Then, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0$ for every $c \leq v \leq d$.*

Proof: Let $\phi(v) = U^A(v, \mathbf{q}) - U^B(v, \mathbf{q})$. By Lemma 4.6 it suffices to prove that there exists $c < z < d$ for which $\phi(z) = 0$. Assume in negation that such z does not exist. Therefore, either $\phi(v) > 0$ for every $c < v < d$ or $\phi(v) < 0$ for every $c < v < d$. Without loss of generality $\phi(v) < 0$ for every $c < v < d$. Note that since $\phi(v) < 0$ for every $v \in (c, d)$, $q^B(v) = 1$ for every such v. Therefore, $Q^B(d, \mathbf{q}) - Q^B(c, \mathbf{q}) = \tilde{Q}^B(\varphi(d, \mathbf{q})) - \tilde{Q}^B(\varphi(c, \mathbf{q})) = \tilde{Q}^B(\varphi(d, \mathbf{q})) - \tilde{Q}^B(\varphi(d, \mathbf{q}) + F(d) - F(c))$. Since $\tilde{Q}^B$ is decreasing in $[0, 1]$,

$$Q^B(d, \mathbf{q}) > Q^B(c, \mathbf{q}). \tag{23}$$

Recall that by Lemma 4.1, $U^B(v, \mathbf{q}) = U^A(c, \mathbf{q}) + \int_c^v Q^A(x, \mathbf{q})dx$ and similarly, $U^B(v, \mathbf{q}) = U^B(c, \mathbf{q}) + \int_c^v Q^B(x, \mathbf{q})dx$. Therefore, $\phi(v) = \int_c^v (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))dx$. We claim that $Q^A(c, \mathbf{q}) \leq Q^B(c, \mathbf{q})$. Indeed, if $Q^A(c, \mathbf{q}) - Q^B(c, \mathbf{q}) > 0$, for sufficiently small $\epsilon > 0$, $Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}) > 0$ for every $x \in [c, c + \epsilon]$. Therefore for every $v \in (c, c + \epsilon]$, $\phi(v) > 0$ contradicting our negation assumption. Similarly, since $\phi(v) = -\int_v^d (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))dx$, $Q^A(d, \mathbf{q}) \geq Q^B(d, \mathbf{q})$. Hence we have:

$$Q^A(c, \mathbf{q}) \leq Q^B(c, \mathbf{q}) < Q^B(d, \mathbf{q}) \leq Q^A(d, \mathbf{q}). \tag{24}$$

However, since $q^A(v) = 0$ for every $c < v < d$, $Q^A(c, \mathbf{q}) = Q^A(d, \mathbf{q})$ contradicting (24). Therefore there exists $c < z < d$ for which $\phi(z) = 0$. This completes the proof. $\square$

Recall that $U^A(0, \mathbf{q}) - U^B(0, \mathbf{q}) = 0$. Define

$$d = \sup\{v \in [0, 1] |\ U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0\}. \tag{25}$$

By continuity ,$U^A(d,\mathbf{q})-U^B(d,\mathbf{q})=0$, and by Lemma 4.7, $U^A(v,\mathbf{q})-U^B(v,\mathbf{q})=0$ for every $0\le v\le d$. We claim that if $d<1$ then $U^A(v,\mathbf{q})>U^B(v,\mathbf{q})$ for every $v>d$. By Lemma 4.7 either $U^A(v,\mathbf{q})>U^B(v,\mathbf{q})$ or $U^A(v,\mathbf{q})<U^B(v,\mathbf{q})$ for every $v>d$. Suppose in negation that $U^A(v,\mathbf{q})<U^B(v,\mathbf{q})$ for all $v>d$. Hence $q^B(v)=1$ for every $v>d$. Therefore, by Lemma 4.2 $Q^B(v,\mathbf{q})=\tilde{Q}^B(\varphi(v,\mathbf{q}))<\tilde{Q}^B(0)=\beta_1$. Therefore, since $Q^A(v,\mathbf{q})=\alpha_1$ for every $v>d$, by Lemma 4.1 $U^A(v,\mathbf{q})=U^A(d,\mathbf{q})+\int_d^v Q^A(x,\mathbf{q})=U^A(d,\mathbf{q})+\alpha_1(v-d)\ge U^B(d,\mathbf{q})+\beta_1(v-d)>U^B(d,\mathbf{q})+\int_d^v Q^B(x,\mathbf{q})=U^B(v,\mathbf{q})$, contradicting our negative assumption. Hence, in order to establish (22), it suffices to prove that $d=v^*$. Before we do it, note that since $U^A(v,\mathbf{q})-U^B(v,\mathbf{q})=0$ for every $0\le v\le d$, the derivative of $U^A(v,\mathbf{q})-U^B(v,\mathbf{q})$ equals 0 almost every where in this interval. Hence, $Q^A(v,\mathbf{q})-Q^B(v,\mathbf{q})=0$ almost everywhere. However, since Q^A,Q^B are continuous, the equality holds everywhere, that is

$$Q^A(v,\mathbf{q})-Q^B(v,\mathbf{q})=0 \ v\in[0,d]. \tag{26}$$

Assume in negation that $v^*<d$. By Lemma 4.5, $Q^A(v,\mathbf{q})-Q^B(v,\mathbf{q})>0$ for every $v>v^*$ contradicting (26).

Assume in negation that $d<v^*$. Let $d<z<v^*$. By Lemma 4.5, $Q(0,z)<0$. On the other hand, $\varphi(z,\mathbf{q})=\int_z^1 q^B(x)dF(x)=0$ since for $x>d$, $q^B(x)=0$. Therefore, $Q(\varphi(z,\mathbf{q}),v)=Q^A(z,\mathbf{q})-Q^B(v,\mathbf{q})=Q(0,z)<0$. Let $d<v<v^*$, $U^A(v,\mathbf{q})-U^B(v,\mathbf{q})=\int_d^v(Q^A(z,\mathbf{q})-Q^B(v,\mathbf{q}))dz<0$ contradicting. This completes the proof of uniqueness.

