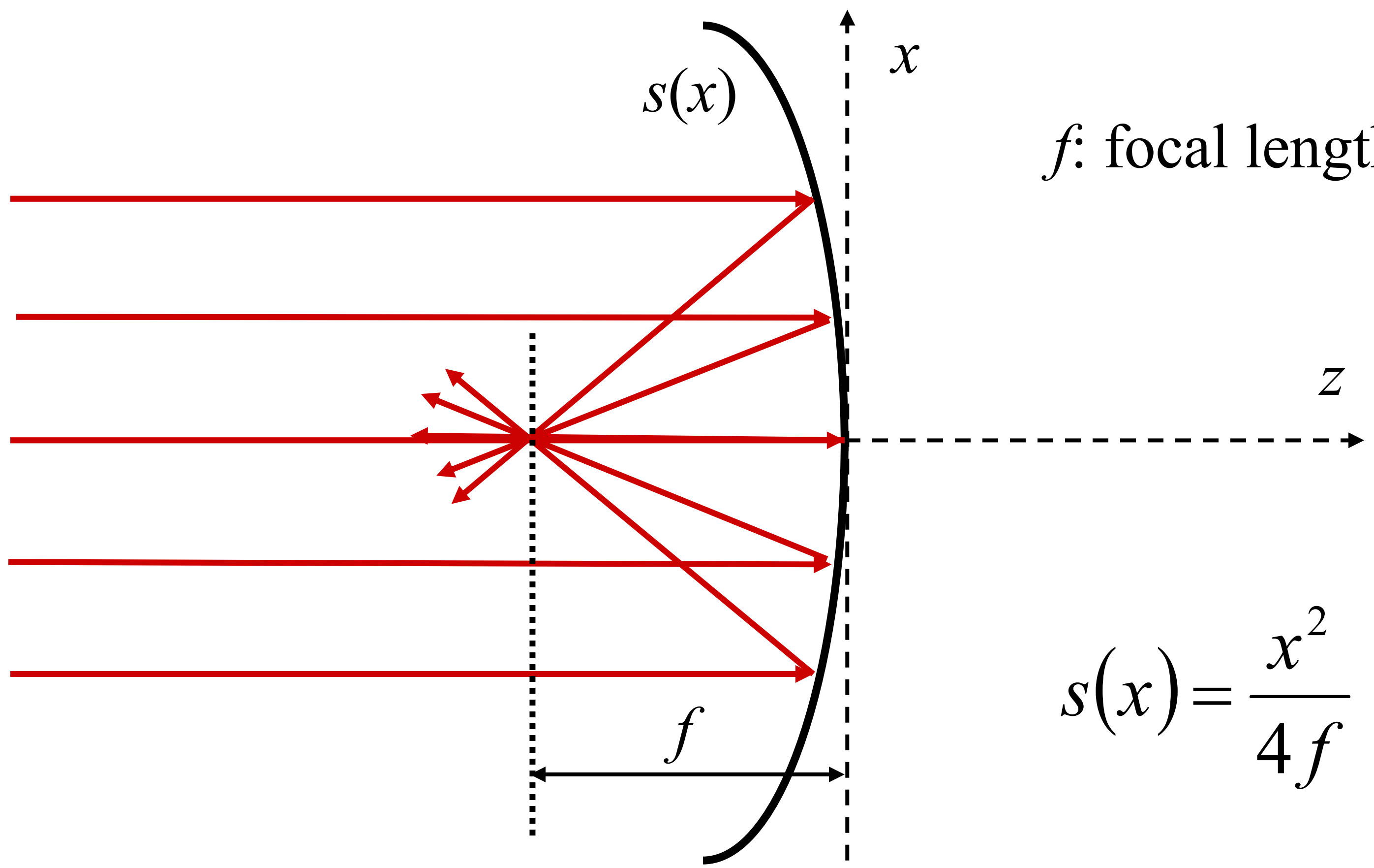
Lenses and Imaging (Part I)

- Why is imaging necessary: Huygen's principle
  - Spherical & parallel ray bundles, points at infinity
- Refraction at spherical surfaces (paraxial approximation)
- Optical power and imaging condition
- Matrix formulation of geometrical optics
- The thin lens
- Surfaces of positive/negative power
- Real and virtual images

Parabloid mirror: perfect focusing

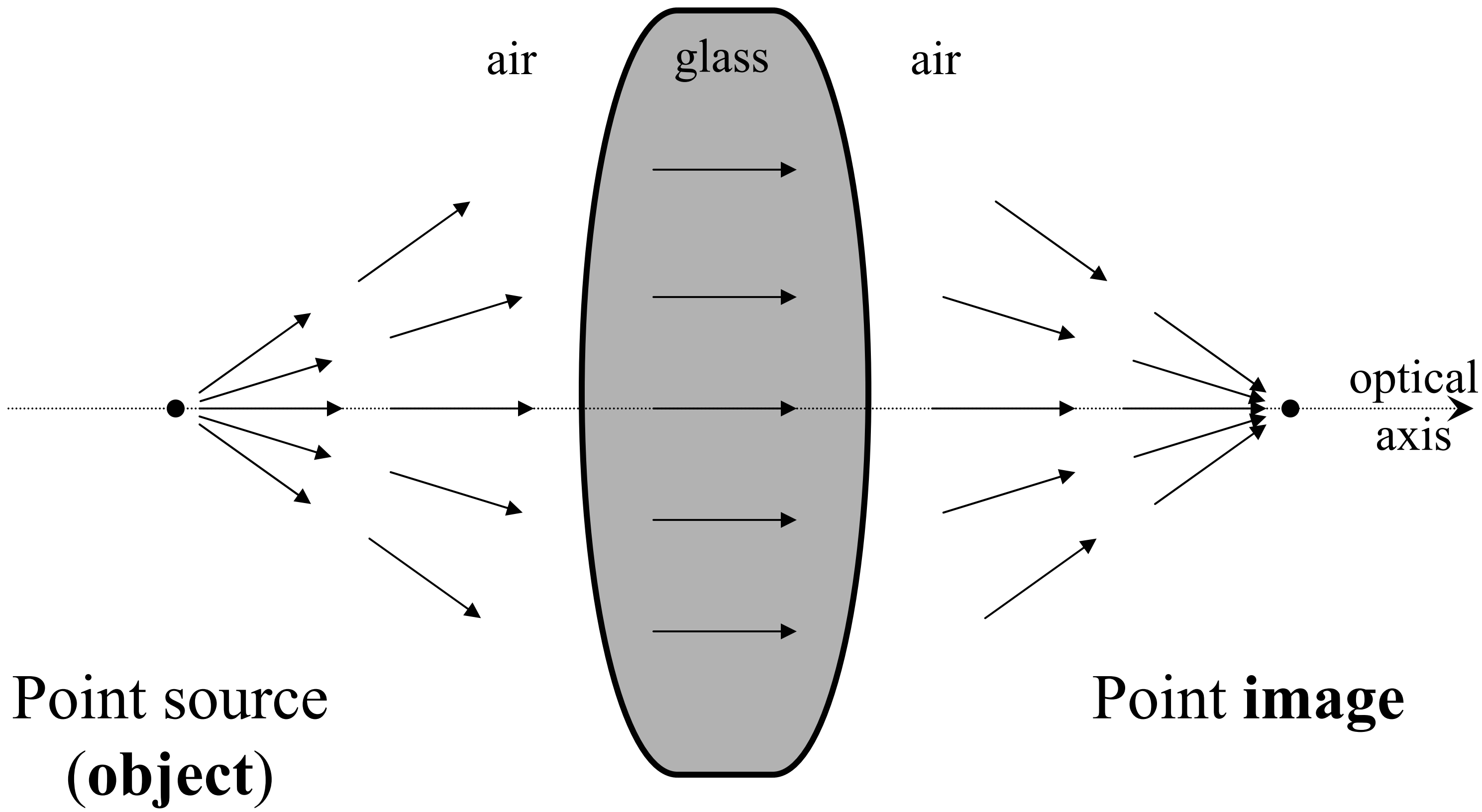
(e.g. satellite dish)



f: focal length

$$s(x) = \frac{x^2}{4f}$$

Lens: main instrument for image formation

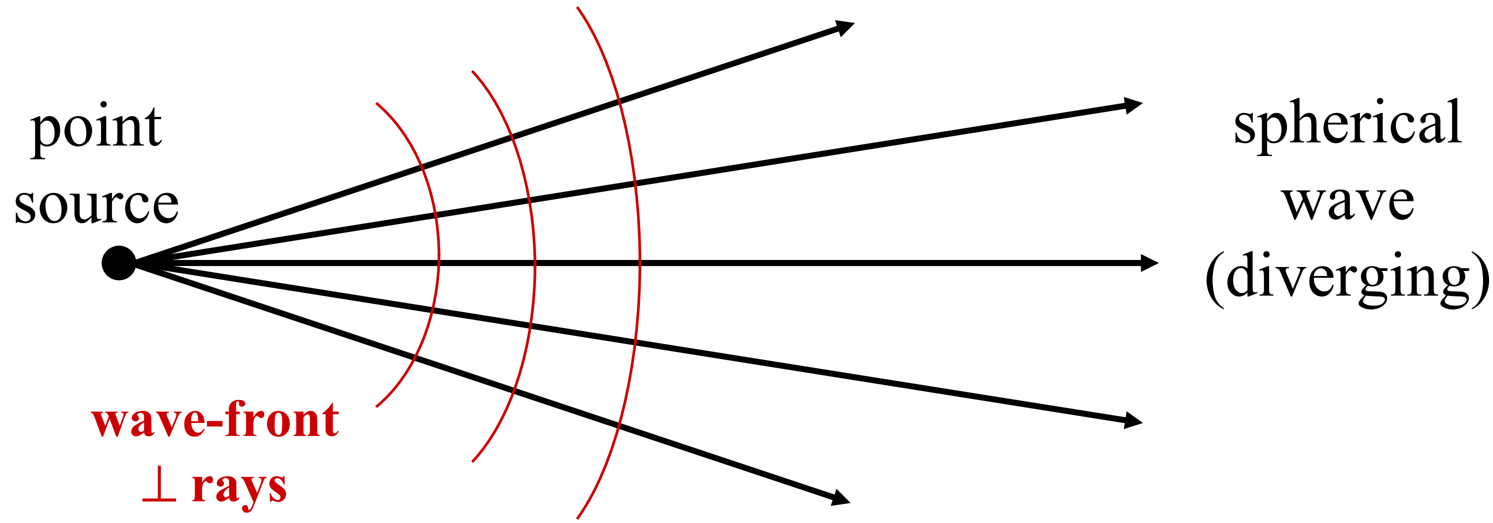


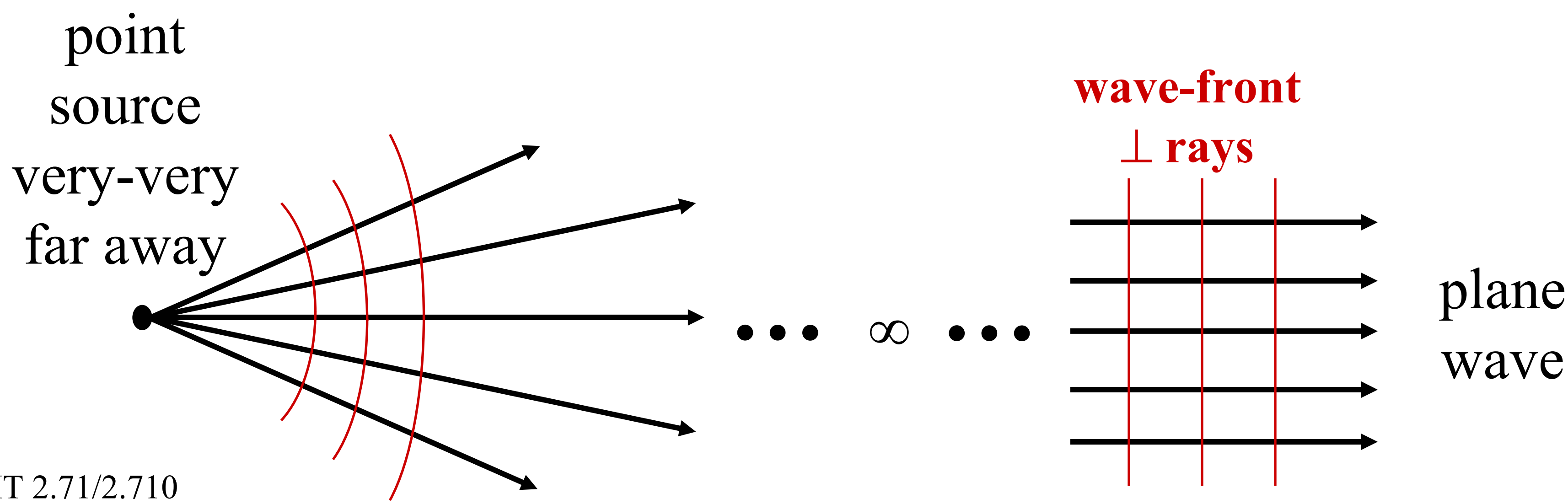
The curved surface makes the rays bend proportionally to their distance from the "optical axis", according to Snell's law. Therefore, the divergent wavefront becomes convergent at the right-hand (output) side.

Why are focusing instruments necessary?