4.4 Degenerate competition

Recall that the competition is degenerate when $\alpha_n\ge\beta_1$ and that in such case $n=k^A$. In this case it can be easily seen that the strategy profile in which every bidder chooses A with probability 1 is in equilibrium. Indeed, If bidder i with valuation v_i deviates to B he receives a utility of $\beta_1 v_i$, while if he stays in A, his minimal utility equals $\alpha_n v_i$. Therefore, the existence assertion follows by letting $v^*=0$. Regarding essential uniqueness, if $\mathbf{q}=(q^A,q^B)$ is a symmetric equilibrium and that $\varphi(0,\mathbf{q})>0$, there exists $0<c<1$ such that $q^B(c)>0$ and $Prob_F(v\in(c,1],q^B(v)>0)>0$. Therefore, $\varphi(c,\mathbf{q})>0$ and therefore $F^B_{\mathbf{q}}(c)<1$. Let i be an arbitrary bidder, and let $v_i=c$. When every other bidder uses $\mathbf{q}$, the utility of i cannot exceed $\beta_1 v_i F^B_q(c)<\beta_1 v_i\le\alpha_n v_i$. Hence, bidder i is better off deviating to A, contradicting

$q^B(c) > 0$. Hence, in every symmetric equilibrium, $\mathbf{q}$, $q^B(v) = 0$ for almost every $v \in [0, 1]$. $\square$

4.5 Examples for Special Cases

Example 1 ($k^A = k^B = 2$) First we consider two ad auctions, each with two positions, that is, $k^A = k^B = 2$. In addition it is assumed that $\alpha_1 > \beta_1 > \alpha_2$. When $n = 2$ the structure of equilibrium is revealed analytically.

By (12) and (13), $\tilde{Q}^A(x, y) = \alpha_1(x + y) + \alpha_2(1 - x - y)$, and $\tilde{Q}^B(x) = \beta_1(1 - x) + \beta_2(x)$. Recall that $Q(x, y) = \tilde{Q}^A(x, F(y)) - \tilde{Q}^B(x)$, and that v^* is the unique solution of $Q(0, v^*) = 1$. Hence, $v^* = F^{-1}(\frac{\beta_1 - \alpha_2}{\alpha_1 - \alpha_2})$. The function $h(v)$ is determined by $Q(h(v), v) = 0$. Hence, $h(v) = \frac{\beta_1 - \alpha_2 - (\alpha_1 - \alpha_2)F(v)}{\alpha_1 - \alpha_2 + \beta_1 - \beta_2}$. Since, $\tilde{q}^B(v) = -\frac{h'(v)}{f(v}$, the essentially unique equilibrium $\tilde{\mathbf{q}}$ satisfies:

$$\tilde{q}^B(v) = \begin{cases} \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_2 + \beta_1 - \beta_2} & 0 < v < v^*; \\ 0 & v^* < v \leq 1. \end{cases} \tag{27}$$

Hence, bidders with high valuations participate with probability 1 in auction A, while a bidder with a low valuation randomizes and assigns a constant probability to each of the auctions at the interval $(0, v^*)$. Note that if $\alpha_1 - \alpha_2 > \beta_1 - \beta_2$, $\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_2 + \beta_1 - \beta_2} > \frac{1}{2}$, that is, a bidder with a low valuation assigns a higher probability to the weaker auction.

When $n = 2$, the function $h(v)$ is determined by a polynomial equation in $h(v)$ of degree 1, and therefore $\tilde{q}^B(v)$ is constant at the first interval. However, for $n > 2$, $h(v)$ is determined by a polynomial equation of degree greater than 1, and $\tilde{q}^B(v)$ is not a constant function.

Example 2 (Strategies are not monotone) It is interesting to note that $\tilde{q}^B(v)$ may be increasing or decreasing in $(0, v^*)$. This is shown in the examples illustrated in Figures 1 and 2, in which the equilibrium is computed by running a computerized method. In both examples $n = 4$, $k^A = k^B = 4$ and F is the uniform distribution. In Figure 1 the click rate vectors of ad auctions A and B are $\alpha = (100, 70, 50, 20)$ and $\beta = (80, 30, 10, 5)$ respectively, and the cutting point is $v^* = 0.76$. In Figure 2 the click rate vectors in ad auctions A and B are $\alpha = (90, 80, 60, 30)$ and $\beta = (85, 70, 40, 510)$ respectively, and the cutting point is $v^* = 0.85$.

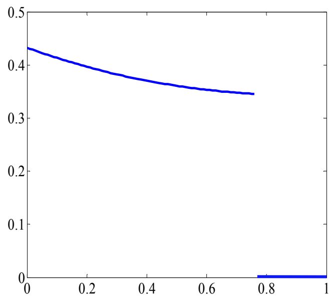

Figure 1.

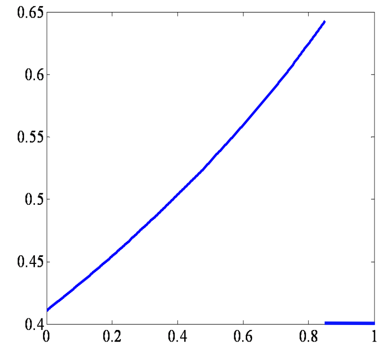

Figure 2.

5 Proof of Theorem 3.5

Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions such that A is stronger than B, and let F be a standard distribution function. Let R^A (R^B) be the expected revenue in auction A (B) at the essentially unique symmetric equilibrium, $\tilde{\mathbf{q}}$ in $H(A, B, F)$. We have to prove that

$$R^A - R^B > 0.$$

Obviously, for $L \in \{A, B\}$, $R^L = n\int_0^1 P^L(v, \tilde{\mathbf{q}})\tilde{q}^L(v)f(v)dv$. In the rest of the proof $\tilde{\mathbf{q}}$ is fixed and therefore omitted from the description of the functions. By the proof of Theorem 3.3 and by (7), $P^A(v) = P^B(v)$ for every $v \in [0, v^*]$, and because, in addition, $\tilde{q}^B(v) = 0, \tilde{q}^A(v) = 1$ for every $v^* \le v \le 1$.

$$\frac{R^A - R^B}{n} = \int_0^{v^*} P^A(v)(1 - 2\tilde{q}^B(v))f(v)dv + \int_{v^*}^1 P^A(v)f(v)dv.$$

Let $\Delta = \frac{R^A - R^B}{n}$. Since for $v > v^*$, $\tilde{q}^B(v) = 0$, $1 = 1 - 2\tilde{q}^B(v)$ for such v, and therefore

$$\Delta = \int_0^1 P^A(v)(1 - 2\tilde{q}^B(v))f(v)dv.$$

Plug in $P^A(v) = vQ^A(v) - U^A(v)$ in the last equality to get

$$\Delta = \int_0^1 vQ^A(v)(1 - 2\tilde{q}^B(v))f(v)dv - \int_0^1 U^A(v)(1 - 2\tilde{q}^B(v))f(v)dv.$$

Since $U^A(v) = \int_0^v Q^A(x)dx$, the second term in the right-hand-side of the last equality is a double integral. By changing the order of the integrals in this second term, and move to the parameter x, we get

$$\Delta = \int_0^1 Q^A(x)[xf(x) - 2x\tilde{q}^B(x)f(x) - 1 + F(x) + 2\varphi(x)]dx. \tag{28}$$