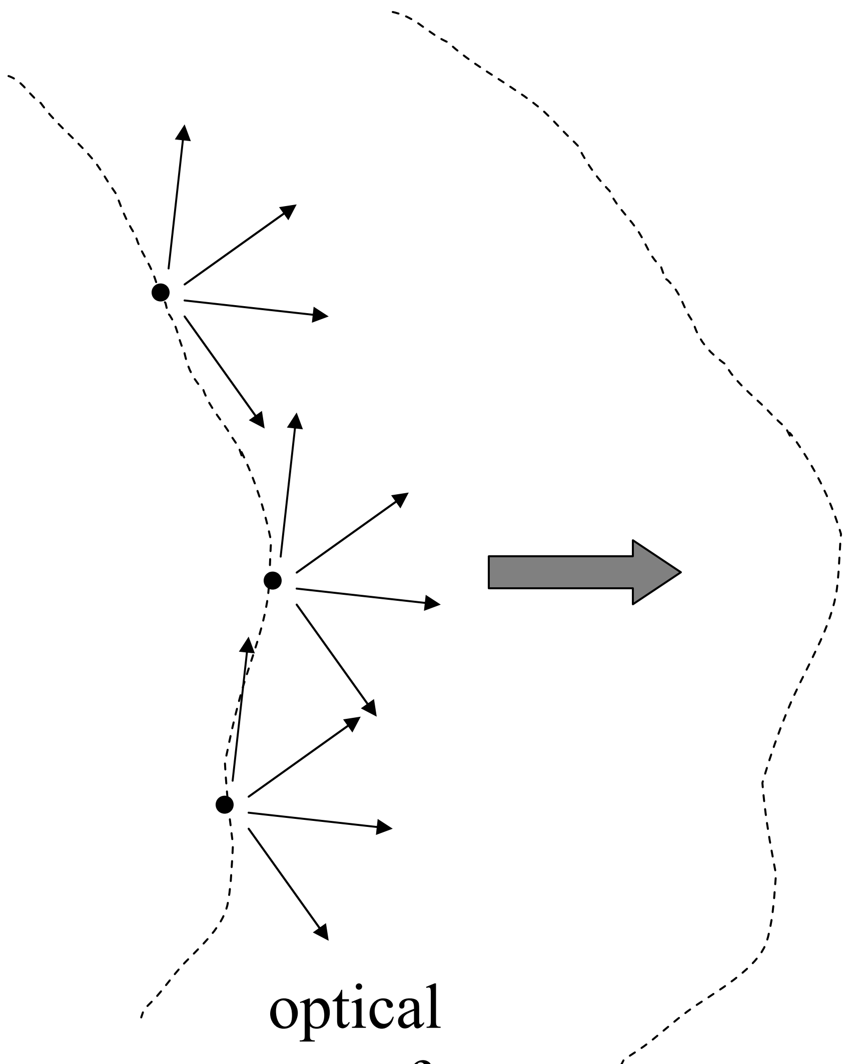
- Ray bundles: spherical waves and plane waves
- Point sources and point images
- Huygens principle and why we can see around us
- The role of classical imaging systems

Ray bundles


point
source
spherical
wave
(diverging)
wave-front
⊥ rays


point
source
very-very
far away
• • • ∞ • • •
wave-front
⊥ rays
plane
wave

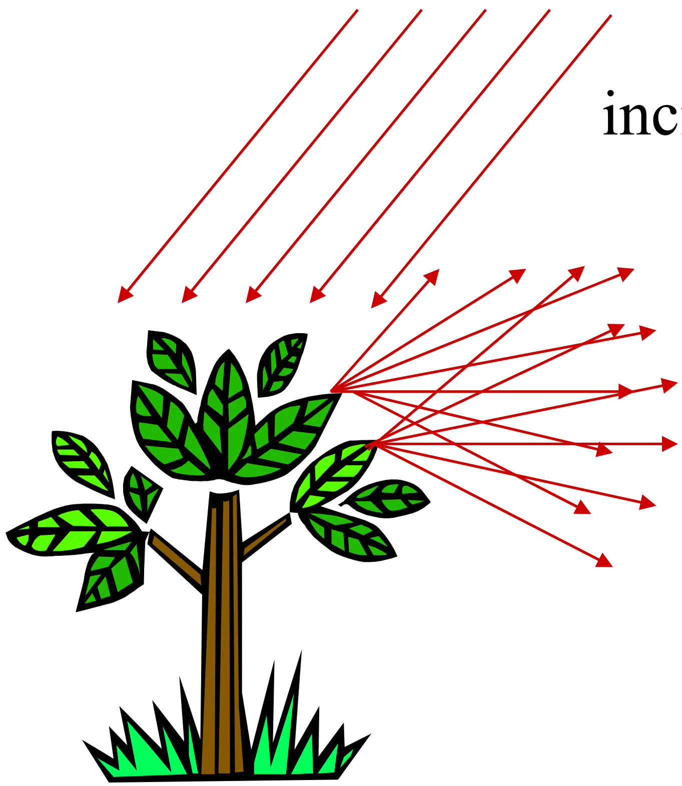
Huygens principle



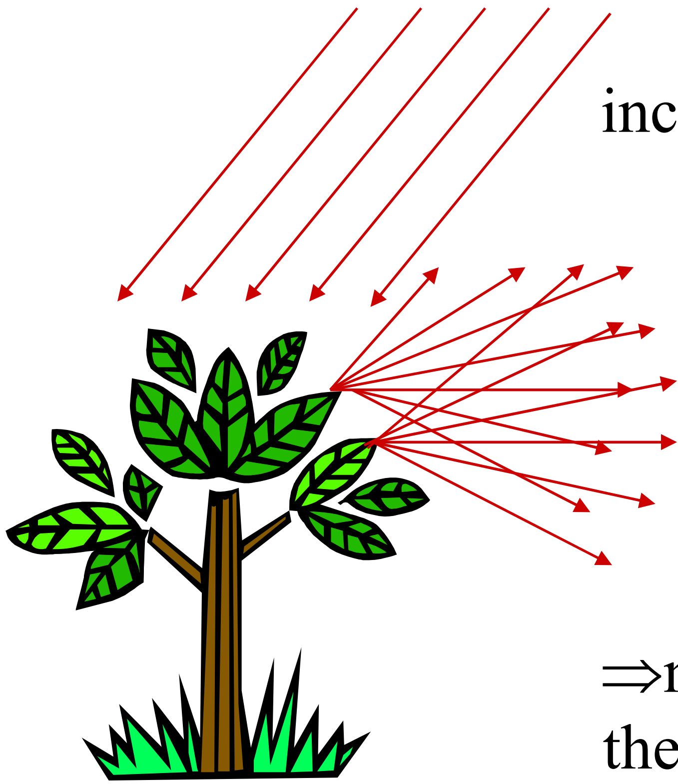
Each point on the wavefront acts as a secondary light source emitting a spherical wave

The wavefront after a short propagation distance is the result of superimposing all these spherical wavelets

Why are focusing instruments necessary?



Why are focusing instruments necessary?



incident light

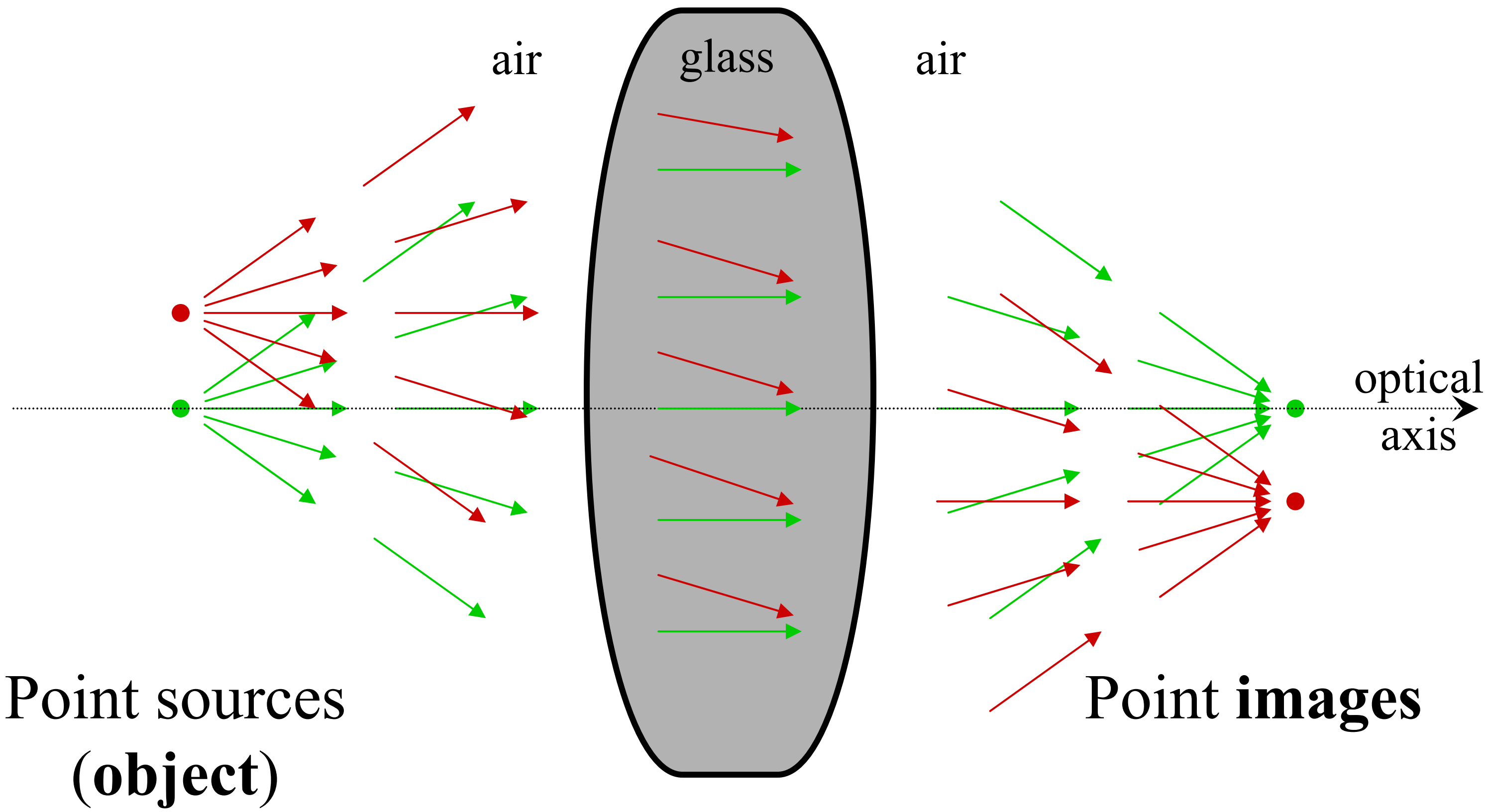
...
is scattered by
the object

⇒need optics to “undo” the effects of scattering, i.e. *focus* the light

object

image

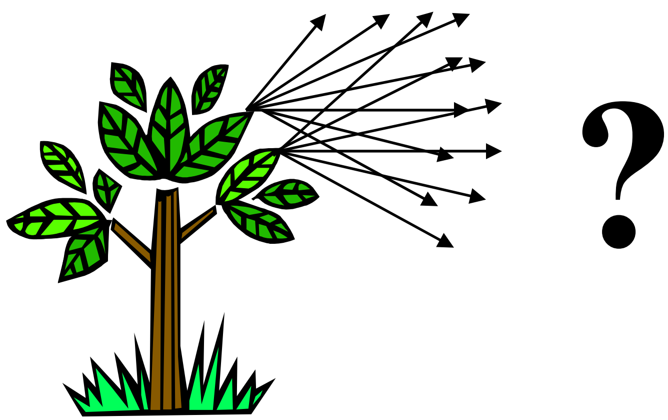
Ideal lens



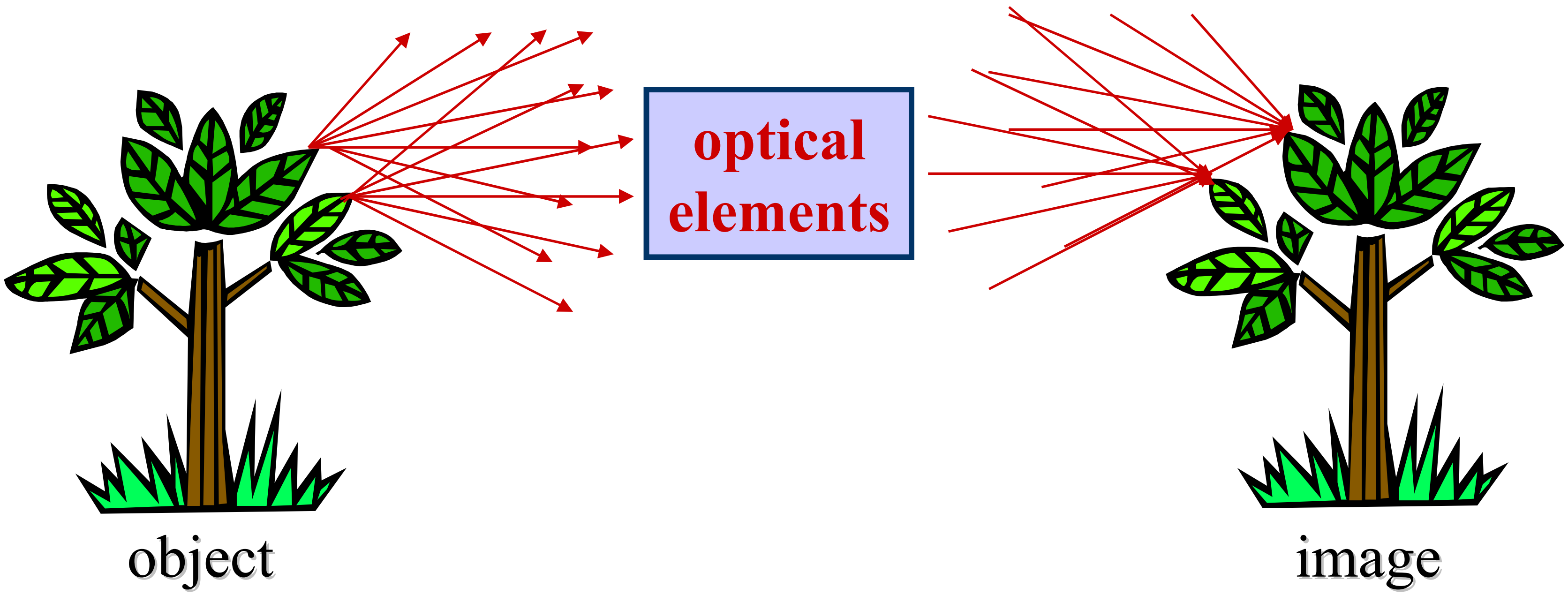
Each point source from the object plane focuses onto a point image at the image plane; NOTE the image inversion

Summary:
Why are imaging systems needed?

- Each point in an object scatters the incident illumination into a spherical wave, according to the Huygens principle.
- A few microns away from the object surface, the rays emanating from all object points become entangled, delocalizing object details.
- To relocalize object details, a method must be found to reassign ("focus") all the rays that emanated from a single point object into another point in space (the "image.")
- The latter function is the topic of the discipline of Optical Imaging.