We are about to apply the method of integration by parts to the right-hand-side of (28). For that matter let $g(x) = xf(x) - 2x\tilde{q}^B(x)f(x) - 1 + F(x) + 2\varphi(x)$. Therefore,

$$\Delta = \int_0^1 Q^A(x)g(x)dx. \tag{29}$$

Let $G(x) = \int_0^x g(t)dt$. We claim that the derivative $Q^A_x(x)$ exists at each of the intervals $(0, v^*)$ and $(v^*, 1)$ and it is bounded at each of these intervals, and that $G(1) = 0$. Therefore, by integration by parts

$$\Delta = -\int_0^1 Q^A_x(x)G(x)dx. \tag{30}$$

Indeed, for $0 < x < v^*$, $Q^A(x) = Q^B(x) = \tilde{Q}^B(h(x))$. Therefore, since h is continuously differentiable at that interval, $Q^A_x(x) = \tilde{Q}^B_x(h(x))h'(x)$. For $v^* < x < 1$, $Q^A(x) = \tilde{Q}^A(0, F(x))$, and therefore $Q^A_x(x) = \tilde{Q}^A_x(0, F(x))f(x)$. It remains to show that $G(1) = 0$. Indeed, it is easily verified that $G(x) = xF(x) + 2x\varphi(x) - x$ for every $0 \le x \le 1$. Hence, $G(1) = 0$. Since by what we showed above $Q^A_x(x) > 0$ except at most three values of x, in order to prove that $\Delta < 0$ it suffices to prove that

$$G(x) = xF(x) + 2x\varphi(x) - x < 0 \text{ for every } 0 < x < 1. \tag{31}$$

For every $v^* \le x < 1$, $G(x) = xF(x) - x < 0$. So, it remains to prove (31) for $0 < x < v^*$. In order to prove it we use the fact that A is stronger than B. By re-arranging the terms in (31), we have to prove that

$$\varphi(x) < \frac{1 - F(x)}{2} \text{ for every } 0 < x < v^*. \tag{32}$$

Let $x \in (0, v^*)$ and let $\tilde{h}(x) = \frac{1-F(x)}{2}$. Note that $\tilde{h}(x) + F(x) = \frac{1+F(x)}{2}$ and $1 - \tilde{h}(x) - F(x) = \frac{1-F(x)}{2}$. Therefore, by (12) and (13)

$$\tilde{Q}^A(\tilde{h}(x), F(x)) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (\frac{1+F(x)}{2})^{n-j} (\frac{1-F(x)}{2})^{j-1} \tag{33}$$

and

$$\tilde{Q}^B(\tilde{h}(x)) = \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} (\frac{1+F(x)}{2})^{n-j} (\frac{1-F(x)}{2})^{j-1}. \tag{34}$$

Therefore, $Q(\tilde{h}(x), x) = \tilde{Q}^A(\tilde{h}(x), F(x)) - \tilde{Q}^B(\tilde{h}(x)) > 0$ since A is stronger than B. Let $0 < x < v^*$. Recall that the function $Q(v, x)$ is increasing in $0 \leq v \leq 1 - F(x)$. As $h(x), \tilde{h}(x)$ are in this interval, and $Q(h(x), x) = 0 < Q(\tilde{h}(x), x)$, $h(x) < \tilde{h}(x)$. Since, in $(0, v^*)$, $h(x) = \varphi(x)$, (32) follows. $\square$

6 Regular Ad Auctions

Consider two ad auctions $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$, Our goal in this section is to provide conditions under which theorems 3.3 and 3.5 hold when A and B are not necessarily VCG ad auctions.

Let $q : [0, 1] \to [0, 1]$ be a measurable function. For every standard distribution function F, regard q as a non-strategic participation function used by each of the n bidders in some ad auction. That is, $q(v)$ is the probability that a bidder participates strategically in the bidding. For technical reasons, in order to keep the assumption that all bidders participate, whenever the realization implies that the bidder does not participate, it is modeled as if this bidder participates but his valuation equals 0. This forms a new ad auction in which the distribution function is

$$F_q(v) = F(v) + \int_v^1 (1 - q(x))dF(x).$$

Let $D = G(k, \alpha, \mathbf{p})$ be an ad auction, and let F be a standard distribution function. For every q let $D(q) = G(k, \alpha, p, F_q)$ be the ad auction with F_q. We say that D is a *regular* ad auction if for every standard distribution function F and for every participation function, q, $D(q)$ possesses a symmetric equilibrium in which every bidder uses a bidding strategy, which is non-decreasing and equals an increasing function almost everywhere with respect to F_q. That is, there exists an increasing strategy in $[0, 1]$ which equals the equilibrium strategy almost everywhere with respect to F_q. In particular, if F_q is constant at a certain interval, the symmetric equilibrium is not necessarily increasing there.

Let $A = G(k^A, \alpha, \mathbf{p}^A)$, $B = G(k^B, \beta, \mathbf{p}^B)$ be regular ad auctions, and let F be a distribution function. When A and B are VCG ad auctions we reduce the strategy set of each bidder to the set of participation strategies by assuming that the bidding strategies are the truth-telling strategies. In general, we replace this assumption with the following one: A strategy for a bidder is a pair $(\mathbf{q}, \mathbf{d})$ in which d^A is a symmetric equilibrium strategy in $A(q^A)$, which is increasing almost everywhere with respect to the probability induced by F_{q^A} and d^B is a symmetric equilibrium strategy in $B(q^B)$, which is increasing almost everywhere with respect to the probability induced by F_{q^B}; Note that because the bidding strategies are increasing, the expected click rate functions in $H(A, B, F)$ are precisely the expected click rate functions, Q^A, Q^B defined at (8) and (9). Moreover, Proposition 4.1 continues to hold with the same proof.[8] Therefore, the following theorem holds:

Theorem 6.1 *Theorems 3.3 and 3.5 are valid for regular ad auctions.*

The question of which auctions are regular is not dealt with in this paper. However we conjecture that the first-price and next-price ad auctions are indeed regular. In order to prove such results it is recommended to consult (Lahaie, 2006), where the existence and uniqueness of a symmetric equilibrium is proved for a first-price ad auction with a standard distribution function, F, but not for F_q, as well as (Lebrun, 2006), where equilibrium in first-price auctions (which can be associated with ad auctions with a single position) are discussed for general distribution functions.

7 Reserve Prices

In this section we extend our study to the context of ad auctions with reserve prices.