The ideal optical imaging system

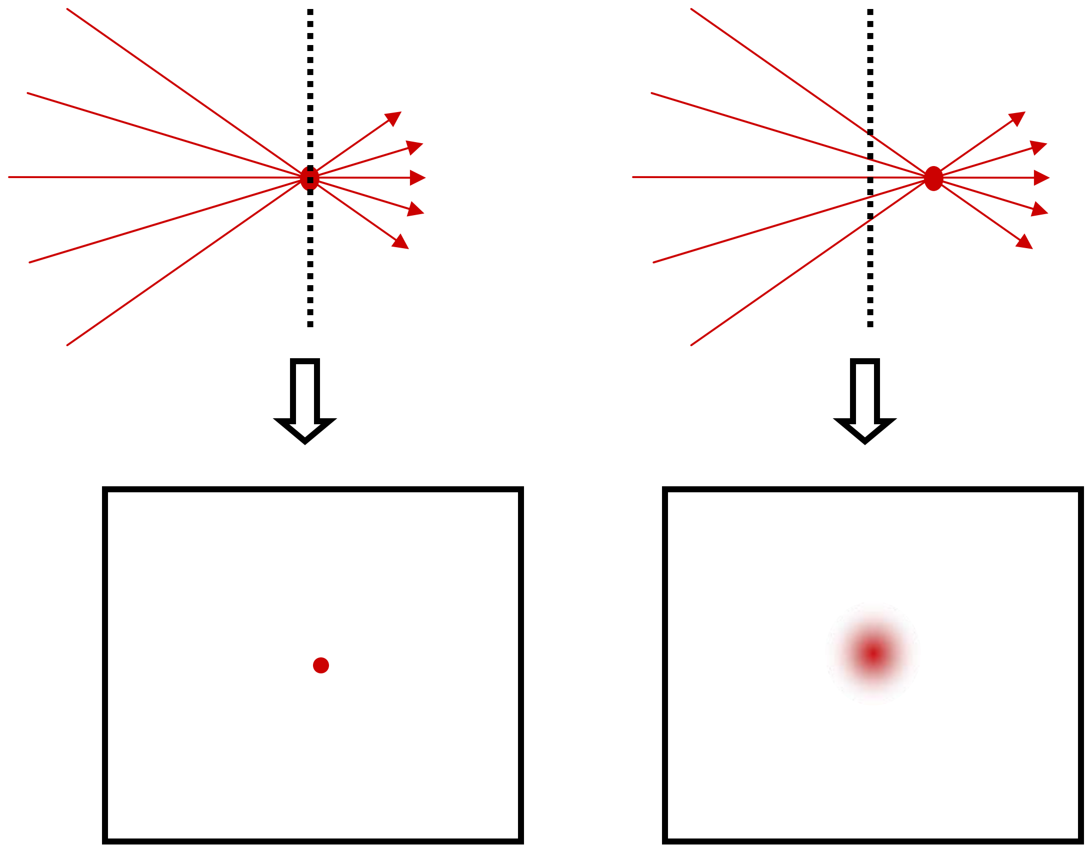


Ideal imaging system:
each point in the object is mapped onto a single point in the image

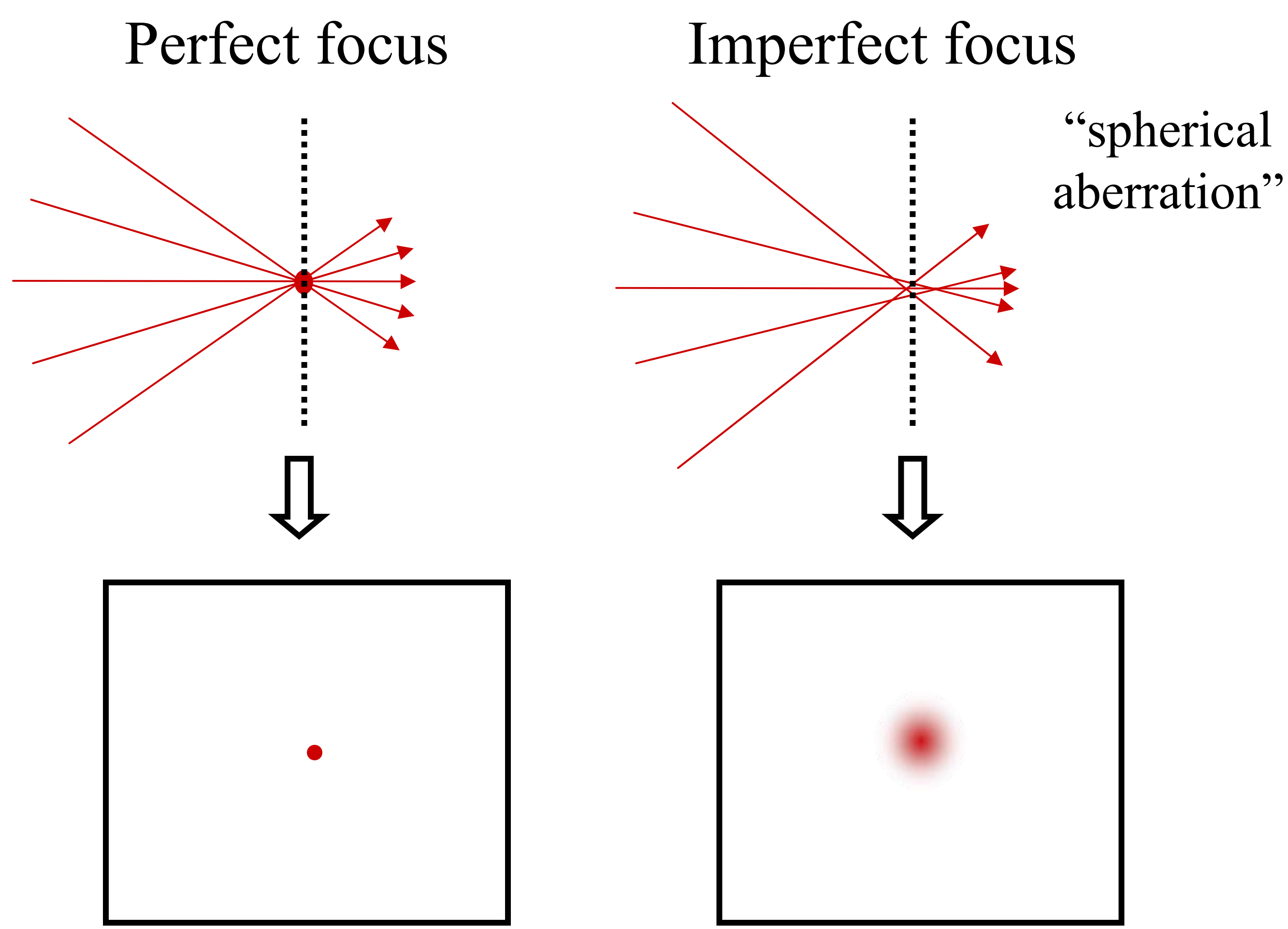
Real imaging systems introduce blur ...

Focus, defocus and blur

Perfect focus　　　　Defocus



Focus, defocus and blur



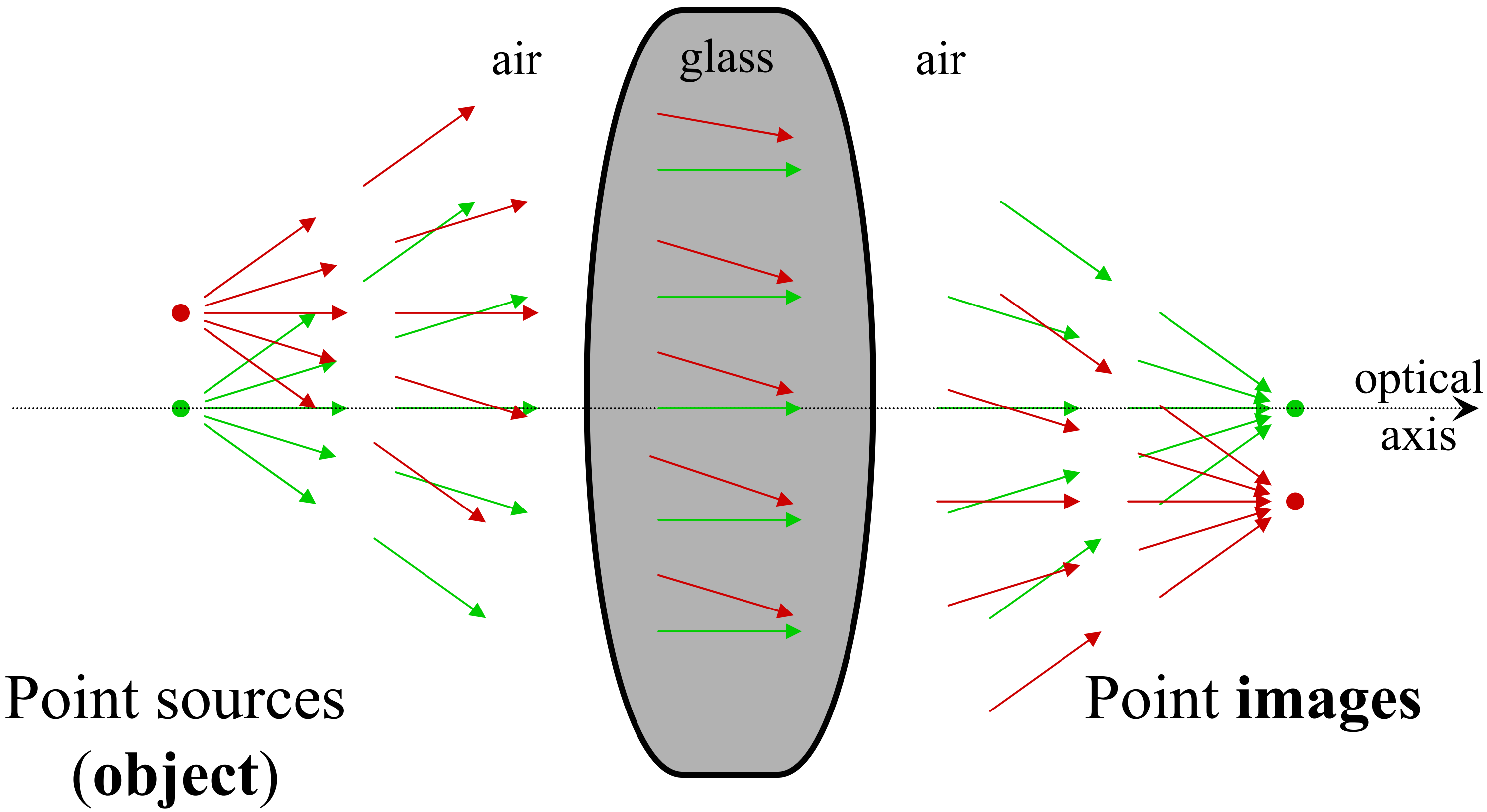
Why optical systems do *not* focus perfectly

- Diffraction
- Aberrations

- However, in the paraxial approximation to Geometrical Optics that we are about to embark upon, optical systems do focus perfectly

- To deal with aberrations, we need non-paraxial Geometrical Optics (higher order approximations)
- To deal with diffraction, we need Wave Optics

Ideal lens

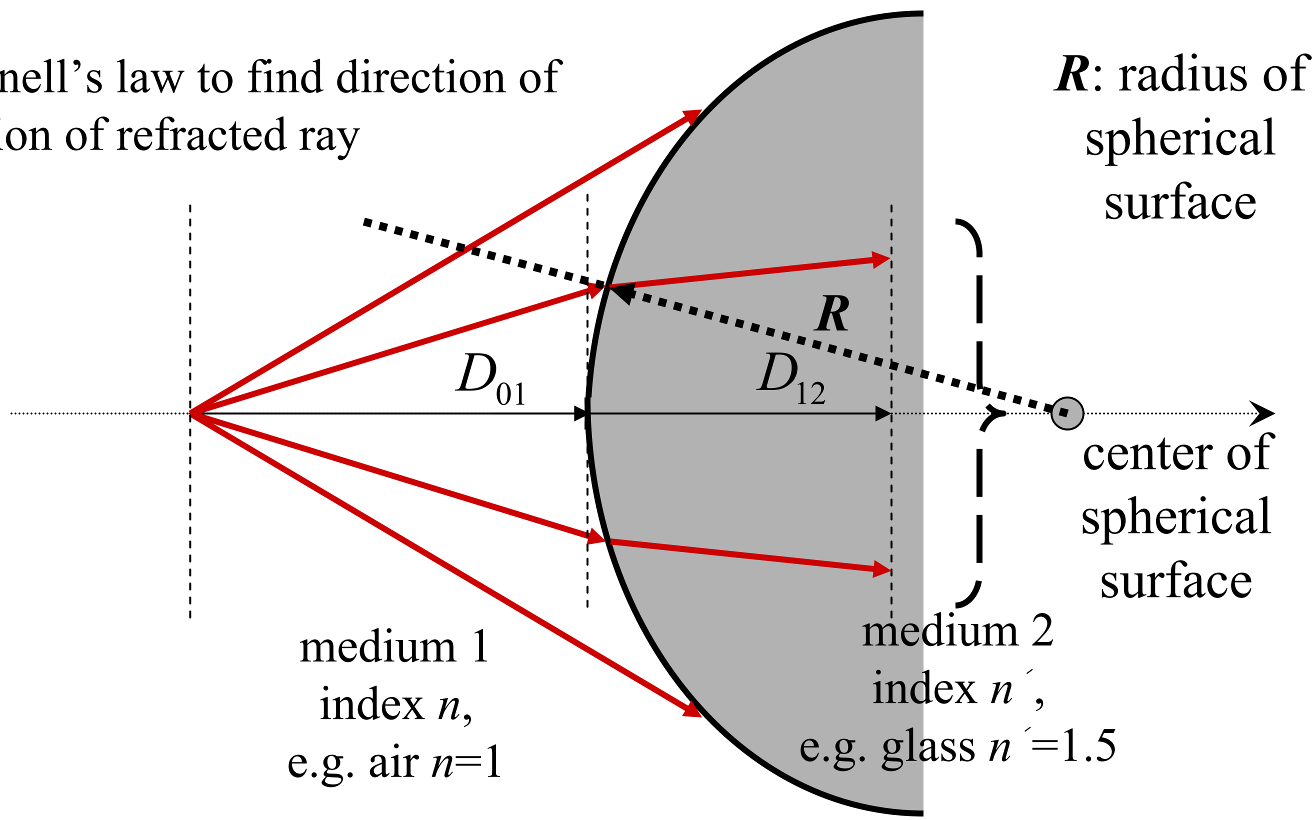


Each point source from the object plane focuses onto a point image at the image plane

Refraction at single spherical surface

for each ray, must calculate

- point of intersection with sphere
- angle between ray and normal to surface
- apply Snell's law to find direction of propagation of refracted ray



Paraxial approximation /1

- In paraxial optics, we make heavy use of the following approximate (1st order Taylor) expressions:

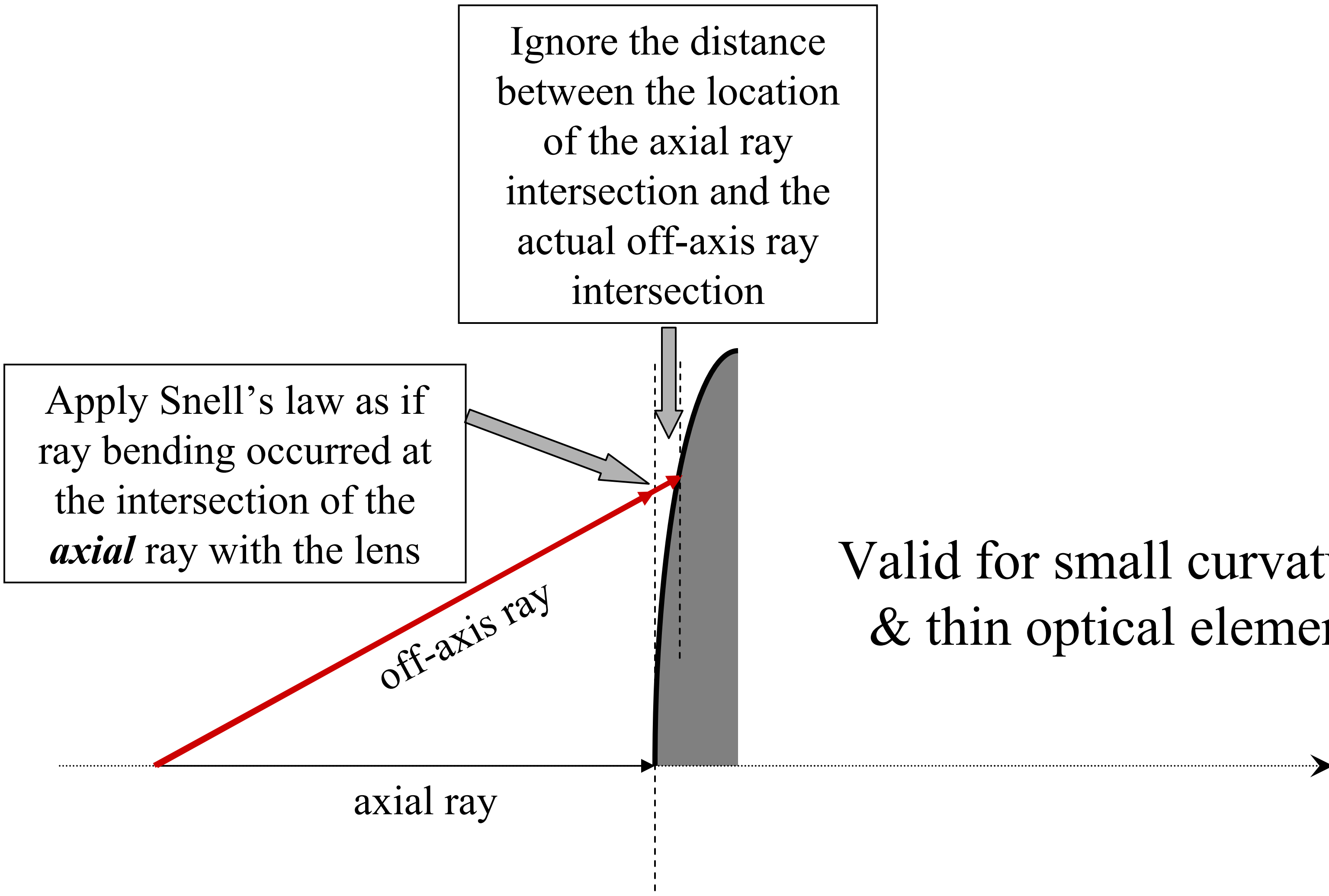
$$\sin\varepsilon \approx \varepsilon \approx \tan\varepsilon \qquad \cos\varepsilon \approx 1$$

$$\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon$$

where ε is the angle between a ray and the optical axis, and is a small number ($\varepsilon \langle\langle 1$ rad). The range of validity of this approximation typically extends up to ~10-30 degrees, depending on the desired degree of accuracy. This regime is also known as "Gaussian optics" or "paraxial optics."

Note the assumption of existence of an optical axis (*i.e.*, perfect alignment!)

Paraxial approximation /2



Valid for small curvatures & thin optical elements

Refraction at spherical surface

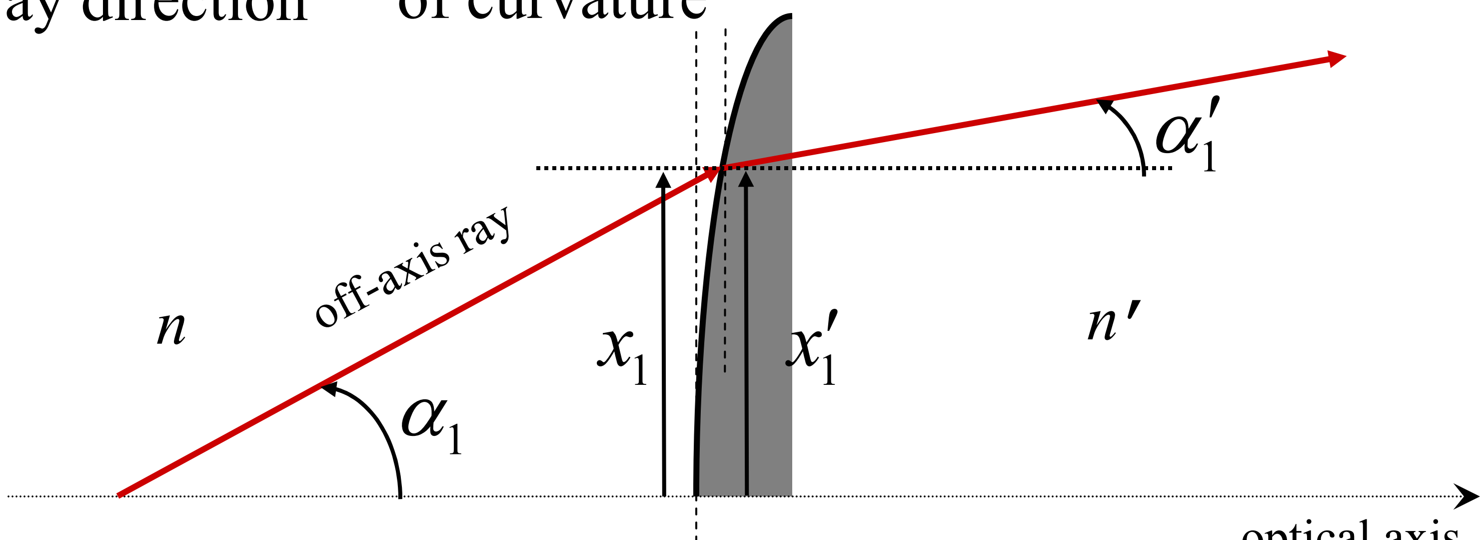
Refraction at positive spherical surface:

$$\begin{cases} x_1' = x_1 \\ \alpha_1' = \dfrac{n}{n'}\alpha_1 - \left[\dfrac{(n'-n)}{n'R}\right]x_1 \end{cases}$$

Power

x: ray height
α: ray direction

R: radius of curvature

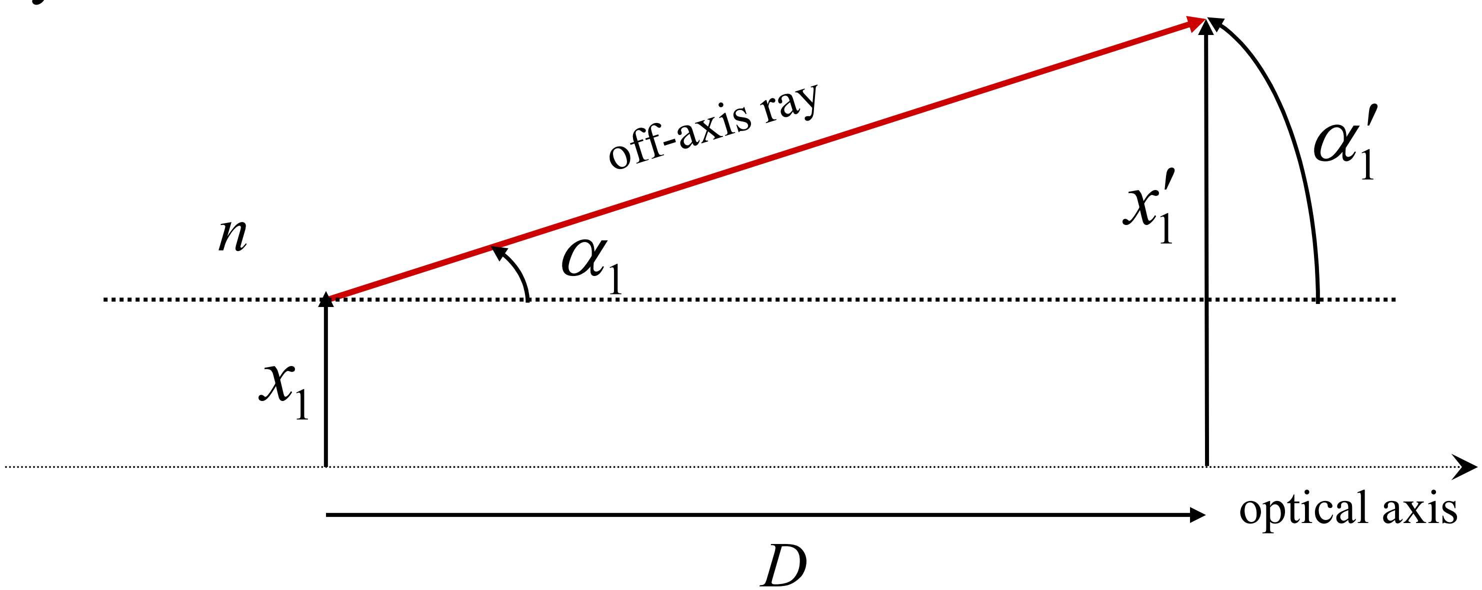


Propagation in uniform space

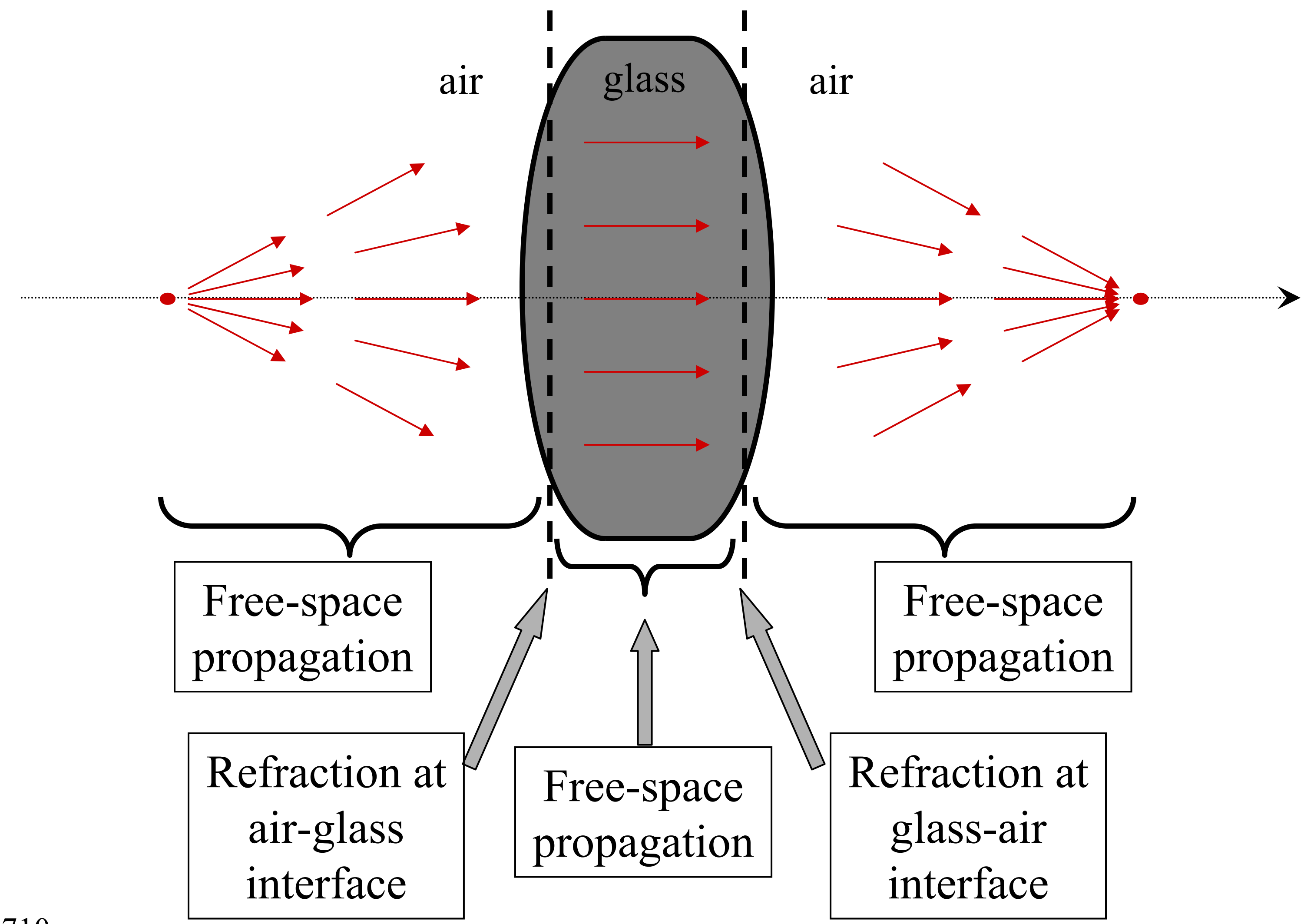
Propagation through distance D: $\begin{cases} x_1' = x_1 + D\alpha_1 \\ \alpha_1' = \alpha_1 \end{cases}$