Every ad auction may have a reserve price r, $0 \leq r < 1$, which is modeled as follows: The allocation rule allocates the non-dummy positions only to the bidders whose bid is at least r, and all other bidders receive the dummy position; A bidder's payment is calculated as if all other bidders whose bid is less than r bid r. An auction $G(k, \alpha, \mathbf{p})$ with a reserve price r is denoted by $G(r, k, \alpha, \mathbf{p})$. Obviously, $G(0, k, \alpha, \mathbf{p})$ is equivalent to $G(k, \alpha, \mathbf{p})$. The competition game with two ad auctions A,B with reserve prices r^A and r^B respectively, and

[8] This proposition is just a version of the Utility Equivalence principle, which is implicitly proved in (Rockafellar, 1970) when dealing with sub-gradients of convex functions, and has been explicitly extended and proved in mechanism design by Holmstrom (1979) (see also (Myerson, 1981), (Hon-Snir, 2005)). The Utility Equivalence principle is used to prove the more famous, Revenue Equivalence principle, see e.g., (Myerson, 1981).

with a distribution function F is denoted by $H(r^A, r^B, A, B, F)$. The definition of essential uniqueness of symmetric equilibrium is naturally changed: The game $H(r^A, r^B, A, B, F)$ possesses an *essentially unique symmetric equilibrium* if it possesses a symmetric equilibrium and for every two symmetric equilibria $\mathbf{q}, \hat{\mathbf{q}}$

$$\mathbf{q}(v) = \hat{\mathbf{q}}(v) \ F\text{−almost everywhere in } [\min\{r^A, r^B\}, 1].$$

The structure of symmetric equilibrium in $H(r^A, r^B, A, B, F)$ is described by an additional cutting point denoted by w^*.

Theorem 7.1 *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions, let $0 \leq r^A, r^B < 1$, and let F be a standard distribution function.*

The game $H(r^A, r^B, A, B, F)$ possesses an essentially unique symmetric equilibrium.

Moreover, If $\alpha_1 \geq \beta_1$ there exist a unique pair of cutting points, $0 \leq v^* \leq 1$, $\max\{r^A, r^B\} \leq w^* \leq 1$ for which there exists a symmetric equilibrium, $\mathbf{q} = (q^A, q^B)$ with the following properties:

$$\text{If } r^A \leq r^B : \begin{cases} q^B(v) = 0 & \text{for every } r^B < v < w^*; \\ 0 < q^B(v) < 1 & \text{for every } \min\{w^*, v^*\} < v < v^*; \\ q^B(v) = 0 & \text{for every } \max\{v^*, w^*\} < v \leq 1. \end{cases} \tag{35}$$

$$\text{If } r^B < r^A : \begin{cases} q^B(v) = 1 & \text{for every } r^A < v < w^*; \\ 0 < q^B(v) < 1 & \text{for every } \min\{w^*, v^*\} < v < v^*; \\ q^B(v) = 0 & \text{for every } \max\{v^*, w^*\} < v \leq 1. \end{cases} \tag{36}$$

Furthermore, $v^* = 0$ if and only if $\alpha_n \geq \beta_1$, $v^* = 1$ if and only if $\alpha_1 = \beta_1$, and $w^* = \max\{r^A, r^B\}$ if and only if $r^A = r^B$.

Theorem 7.1 is proved in Subsection 7.1.

Note that in the case in which $\alpha_1 > \beta_1, \alpha_n < \beta_1$ and $r^A < r^B$, the new parameter w^* satisfies $r^B < w^*$, and a bidder with a valuation between r^B and w^* participates in A with probability 1. Hence, bidders with low valuations above the maximal reserve price and bidders with high valuations participate in A. Only bidders with interim size of valuations randomize. Such a phenomenon cannot happen when $r^A = r^B$, and in particular it cannot happen when both auctions do not have reserve prices. A typical structure of the equilibrium for the case in which $r^A < r^B$ is illustrated in Figure 3.

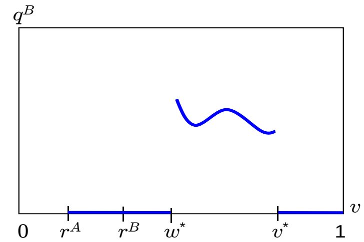

Figure 3.

Regarding revenue, Theorem 3.5 is extended as follows:

Theorem 7.2 *Let $A = G(k^A, \alpha, \mathbf{p}^A)$ and $B = G(k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions, and let F be a standard distribution function. If $r^A \leq r^B$ and A is stronger than B, the expected revenue in A is greater than the expected revenue in B in the essentially unique symmetric equilibrium of the game $H(r^A, r^B, A, B, F)$.*

The proof of Theorem 7.2 is given in Subsection 7.2.

Note one conclusion of Theorem 7.2: When the stronger auction sets a non strategic reserve price 0, even if the weaker auction sets an optimal reserve price, its expected revenue will be less than the one in A, that is, click rates are more influential for revenue than reserve prices; If auction B's organizer wishes to have a higher revenue than the one in A, she must improve her performance and provide better click rates.[9]

The proofs of the theorems in this section combine extensions of the techniques in previous sections with extensions of techniques established in (Burguet and Sakovics, 1999).

A very interesting question to ask is what reserve prices will the machanism designers would choose if they could do so? Unfortunately, there is no sub-game perfect equilibrium (when sellers cannot use mixed strategies) in such a setting as mentioned in the introduction.

[9] This is a complementary principle to the one established by Bulow and Klemperer (1996) according to which, the number of participants in a single-item auction is more influential for revenue than the reserve price.

7.1 Proof of Theorem 7.1

The definitions, theorems and proofs in Subsection 4.1 are valid for the setup with reserve prices with the following modifications: At the presence of reservation prices the formulas for the expected click rate functions are as follows:

$$Q^A(v,\mathbf{q}) = \begin{cases} \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (F_q^A(v))^{n-j}(1-F_q^A(v))^{j-1} & v \geq r^A \\ 0 & v < r^A. \end{cases} \tag{37}$$

$$Q^B(v,\mathbf{q}) = \begin{cases} \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} (F_q^B(v))^{n-j}(1-F_q^B(v))^{j-1} & v \geq r^B \\ 0 & v < r^B. \end{cases} \tag{38}$$

Equalities (i) and (ii) in (15) continue to hold for $v \geq r^A$ and $v \geq r^B$ respectively, and (16) holds for $v \geq max\{r^A, r^B\}$.

Finally, in part 3 in Lemma 4.5 one should require that the inequality holds for every $v > max\{v^*, r^A\}$.

We proceed with the proof of the theorem assuming $r^A \leq r^B$. The proof for the other case is very similar and therefore it is omitted.