x: ray height

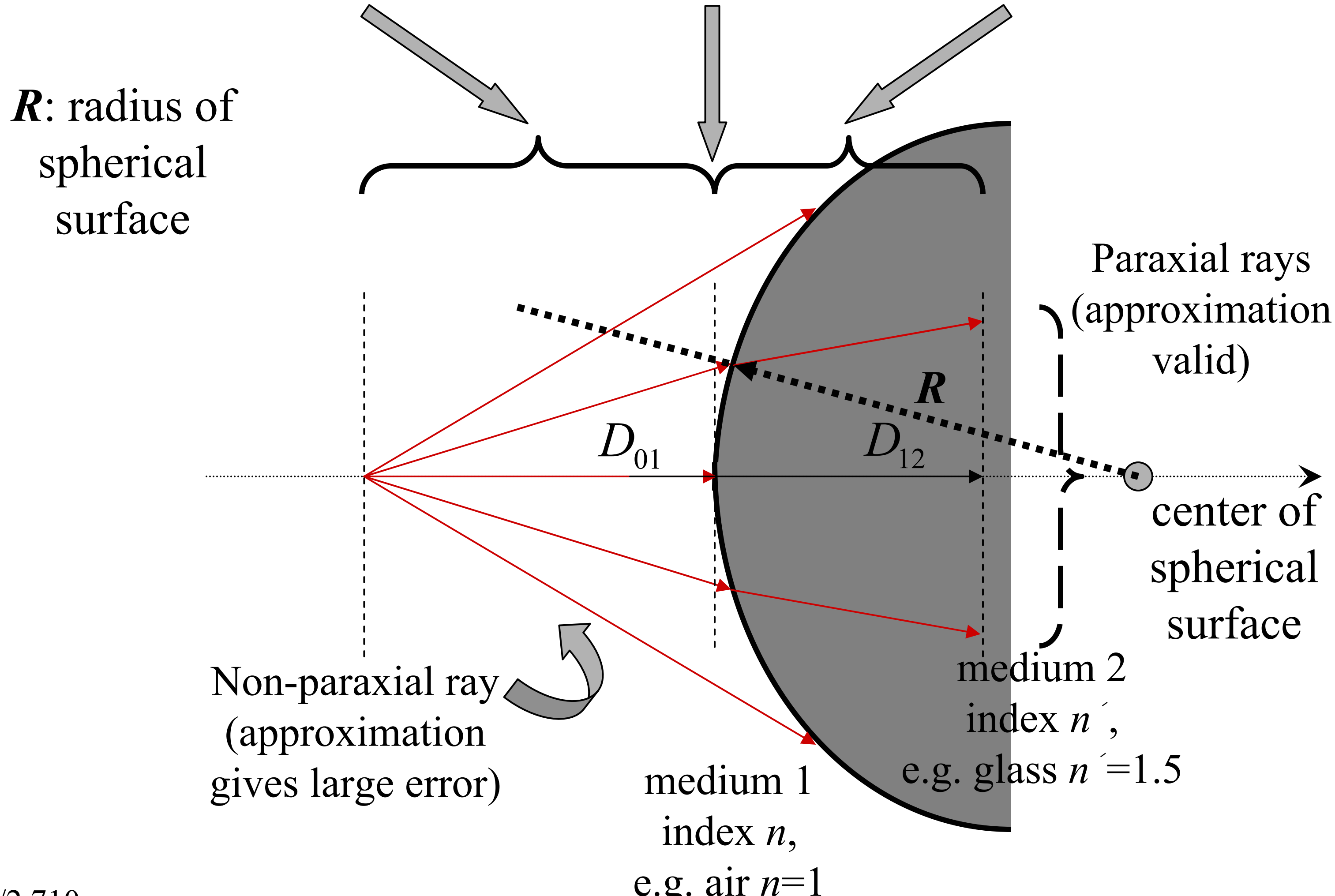
α: ray direction



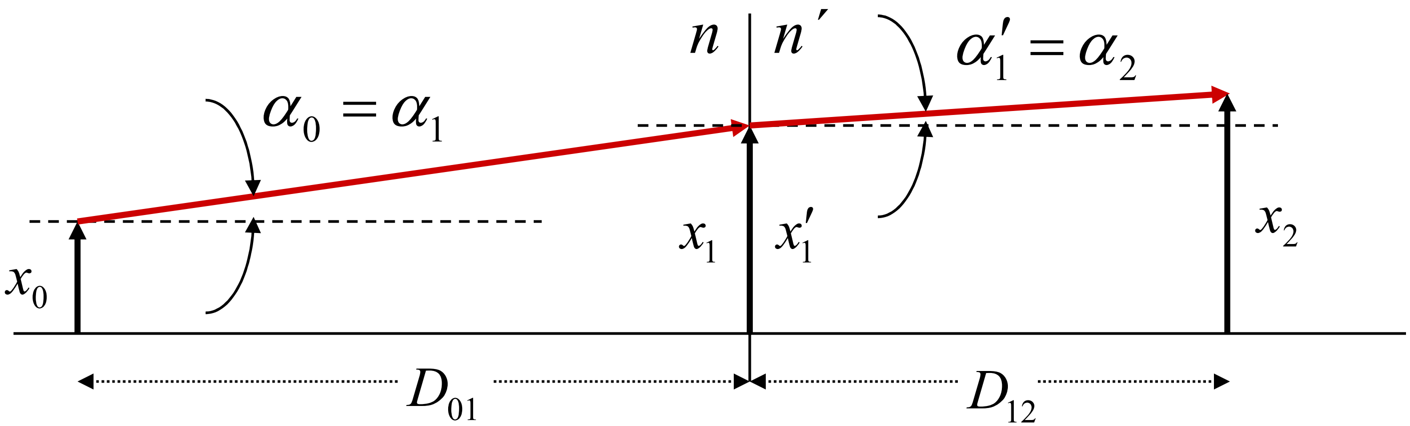
Paraxial ray-tracing



Example: one spherical surface, translation+refraction+translation



Translation+refraction+translation /1

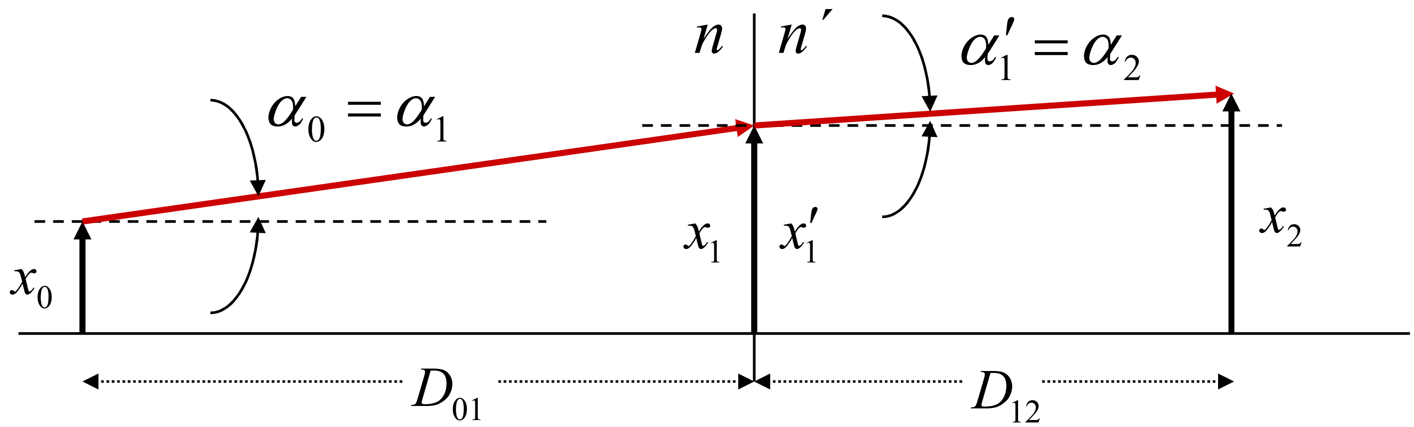


Starting ray: location x_0 direction α_0

Translation through distance D_{01} (+ direction): $\begin{cases} x_1 = x_0 + D_{01}\alpha_0 \\ \alpha_1 = \alpha_0 \end{cases}$

Refraction at positive spherical surface: $\begin{cases} x_1' = x_1 \\ \alpha_1' = \dfrac{n}{n'}\alpha_1 - \left[\dfrac{(n'-n)}{n'R}\right]x_1 \end{cases}$

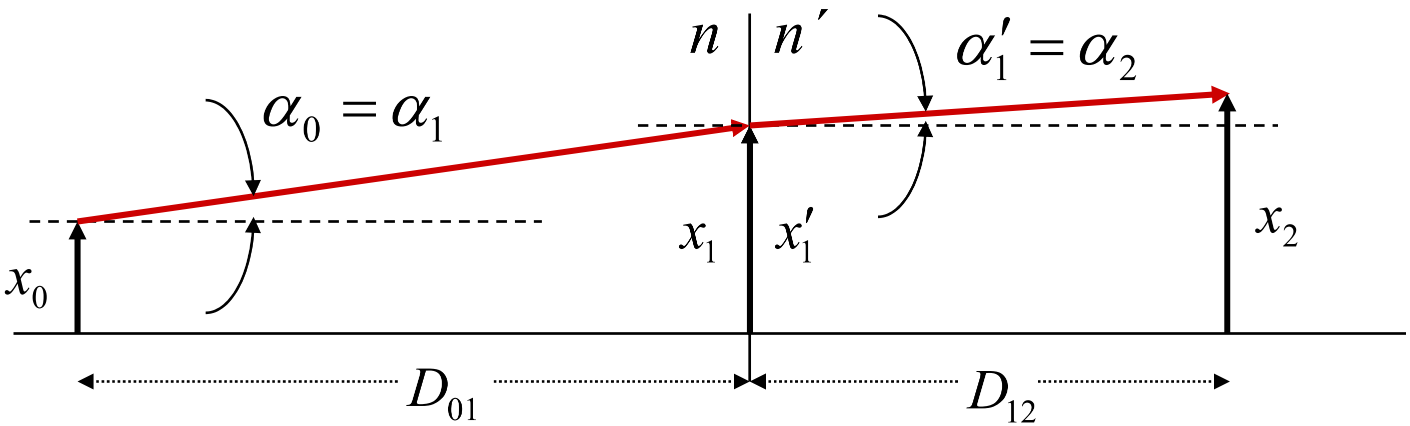
Translation+refraction+translation /2



Translation through distance D_{12} (+ direction): $\begin{cases} x_2 = x_1 + D_{12}\alpha_1' \\ \alpha_2 = \alpha_1' \end{cases}$

Put together:

Translation+refraction+translation /3

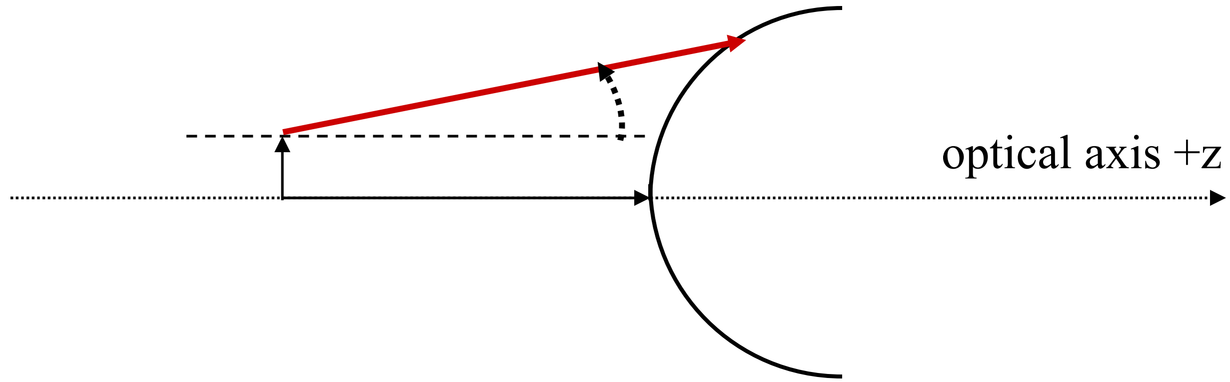


$$x_2 = \left[\frac{(n-n')D_{12}}{n'R} + 1\right]x_0 + \left[D_{01} + \frac{nD_{12}}{n'} + \frac{(n-n')D_{01}D_{12}}{n'R}\right]\alpha_0$$

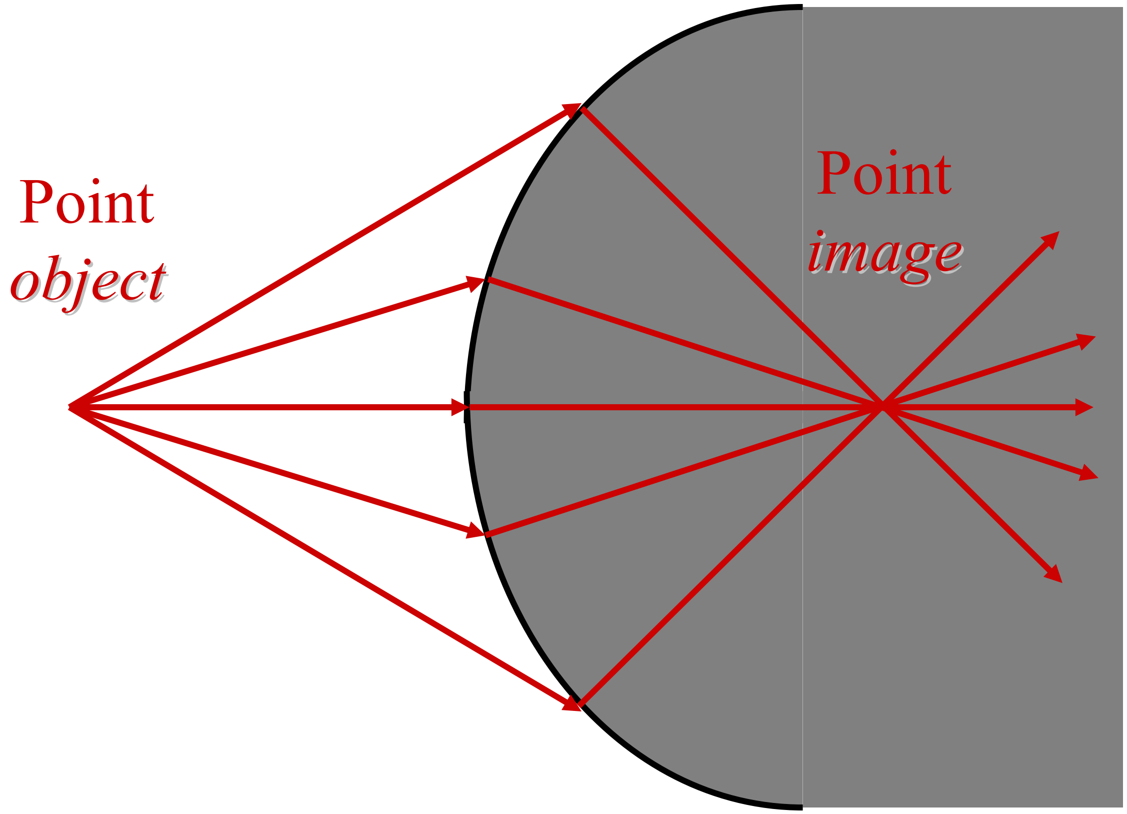
$$\alpha_2 = \left[\frac{n-n'}{n'R}\right]x_0 + \left[\frac{n}{n'} + \frac{(n-n')D_{01}}{n'R}\right]\alpha_0$$

Sign conventions for refraction

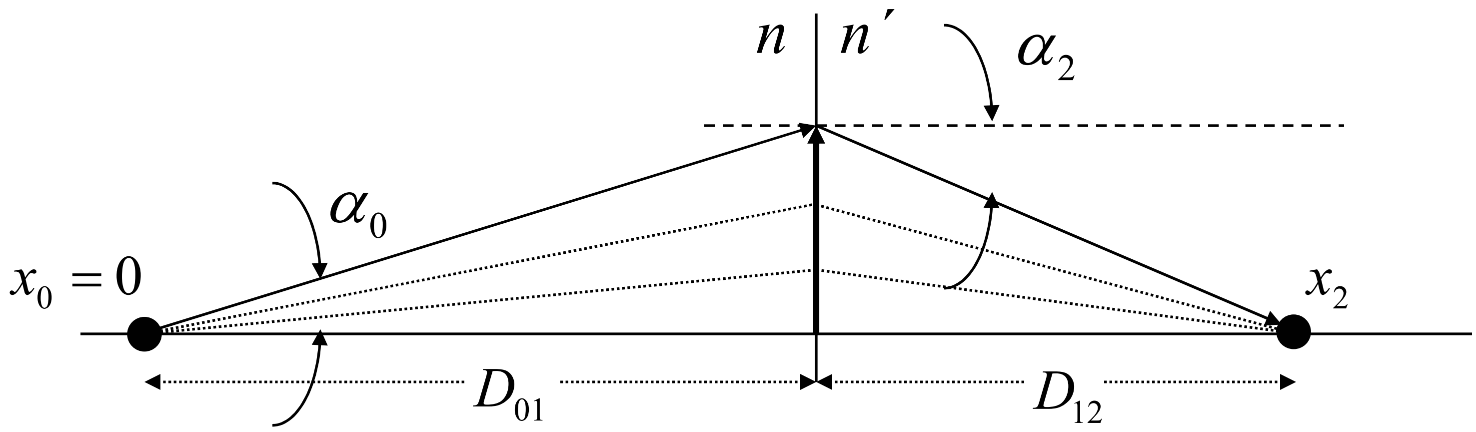
- Light travels from left to right
- A radius of curvature is positive if the surface is convex towards the left
- Longitudinal distances are positive if pointing to the right
- Lateral distances are positive if pointing up
- Ray angles are positive if the ray direction is obtained by rotating the +z axis counterclockwise through an acute angle



On-axis image formation



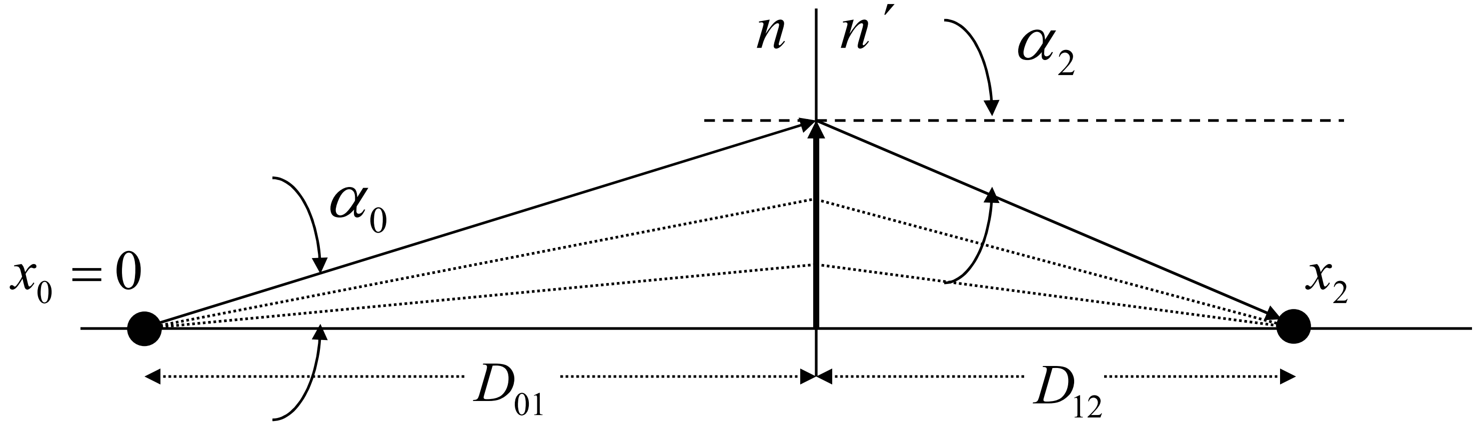
On-axis image formation



All rays emanating at x_0 arrive at x_2 irrespective of departure angle α_0 $\Rightarrow$ $\dfrac{\partial x_2}{\partial \alpha_0} = 0$

$$x_2 = \left[\cdots\right] x_0 + \left[D_{01} + \frac{n D_{12}}{n'} - \frac{(n'-n) D_{01} D_{12}}{n' R} \right] \alpha_0$$

On-axis image formation

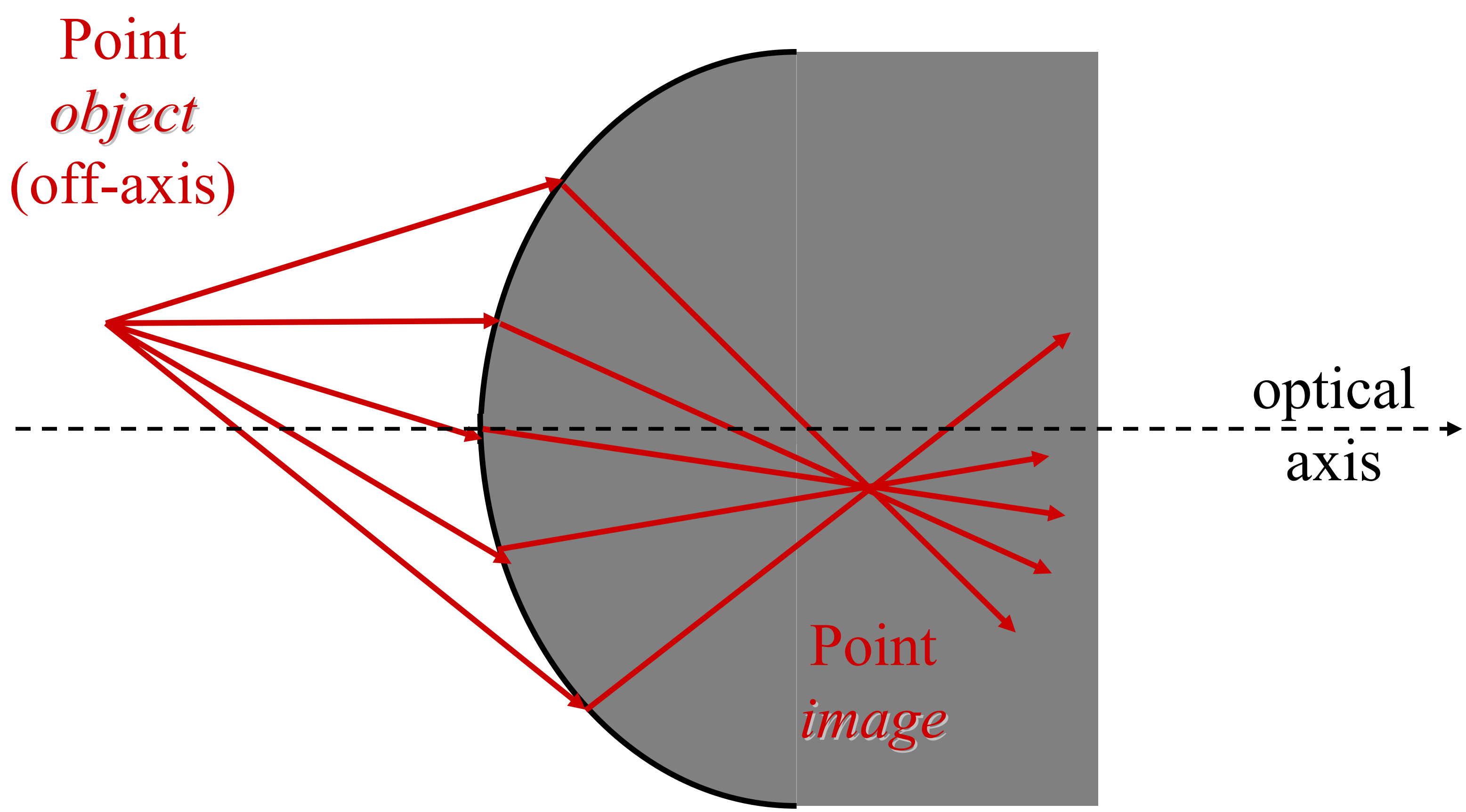


All rays emanating at x_0 arrive at x_2 irrespective of departure angle α_0 ⇒ $\dfrac{\partial x_2}{\partial \alpha_0} = 0$ ⇒

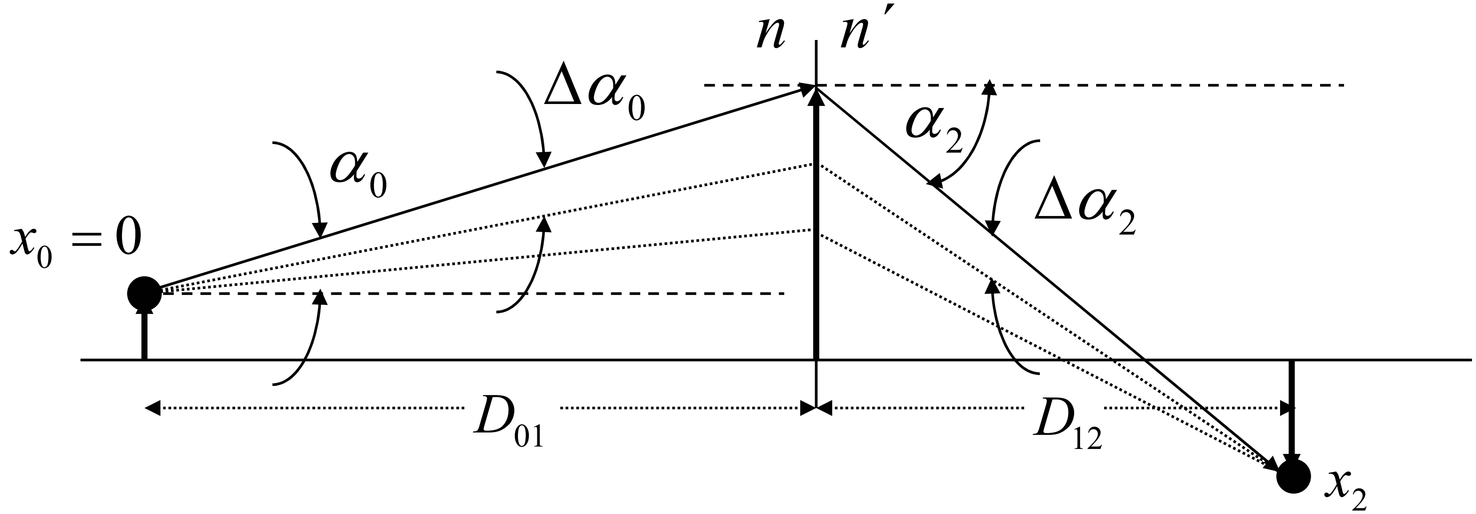
$$\frac{n'}{D_{12}} + \frac{n}{D_{01}} = \frac{n' - n}{R}$$

"Power" of the spherical surface [units: diopters, 1D=1 m^{-1}]

Off-axis image formation



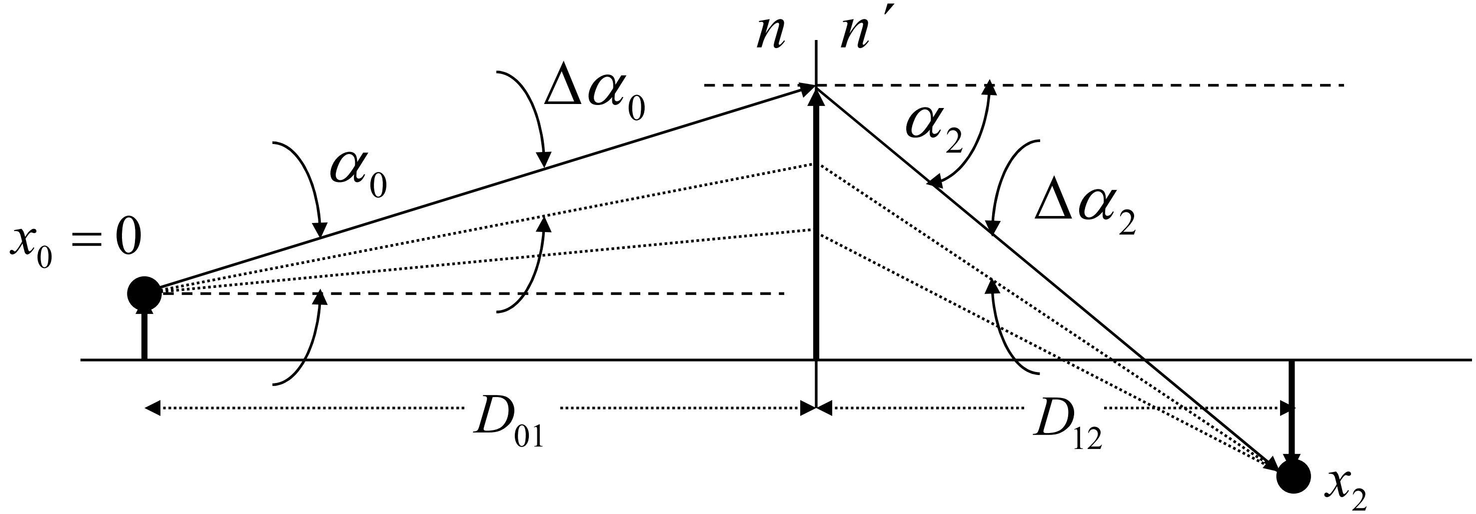
Magnification: lateral (off-axis), angle



Lateral $$m_x = \frac{x_2}{x_0} = \frac{n-n'}{R}\frac{D_{12}}{n'} + 1 = \ldots\ldots = -\frac{n}{n'}\frac{D_{12}}{D_{01}}$$

Angle $$m_\alpha = \frac{\Delta\alpha_2}{\Delta\alpha_0} = -\frac{D_{01}}{D_{12}}$$