7.1.1 Existence

Define a family of strategies $\tilde{\mathbf{q}}_w = (\tilde{q}^A_w, \tilde{q}^b_w)$, parameterized by $w \in [0,1]$, as follows: The values $\tilde{q}^B_w(0)$,$\tilde{q}^B_w(w)$ and perhaps $\tilde{q}^B_{v^*}(v^*)$ are left unspecified, and for any other $0 < v \leq 1$

$$\begin{cases} \tilde{q}^B_w(v) = 0 & \text{for every } 0 < v < w; \\ -\frac{h'(v)}{f(v)} & \text{for every } \min\{w, v^*\} < v < v^*; \\ \tilde{q}^B_w(v) = 0 & \text{for every } \max\{v^*, w\} < v \leq 1, \end{cases} \tag{39}$$

where the function h is the unique function on $[0, v^*]$ established in Proposition 4.4. Note that $\tilde{\mathbf{q}}_0$ coincides with the symmetric equilibrium, $\tilde{\mathbf{q}}$ of the game $H(A, B, F)$, except for at most three points. Moreover, by similar arguments to those applied in Subsection 4.2, for every $w \leq v^*$

$$Q^A(v, \tilde{\mathbf{q}}_w) = Q^B(v, \tilde{\mathbf{q}}_w) \text{ for every } w \leq v \leq v^*. \tag{40}$$

Let $D = \{w \in [0,1] : \; r^B \leq w \leq 1, U^A(w, \mathbf{q}_w) \leq U^B(w, \mathbf{q}_w)\}$. Define

$$w^* = \begin{cases} \inf D & D \neq \phi \\ 1 & D = \phi. \end{cases} \tag{41}$$

We argue that $\tilde{\mathbf{q}}_{w^*}$ is a symmetric equilibrium. To show this, it suffices to prove the following three claims:

(a) $U^A(v, \tilde{\mathbf{q}}_{w^*}) - U^B(v, \tilde{\mathbf{q}}_{w^*}) = 0$ for every $\min\{w^*, v^*\} < v < v^*$;

(b) $U^A(v, \tilde{\mathbf{q}}_{w^*}) - U^B(v, \tilde{\mathbf{q}}_{w^*}) \geq 0$ for every $v \geq \max\{v^*, w^*\}$.

(c) $U^A(v, \tilde{\mathbf{q}}_{w^*}) - U^B(v, \tilde{\mathbf{q}}_{w^*}) \geq 0$ for every $r^B \leq v \leq w^*$.

Note that

$$\varphi(v, \tilde{\mathbf{q}}_{w^*}) = \varphi(v, \tilde{\mathbf{q}}) \text{ for every } v \geq w^*. \tag{42}$$

Therefore, the proofs for (a) and (b) are established by using similar arguments to those used at the analogous proofs in Subsection 4.2. We proceed to prove (c). By (41) $w^* \geq r^B$. If $w^* = r^B$, then since $U^B(r^B, \tilde{\mathbf{q}}_{w^*}) = 0$ we are done. Therefore, we assume that $r^B < w^*$. We distinguish two cases.

Case 1: $w^* \leq v^*$. Note that

$$U^A(r^B, \tilde{\mathbf{q}}_{w^*}) \geq U^B(r^B, \tilde{\mathbf{q}}_{w^*}) = 0, \tag{43}$$

and by (a) and the continuity of U^A, U^B in v,

$$U^A(w^*, \tilde{\mathbf{q}}_{w^*}) = U^B(w^*, \tilde{\mathbf{q}}_{w^*}). \tag{44}$$

We are about to prove that

$$Q^B(v, \tilde{\mathbf{q}}_{w^*}) \geq Q^A(v, \tilde{\mathbf{q}}_{w^*}) \quad \forall v \in (r^B, w^*). \tag{45}$$

However, in latter proofs we will need a stronger result, which is proved at the next lemma.

Lemma 7.3 *Let* $0 \leq w \leq 1$*. For every* $r^B \leq v < min\{v^*, w\}$, $Q^B(v, \tilde{\mathbf{q}}_w) \geq Q^A(v, \tilde{\mathbf{q}}_w)$.

Proof: Since $\tilde{q}^B_w(v) = 0$ for every $r^B < v < w$, $\varphi(v, \tilde{\mathbf{q}}_w) = \varphi(w, \tilde{\mathbf{q}}_w)$ for every such v. Therefore

$$Q^B(v, \tilde{\mathbf{q}}_w) = Q^B(w, \tilde{\mathbf{q}}_w) \quad \forall v \in (r^B, min\{v^*, w\}). \tag{46}$$

We consider the following two cases:

1. $w \leq v^*$: Since Q^A is non-decreasing in $v \in [0, 1]$

$$Q^A(w, \tilde{\mathbf{q}}_w) \geq Q^A(v, \tilde{\mathbf{q}}_w) \quad \forall v \in [r^B, w). \tag{47}$$

   By (40) $Q^B(w, \tilde{\mathbf{q}}_w) = Q^A(w, \tilde{\mathbf{q}}_w)$. Therefore, together with (46) and (47) the result follows.

2. $w > v^*$: Since Q^A is non-decreasing in $v \in [0, 1]$

$$Q^A(v^*, \tilde{\mathbf{q}}_w) \geq Q^A(v, \tilde{\mathbf{q}}_w) \quad \forall v \in [r^B, v^*). \tag{48}$$

   Moreover, by (46) we have that

$$Q^B(v, \tilde{\mathbf{q}}_w) = Q^B(v^*, \tilde{\mathbf{q}}_w). \quad \forall v \in [r^B, v^*). \tag{49}$$

   By (48) and (49) it remains to show that

$$Q^B(v^*, \tilde{\mathbf{q}}_w) \geq Q^A(v^*, \tilde{\mathbf{q}}_w). \tag{50}$$

   By (40) we have that $Q^B(v^*, \tilde{\mathbf{q}}_{v^*}) = Q^A(v^*, \tilde{\mathbf{q}}_{v^*})$. Hence, $Q(\varphi(v^*, \tilde{\mathbf{q}}_{v^*}), v^*) = 0$. Since $v^* < w$, by (39) $\varphi(v^*, \tilde{\mathbf{q}}_w) < \varphi(v^*, \tilde{\mathbf{q}}_{v^*})$. Therefore, by Lemma 4.2 $Q(\varphi(v^*, \tilde{\mathbf{q}}_w), v^*) < 0$, which implies that inequality (50) holds.

□

We proceed with the main proof. By (43), (45), and Proposition 4.1, if $U^B(v', \tilde{\mathbf{q}}_{w^*}) > U^A(v', \tilde{\mathbf{q}}_{w^*})$ for some $v' \in (r^B, w^*)$, $U^B(v, \tilde{\mathbf{q}}_{w^*}) > U^A(v, \tilde{\mathbf{q}}_{w^*})$ for every $v' \leq v \leq w^*$, contradicting (44). Therefore (c) follows for the case $w^* \leq v^*$.