Object-image transformation



$$x_2 = m_x x_0$$

$$\alpha_2 = -\frac{1}{f'} x_0 + m_\alpha \alpha_0$$

Ray-tracing transformation (paraxial) between object and image points

Image of point object at infinity

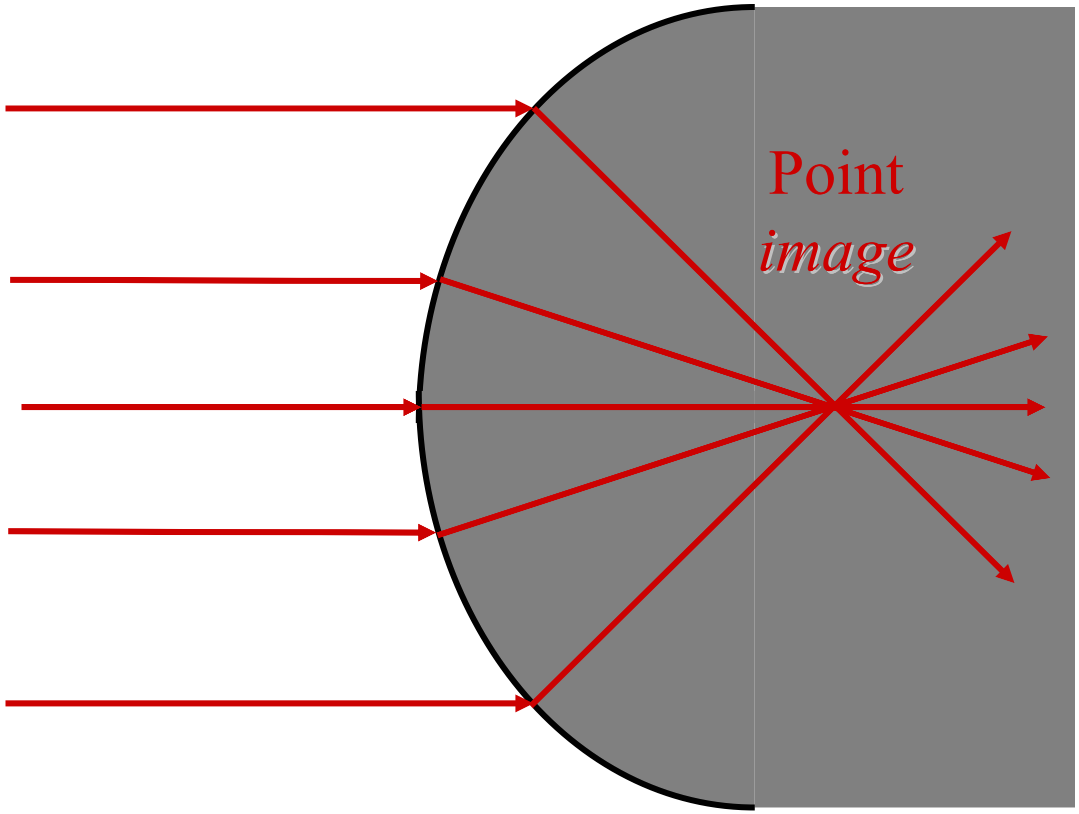
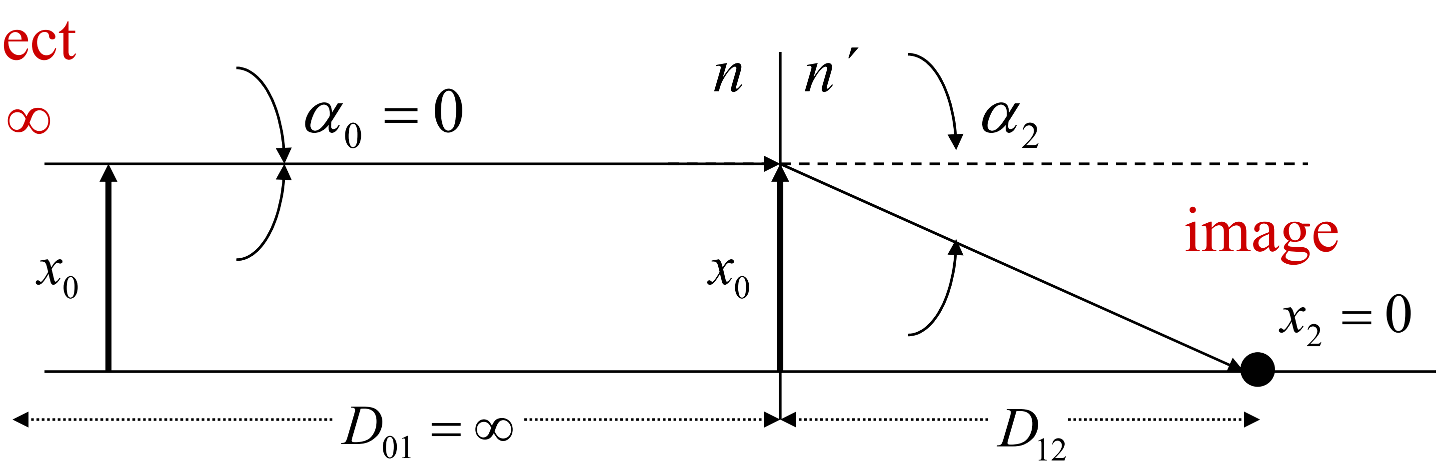


Image of point object at infinity



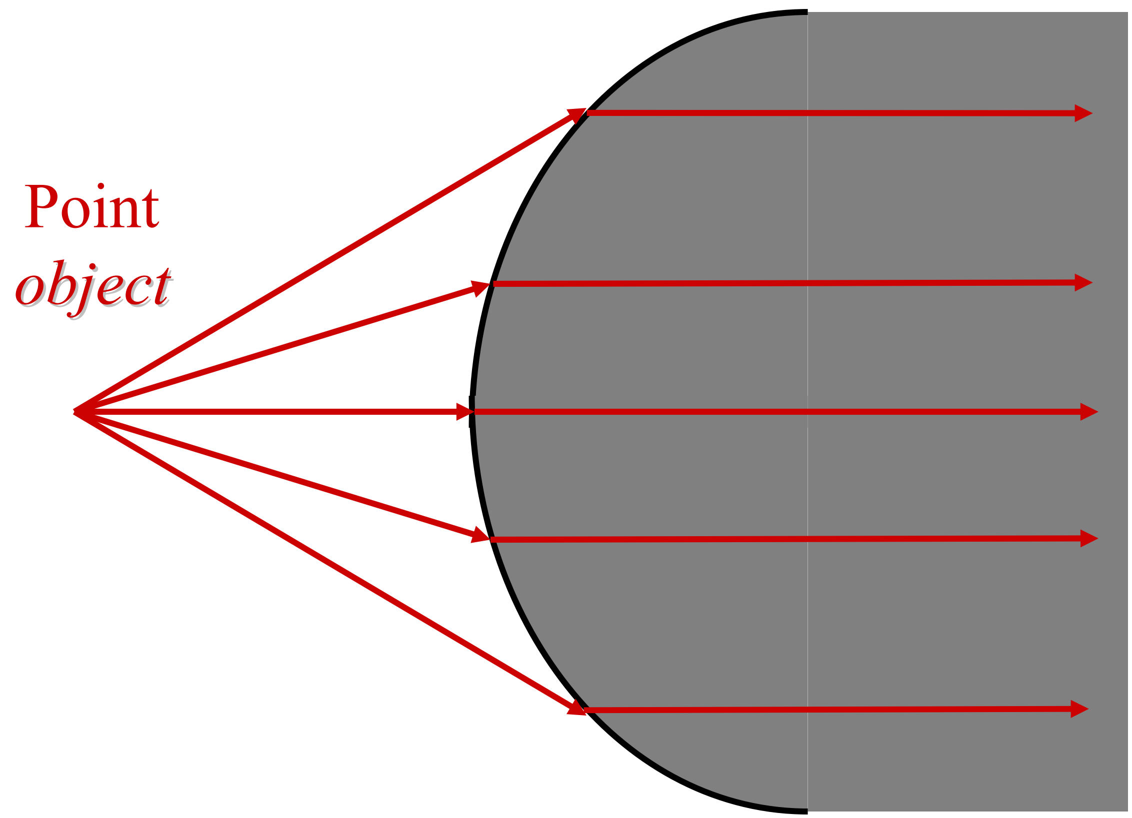
$$\frac{n'}{D_{12}} = \frac{n'-n}{R} \Rightarrow D_{12} = \frac{n'R}{n'-n} \equiv f' : \text{image focal length}$$

Note: $$f' = n' \times \frac{R}{n'-n} \equiv n' \times \frac{R}{n'-n}$$

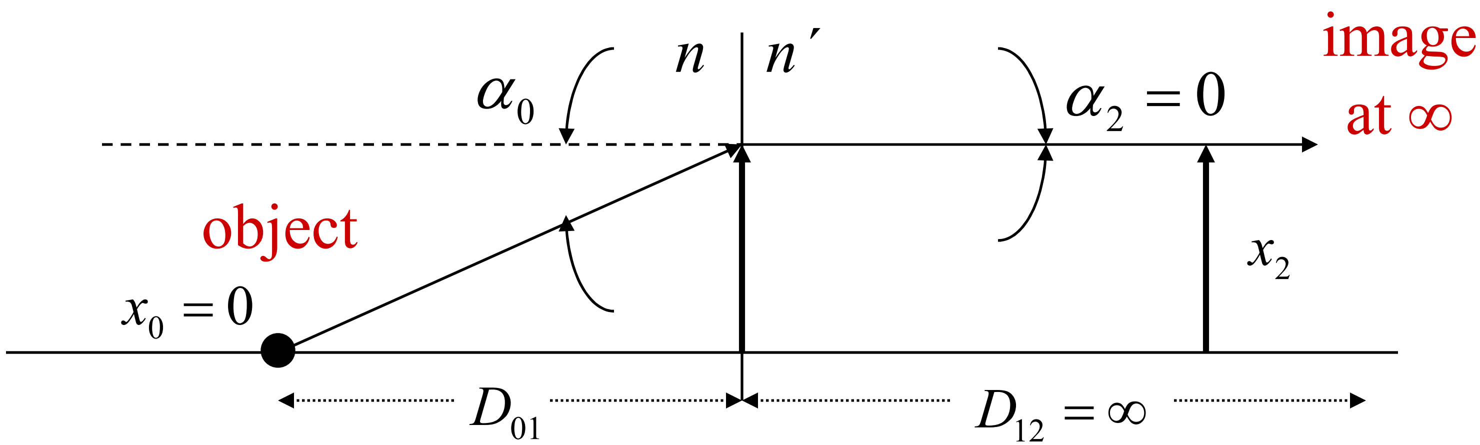
ambient refractive index at space of point image (n')

1/Power ($\frac{R}{n'-n}$)

Point object imaged at infinity



Point object imaged at infinity



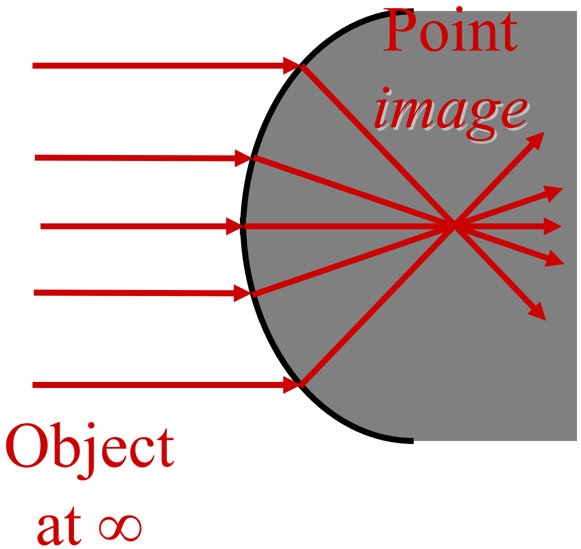
$$\frac{n}{D_{01}} = \frac{n' - n}{R} \Rightarrow D_{01} = \frac{n\,R}{n' - n} \equiv f : \text{object focal length}$$

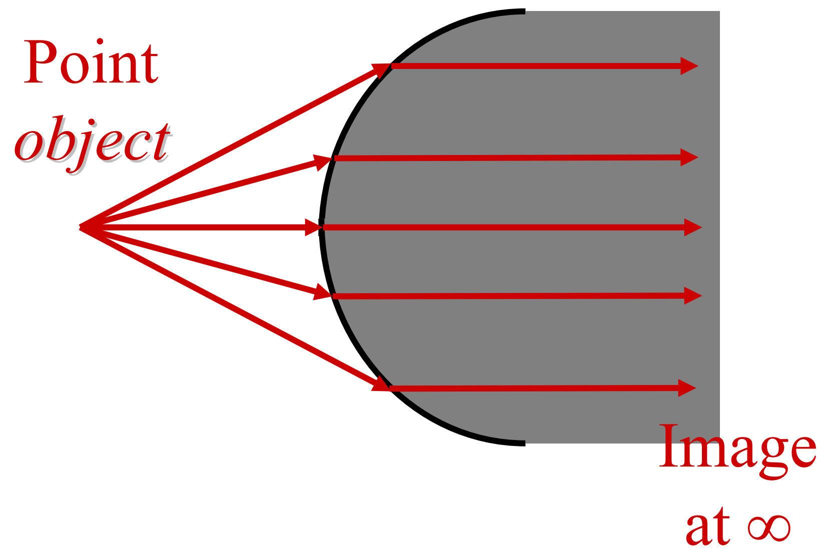
Note: $$f = n \times \frac{R}{n' - n} \equiv n \times \frac{R}{n' - n}$$

ambient refractive index at space of point object (n)

1/Power ($\frac{R}{n'-n}$)

Image / object focal lengths





$$f' = n' \times \frac{R}{n' - n}$$

$$= n' \times \frac{1}{\text{Power}}$$

$$f = n \times \frac{R}{n' - n}$$

$$= n \times \frac{1}{\text{Power}}$$

Matrix formulation /1

$$x_1 = x_0 + D_{01}\alpha_0$$

$$\alpha_1 = \alpha_0$$

translation by distance D_{01}

$$x_1' = x_1$$

$$\alpha_1' = \frac{n}{n'}\alpha_1 + \left[\frac{(n - n')}{n'R}\right]x_1$$

refraction by surface with radius of curvature R

$$x_2 = m_x x_0$$

$$\alpha_2 = -\frac{1}{f'}x_0 + m_\alpha \alpha_0$$

ray-tracing object-image transformation

$$\alpha_{\text{out}} = M_{11}\alpha_{\text{in}} + M_{12}x_{\text{in}}$$

$$x_{\text{out}} = M_{21}\alpha_{\text{in}} + M_{22}x_{\text{in}}$$

form common to all

Matrix formulation /2

$$\alpha_{\text{out}} = M_{11}\alpha_{\text{in}} + M_{12}x_{\text{in}}$$
$$x_{\text{out}} = M_{21}\alpha_{\text{in}} + M_{22}x_{\text{in}}$$

$$\begin{pmatrix} n_{\text{out}}\alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} n\alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix}$$

<u>Refraction by spherical surface</u>

$$x_1' = x_1$$
$$\alpha_1' = \frac{n}{n'}\alpha_1 + \left[\frac{(n-n')}{n'R}\right]x_1$$