Case 2: $w^* > v^*$. By (41), $U^A(v^*, \tilde{\mathbf{q}}_{v^*}) > U^B(v^*, \tilde{\mathbf{q}}_{v^*})$. Therefore, since $\tilde{q}^B_{w^*}$ equals $\tilde{q}^B_{v^*}$ almost everywhere at $[r^A, 1]$,

$$U^A(v^*, \tilde{\mathbf{q}}_{w^*}) > U^B(v^*, \tilde{\mathbf{q}}_{w^*}). \tag{51}$$

In addition, by Lemma 7.3,

$$Q^B(v, \tilde{\mathbf{q}}_{w^*}) \geq Q^A(v, \tilde{\mathbf{q}}_{w^*}) \quad \forall v \in [r^B, v^*). \tag{52}$$

By Proposition 4.1 for every $L \in \{A, B\}$ and for every $r^B \leq v < v^*$ $U^L(v, \tilde{\mathbf{q}}_{w^*}) = U^L(v^*, \tilde{\mathbf{q}}_{w^*}) - \int_v^{v^*} Q^L(v, \tilde{\mathbf{q}}_{w^*})$. Therefore, by (51) and (52) $U^A(v, \tilde{\mathbf{q}}_{w^*}) > U^B(v, \tilde{\mathbf{q}}_{w^*})$ for every $r^B < v < v^*$. Moreover, by Lemma 4.5 $Q^A(v, \tilde{\mathbf{q}}_{w^*}) \geq Q^B(v, \tilde{\mathbf{q}}_{w^*})$ for every $v \geq v^*$. Therefore, by Proposition 4.1 and (51) we obtain that $U^A(v, \tilde{\mathbf{q}}_{w^*}) \geq U^B(v, \tilde{\mathbf{q}}_{w^*})$ for every $v \geq v^*$, which completes the proof of part (c). □

7.1.2 Uniqueness

We prove that $\tilde{\mathbf{q}}_{w^*}$ is essentially unique. Let $\mathbf{q}$ be a symmetric equilibrium. As in the uniqueness proof for the case with no reserve prices given in Subsection 4.3, it suffices to show that

$$\begin{cases} U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0 & \text{for every } \min\{w^*, v^*\} < v < v^*; \\ U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) > 0 & \text{for every } v > \max\{v^*, w^*\}; \\ U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) > 0 & \text{for every } r^A < v < w^*. \end{cases} \tag{53}$$

We will use the following Lemma, whose proof is clear and therefore omitted.

Lemma 7.4 *For every $v \in [r^A, r^B]$,*

$$U^A(v, \mathbf{q}) \geq 0 \text{ and } U^B(v, \mathbf{q}) = 0.$$

Moreover, $U^A(v, \mathbf{q}) > 0$ if and only if $r^A < r^B$.

We deal first with the simple case in which $r^B \geq v^*$. By (41) $w^* \geq r^B$ and therefore $w^* \geq v^*$. Hence, we need to show only the two strict inequalities in (53). By Lemma 7.4, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) > 0$ for every $r^A < v \leq r^B$. Therefore, it is enough to show that

$$U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) > 0 \text{ for every } v > r^B. \tag{54}$$

By Lemma 7.4, $U^A(r^B, \mathbf{q}) \geq U^B(r^B, \mathbf{q})$. Moreover, by Lemma 4.5, $Q^A(v, \mathbf{q}) > Q^B(v, \mathbf{q})$ for every $v \geq r^B$. Therefore, (54) follows from Proposition 4.1.

Henceforth, we assume that $r^B < v^*$. Lemma 4.7, which plays a key role in the proof of uniqueness for the case $r^A = r^B = 0$ should be slightly modified. The proof of the modified lemma mimics the proof of the original lemma, and therefore it is omitted:

Lemma 7.5 *Let $r^A \leq c < d \leq 1$ be two valuations for which $U^A(c, \mathbf{q}) - U^B(c, \mathbf{q}) = 0$ and $U^A(d, \mathbf{q}) - U^B(d, \mathbf{q}) = 0$. Then, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0$ for every $c \leq v \leq d$.*

Let $T = \{v : v \geq r^A, U^B(v, \mathbf{q}) \geq U^A(v, \mathbf{q})\}$ and define

$$t = \begin{cases} \inf T & T \neq \phi \\ 1 & T = \phi. \end{cases} \tag{55}$$

By similar arguments to the case with no reserve prices and by (55), we have

Lemma 7.6

(a) $U^A(v, \mathbf{q}) > U^B(v, \mathbf{q})$ *for every* $r^A < v < t$.

(b) $U^A(v, \mathbf{q}) = U^B(v, \mathbf{q})$ *for every* $\min\{t, v^*\} < v < v^*$.

(c) $U^A(v, \mathbf{q}) > U^B(v, \mathbf{q})$ *for every* $v > \max\{t, v^*\}$.

Since $\mathbf{q}$ is an equilibrium, $t \geq r^B$. Hence, by Lemma 7.6, q^B equals $\tilde{q}_t^B$ almost everywhere at $[r^A, 1]$. In particular $\tilde{\mathbf{q}}_t$ is a symmetric equilibrium. By definition of w^*, (41), $t \geq w^*$. Therefore, to complete the proof, it remains to show that if $w^* < v^*$, $t = w^*$. Suppose in negation that $w^* < v^*$ and $t > w^*$. We prove that

$$U^A(w^*, \tilde{\mathbf{q}}_t) < U^B(w^*, \tilde{\mathbf{q}}_t). \tag{56}$$

Since $\tilde{q}_t^B(w^*) = 0$ and $\tilde{\mathbf{q}}_t$ is a symmetric equilibrium, (56) implies the desired contradiction.

Note that $U^A(r^A, \mathbf{q}_{w^*}) = U^A(r^A, \mathbf{q}_t) = 0$ and that $U^B(r^B, \mathbf{q}_{w^*}) = U^B(r^B, \mathbf{q}_t) = 0$. Since $\varphi(v, \tilde{\mathbf{q}}_{w^*}) > \varphi(v, \tilde{\mathbf{q}}_t)$ for every $r^A < v \leq w^*$, by Lemma 4.2 and (15)

$$Q^A(v, \tilde{\mathbf{q}}_{w^*}) > Q^A(v, \tilde{\mathbf{q}}_t)$$

and

$$Q^B(v, \tilde{\mathbf{q}}_{w^*}) < Q^B(v, \tilde{\mathbf{q}}_t)$$

for every $r^A < v \leq w^*$.

Therefore, by Proposition 4.1, $U^B(w^*, \tilde{\mathbf{q}}_t) > U^B(w^*, \tilde{\mathbf{q}}_{w^*})$ and $U^A(w^*, \tilde{\mathbf{q}}_t) < U^A(w^*, \tilde{\mathbf{q}}_{w^*})$. Moreover, since $w^* < v^*$ $U^A(w^*, \tilde{\mathbf{q}}_{w^*}) = U^B(w^*, \tilde{\mathbf{q}}_{w^*})$ and therefore (56) holds. This completes the uniqueness part. □

7.1.3 Proving the if and only if parts

Proving that $v^* = 0$ if and only if $\alpha_n \geq \beta_1$ and that $v^* = 1$ if and only if $\alpha_1 = \beta_1$ is similar to the analogous proof in the case with no reserve prices. We proceed to prove that $w^* = \max\{r^A, r^B\}$ if and only if $r^A = r^B$. Indeed, for every $w \in [0,1]$, $U^A(r^A, \tilde{\mathbf{q}}_w) = U^B(r^B, \tilde{\mathbf{q}}_w) = 0$. Therefore, if $r^A = r^B$, by (41) we have $w^* = r^A = r^B$. Conversely, if $r^A > r^B$, $U^A(r^A, \tilde{\mathbf{q}}_{r^B}) > 0$ and $U^A(r^B, \tilde{\mathbf{q}}_{r^B}) = 0$ implying $w^* > r^B$. □

7.2 Proof of Theorem 7.2

Let $A = G(r^A, k^A, \alpha, \mathbf{p}^A)$ and $B = G(r^B, k^B, \beta, \mathbf{p}^B)$ be VCG ad auctions with $r^A \leq r^B$ such that A is stronger than B, and let F be a standard distribution function. Let R^A,R^B be the expected revenues in auction A and B respectively at the essentially unique symmetric equilibrium, $\tilde{\mathbf{q}}_{w^*}$ of $H(r^A, r^B, A, B, F)$. We have to prove that

$$R^A - R^B > 0.$$

We distinguish two cases. Case 1: $w^* \geq v^*$. In this case, for every $v \geq r^B$ except for at most two values, $\tilde{q}^B_{w^*}(v) = 0$. Therefore, $R^A > R^B$.

Case 2: $w^* < v^*$. Whenever a strategy is omitted from the description of functions we assume it to be the strategy $\tilde{\mathbf{q}}_{w^*}$. By Theorem 7.1 and (7), $P^A(v) = P^B(v)$ for every $v \in [w^*, v^*]$, $P^A(v) = 0$ for every $v < r^A$, and $\tilde{q}^B_{w^*}(v) = 0$ for every $r^A < v < w^*$ and for every $v^* < v \leq 1$. Therefore,

$$\Delta = \int_0^1 P^A(v)(1 - 2\tilde{q}^B_{w^*}(v))f(v)dv,$$

where $\Delta = \frac{R^A - R^B}{n}$. The proof continues similarly to the proof of Theorem 3.5 to the point in which

$$\Delta = -\int_0^1 Q^A_x(x)G(x)dx, \tag{57}$$

where in this case $G(x) = \int_0^x [tf(t) - 2t\tilde{\mathbf{q}}^B_{w^*}(t)f(t) - 1 + F(t) + 2\varphi(t)]dt$.

Since $Q^A_x(v) = 0$ for all $v < r^A$,

$$\Delta = -\int_{r^A}^1 Q^A_x(x)G(x)dx. \tag{58}$$

As in the proof of Theorem 3.5, $Q_x(v) > 0$ for every $r^A < v < v^*$. It remains to show that $G(x) < 0$ for every $r^A < x < v^*$, which is equivalent to proving that

$$\varphi(x, \tilde{\mathbf{q}}_{w^*}) < \frac{1 - F(x)}{2} \text{ for every } r^A < x < v^*. \tag{59}$$

We showed in the proof of Theorem 3.5 that with no reserve prices $\varphi(v, \tilde{\mathbf{q}}) < \frac{1-F(x)}{2}$ for every $0 < x < v^*$. Since $\varphi(x, \tilde{\mathbf{q}}) \geq \varphi(x, \tilde{\mathbf{q}}_{w^*})$ for $x < v^*$, (59) follows. $\Box$

8 Multi-Campaign Advertisers: Sensitivity Analysis of the Revenue in Monopolistic Setups

Our main goal in this section is to derive the general formula for the expected revenue realized by the seller in the VCG auction, and to use it to show that the expected revenue of the VCG ad auction is not necessarily positively influenced by an increase in its click rates. For symmetric equilibria, these results can be extended via the revenue equivalence principle to regular ad auctions.[10] In addition, without any reference to the revenue equivalence theorem, these results are applied below to the most common model in the literature of ad auctions, which involves the next-price ad auction with complete information. Our results have immediate implications for the simultaneous ad auctions with multi-campaign advertisers setup in which both auctions attract all advertisers.

Let $A = G(k, \alpha, \mathbf{p}, \mathbf{F})$ be the standard VCG ad auction. Denote by $R^A = R(k, \alpha, \mathbf{p}, \mathbf{F})$ the expected revenue realized by the seller in A calculated under the assumption that every bidder uses the truth-telling strategy.[11]

In this section we address the natural question:

$$\text{Do more clicks or/and more positions yield more revenue?} \tag{60}$$

[10] See Section 6.

[11] Recall that truth telling is a weakly dominant strategy for every bidder.

Let $\tilde{v}_{(t)}$ be the t^{th} reverse-order statistics generated by the random variables $\tilde{v}_1, \ldots, \tilde{v}_n$. In particular, $\tilde{v}_{(1)} \geq \tilde{v}_{(2)} \geq \ldots \geq \tilde{v}_{(n)}$. $\tilde{v}_{(t)}$ is identically zero for $t > n$. By (2), the expected total payment realized in position l equals

$$\sum_{j=l}^{k} (\alpha_j - \alpha_{j+1}) E[\tilde{v}_{(j+1)}], \tag{61}$$

where $E[\cdot]$ is the expectation operator with respect to the joint distribution of profiles of valuations. Therefore,

$$R^A = \sum_{l=1}^{k} \sum_{j=l}^{k} (\alpha_j - \alpha_{j+1}) E[\tilde{v}_{(j+1)}],$$

implying that

$$R^A = \sum_{j=1}^{k} (\alpha_j - \alpha_{j+1}) j E[\tilde{v}_{(j+1)}], \tag{62}$$

Hence, by (62), increasing all click rates by a positive multiplicative constant increases revenue by the same multiplicative constant. However, as is shown in the next theorem, if the click rates change in a non-linear way, the influence on the revenue is not necessarily positive.

Theorem 8.1 *Let $A = G(k, \alpha, \mathbf{p}, \mathbf{F})$ be the standard VCG ad auction with an arbitrary vector of distribution functions.*

1. *The revenue, $R(k, \alpha, \mathbf{p}, \mathbf{F}) = R(k, (\alpha_1, \alpha_2, \ldots, \alpha_k), \mathbf{p}, \mathbf{F})$, is non-decreasing in α_1; It is increasing in α_1 if and only if $E[\tilde{v}_{(2)}] > 0$.*

2. *For every position j, $2 \leq j \leq k$, The revenue, $R(k, (\alpha_1, \alpha_2, \ldots, \alpha_k), \mathbf{p}, \mathbf{F})$, can be non-decreasing or non-increasing in α_j.*

Proof: By (62)

$$R(k, \alpha, \mathbf{p}, \mathbf{F}) = \alpha_1 E[\tilde{v}_{(2)}] + \sum_{j=2}^{k} \alpha_j \left(j E[\tilde{v}_{(j+1)}] - (j-1) E[\tilde{v}_{(j)}] \right), \tag{63}$$

which proves the theorem. $\square$

So, increasing the click rate of position 1 increases revenue. However, the influence of incresing/decreasing other positions' click rates on the revenue depends on the distribution of valuations. To complete the answer to question (60), note that adding a position is equivalent to increasing the click rate in the current dummy position. Hence, the value of such a policy depends on its effects on the other click rates and on the distribution functions.

In the following example we consider a special important case.

Example 3 (Symmetric model, uniform distribution) Consider the classical model in which, $n > k$, $r = 0$ and each F_i is the uniform distribution over $[0, 1]$. In this model, $E[\tilde{v}_{(j+1)}] = \frac{n-j}{n+1}$, and therefore, for $j \geq 2$,

$$jE[\tilde{v}_{(j+1)}] - (j-1)E[\tilde{v}_{(j)}] > 0 \text{ if and only if } j < \frac{n+1}{2}.$$

Hence, if for example $n = 4$ and $k = 3$, an increase of α_3 reduces the revenue.

Although the VCG ad auction is currently not used in practice, it may be used in the future because of its compelling feature of having dominant strategies. Furthermore, it was shown by Ashlagi et al. (2008) that a reliable mediator can transform the next-price ad auction, which is used in practice to a VCG ad auction, which makes the analysis of the VCG auction relevant. Moreover, Varian (2007) and Edelman et al. (2007) showed that with complete information, the VCG outcome is obtained in an equilibrium of the next-price ad auction. In addition, it was proved in (Edelman et al., 2007) that this equilibrium is locally envy-free, and in (Varian, 2007) it is claimed that this equilibrium is consistent with the empirical data. Hence, as we show in the next example, results about the revenue in the VCG auction can be used for existing systems.

Example 4 (Next-price auctions with complete information) In existing systems, auctions' organizers run variants of the next-price ad auction. This auction has the welfare maximizer allocation rule and its payment scheme is given in (1). In a model with complete information bidder i has a commonly known valuation v_i, and without loss of generality it is assumed that

$$v_1 > v_2 > \cdots > v_n > 0, \quad n > k.$$

Varian (2007) and Edelman et al. (2007) proved that in this game there exist multiple Nash equilibria, and that one of these equilibria generates the VCG outcome. In addition, it was empirically claimed by Varian (2007) that in practice the equilibrium that generates the VCG outcome is likely to be played. Hence, the revenue of the organizer is the same revenue as obtained in a VCG ad auction. The VCG ad auction with complete information is a special case of our study, in which the distribution F_i gives probability 1 to v_i. As in this case $E[\tilde{v}_{j+1}] = v_{j+1}$, the revenue in the next-price ad auction is given by the following formula:

$$R = \alpha_1 v_2 + \sum_{j=2}^{k} \alpha_j \left(j v_{j+1} - (j-1) v_j \right). \tag{64}$$

Formula (64) enables one to determine the effects on the revenue of changing the click rates and/or adding/removing positions. *As one can see the revenue can be either increasing or decreasing with the click-rates as determined by the above equation.*

The implications of our findings for a search engine are as follows: a search engine should take into consideration the characteristics of the positions in the search results, as these can change the click rates and improve revenue. Moreover, by attracting advertisers and optimizing the mechanism it may be possible for an ad auction to outperform the revenue of a stronger ad auction with higher click-rates.

References

Ashlagi, I., Monderer, D., and Tennenholtz, M. (2008): "Mediators in Position Auctions." To appear in Games and Economic Behavior. A shorter version appears in the Proceedings of the 8th ACM conference on Electronic Eommerce.

Athey, S. and Ellison, G. (2007): *Position Auctions and Consumer Search*. Working paper. Harvard University.

Borgs, C., Chayes, J. T., Etesami, O., Immorlica, N., and Mahdian, M. (2007): "Dynamics of Bid Optimization in Online Advertisement Auctions." In *Proceedings of the 16th international conference on World Wide Web*, pages 531–540.

Bulow, J. and Klemperer, P. (1996): "Auctions versus Negotiations." *American Economic Review*, 86(1): 180–194.

Burguet, R. and Sakovics, J. (1999): “Imperfect Competition in Auction Designs.” *International Economic Review*, 40(1): 231–247.

Cheng, J. (2008): “European Commission: Google-Yahoo ad deal needs closer look.” Http://arstechnica.com/news.ars/post/20080915-european-commission-google-yahoo-ad-deal-needs-closer-look.html.

Edelman, B. (2008a): “Google-Yahoo Ad Deal is Bad for Online Advertising.” Http://hbswk.hbs.edu/item/5995.html.

Edelman, B. (2008b): “PPC Platform Competition and Google’s ”May Not Copy” Restriction.” Http://www.benedelman.org/news/062708-1.html.

Edelman, B., Ostrovsky, M., and Schwarz, M. (2007): “Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords.” *American Economic Review*, 97(1): 242–259.

Holmstrom, B. (1979): “Groves’ Scheme on Restricted Domains.” *Econometrica*, 47(5): 1137–1144.

Hon-Snir, S. (2005): “Utility Equivalence in Auctions.” *Contributions to Theoretical Economics*, 5: 65–88.

Lahaie, S. (2006): “An analysis of alternative slot auction designs for sponsored search.” In *Proceedings of the 7th ACM conference on Electronic commerce*, pages 218–227.

Lebrun, B. (2006): “Uniqueness of the equilibrium in first-price auctions.” *Game and Economic Behavior*, 55(1): 131–151.

McAfee, P. (1993): “Mechanism Design by Competing Sellers.” *Econometrica*, 61: 1281–1312.

Mehta, A., Saberi, A., Vazirani, V., and Vazirani, U. (2007): “Adwords and Generalized Online Matching.” *Journal of the ACM*, 54(5):22.

Monderer, D. and Tennenholtz, M. (2004): "K-price auctions: Revenue Inequalities, Utility Equivalence, and Competition in Auction Design." *Economic Theory*, 24(2): 255–270.

Myerson, R. B. (1981): "Optimal Auction Design." *Mathematics of Operations Research (6)*, 21: 58–73.

Peters, M. (1997): "A Competitive Distribution of Auctions." *Joutnal of Ecoonimc Theory*, 64: 97–123.

Peters, M. and Severinov, S. (1997): "Competition Among Sellers Who Offer Auctions Instead of Prices." *Joutnal of Ecoonimc Theory*, 75: 141–179.

Rastegari, B., Condon, A., and Leyton-Brown, K. (2007): "Revenue Monotonicity in Combinatorial Auctions." In *Association for the Advancement of Artificial Intelligence (AAAI)*, pages 122–127.

Rockafellar, R. (1970): *Convex analysis.* Princeton Univ. Press.

Varian, H. (2007): "Position Auctions." *International Journal of Industrial Organization*, 25: 1163–1178.