Power

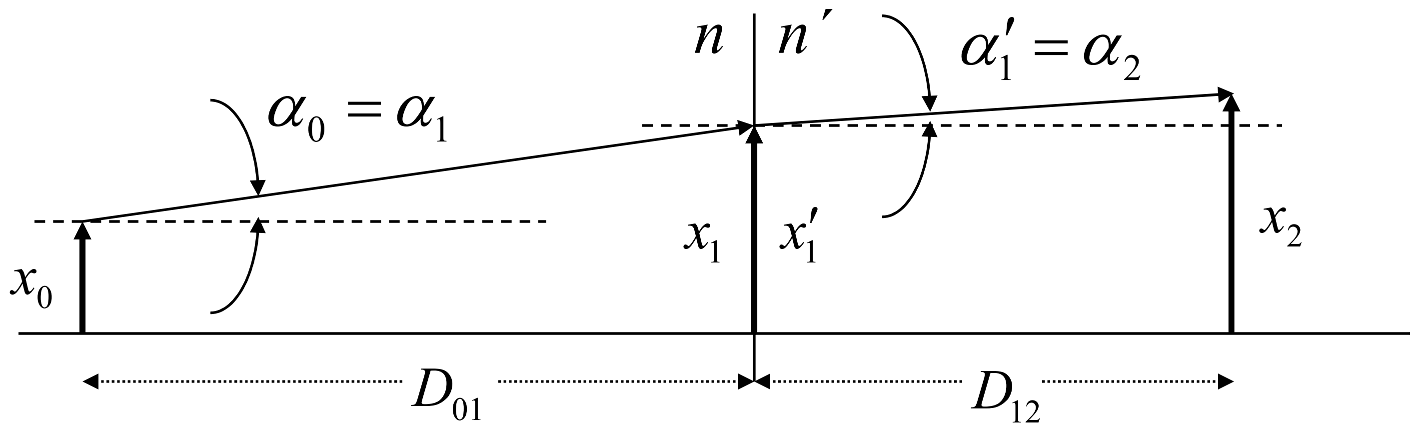
$$\begin{pmatrix} n'\alpha_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n'-n}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix}$$

<u>Translation through uniform medium</u>

$$x_1 = x_0 + D_{01}\alpha_0$$
$$\alpha_1 = \alpha_0$$

$$\begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{D_{01}}{n} & 1 \end{pmatrix} \begin{pmatrix} n\alpha_0 \\ x_0 \end{pmatrix}$$

Translation+refraction+translation



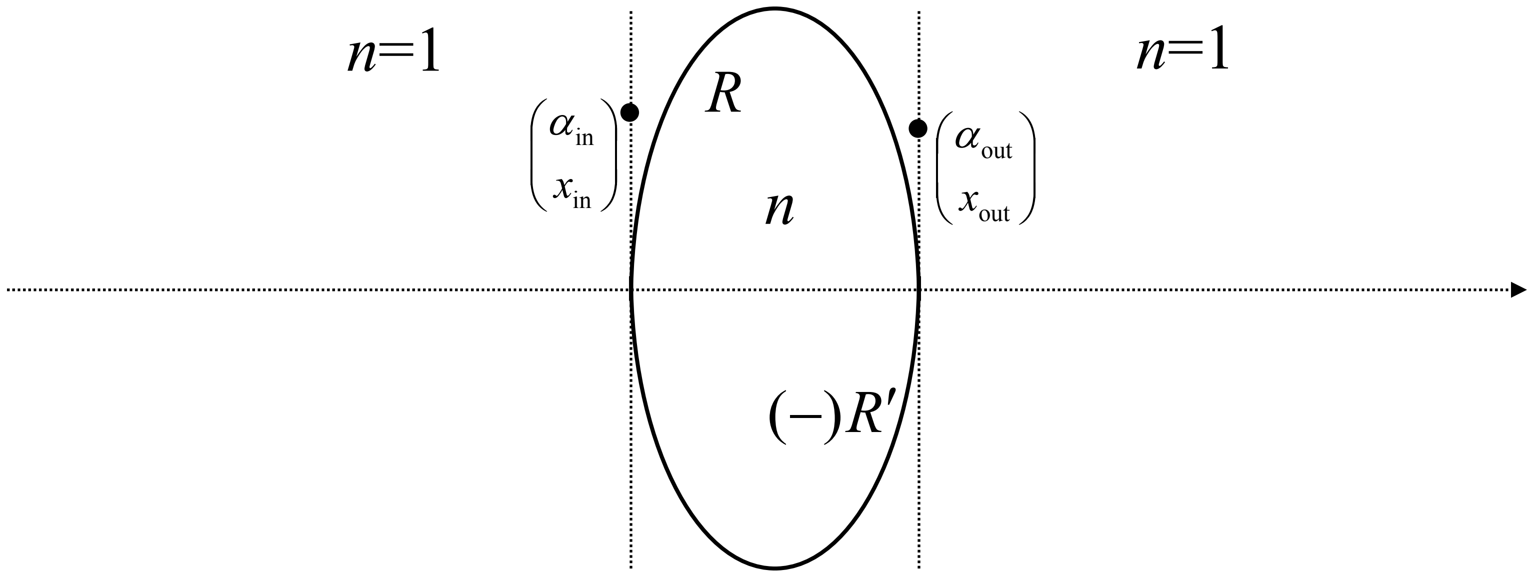
$$\begin{pmatrix} n'\alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{translation} \\ \text{by } D_{12} \end{pmatrix} \begin{pmatrix} \text{refraction} \\ \text{by r.curv. } R \end{pmatrix} \begin{pmatrix} \text{translation} \\ \text{by } D_{01} \end{pmatrix} \begin{pmatrix} n\alpha_0 \\ x_0 \end{pmatrix}$$

result…

$$n'\alpha_2 = \left[\frac{n-n'}{R}\right]x_0 + \left[n + \frac{(n-n')D_{01}}{R}\right]\alpha_0$$

$$x_2 = \left[\frac{(n-n')D_{12}}{n'R} + 1\right]x_0 + \left[D_{01} + \frac{nD_{12}}{n'} + \frac{(n-n')D_{01}D_{12}}{n'R}\right]\alpha_0$$

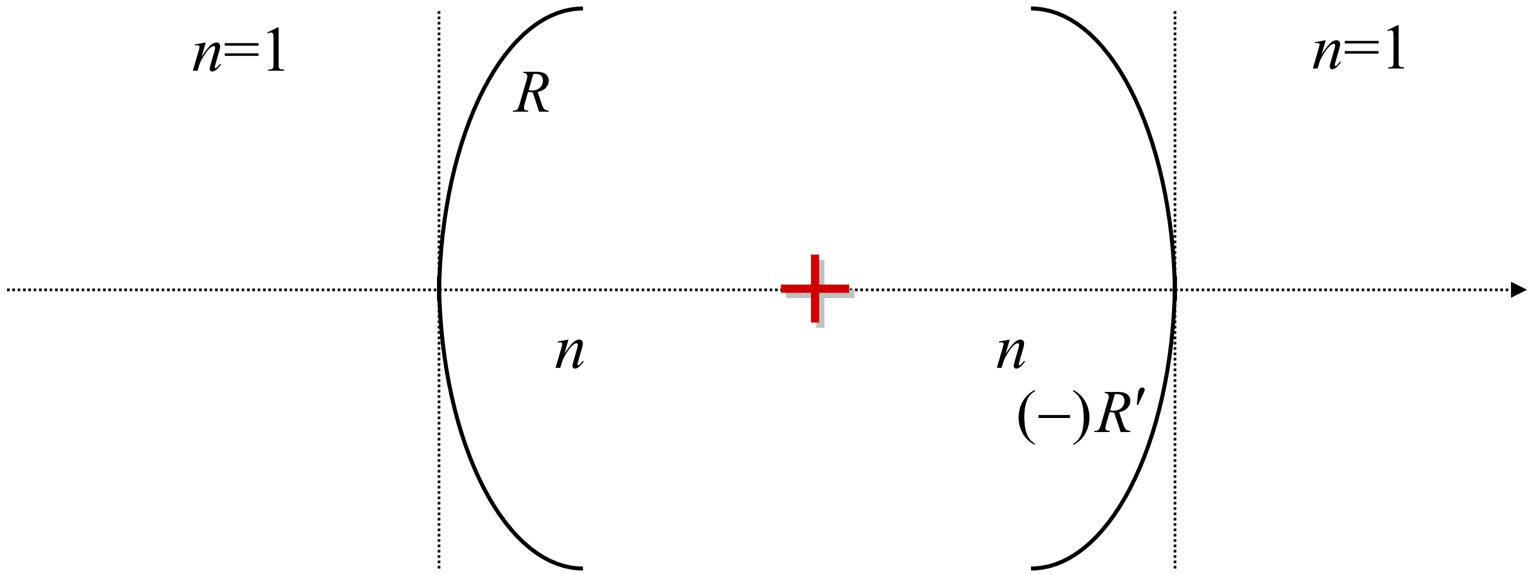
Thin lens in air



$$\begin{pmatrix}\alpha_{\text{out}}\\ x_{\text{out}}\end{pmatrix} = \begin{pmatrix}? & ?\\ ? & ?\end{pmatrix}\begin{pmatrix}\alpha_{\text{in}}\\ x_{\text{in}}\end{pmatrix}$$

Objective: specify input-output relationship

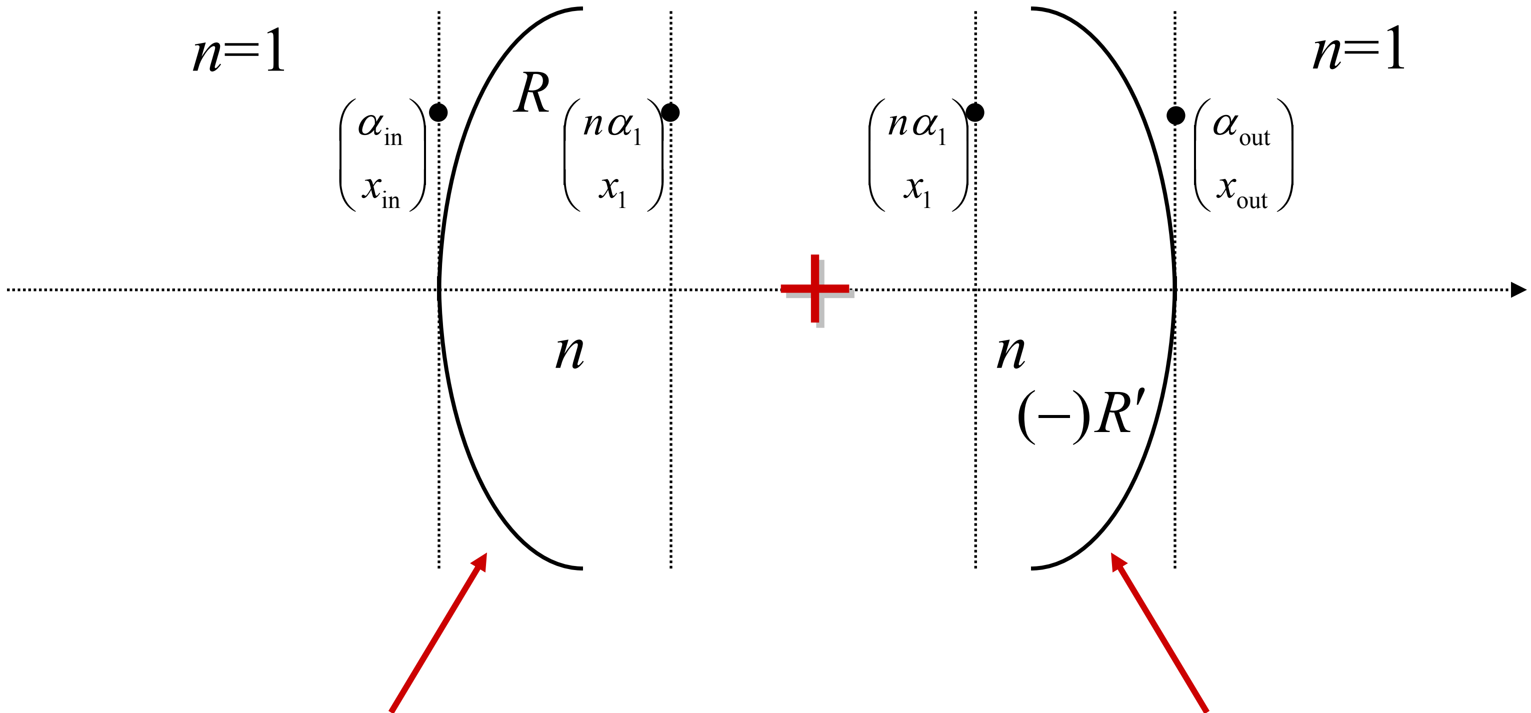
Thin lens in air



Model: refraction from first (positive) surface
+ refraction from second (negative) surface

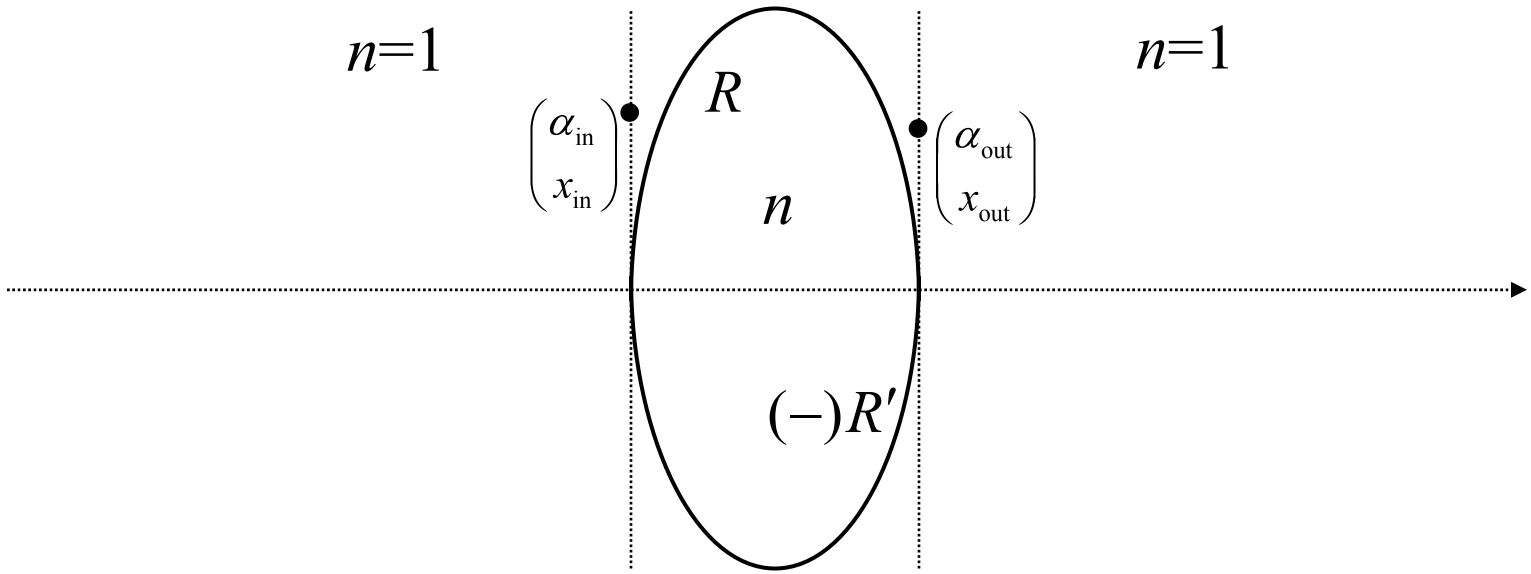
Ignore space in-between (*thin* lens approx.)

Thin lens in air



$$\begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{n-1}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix} \qquad \begin{pmatrix} \alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-n}{R'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix}$$

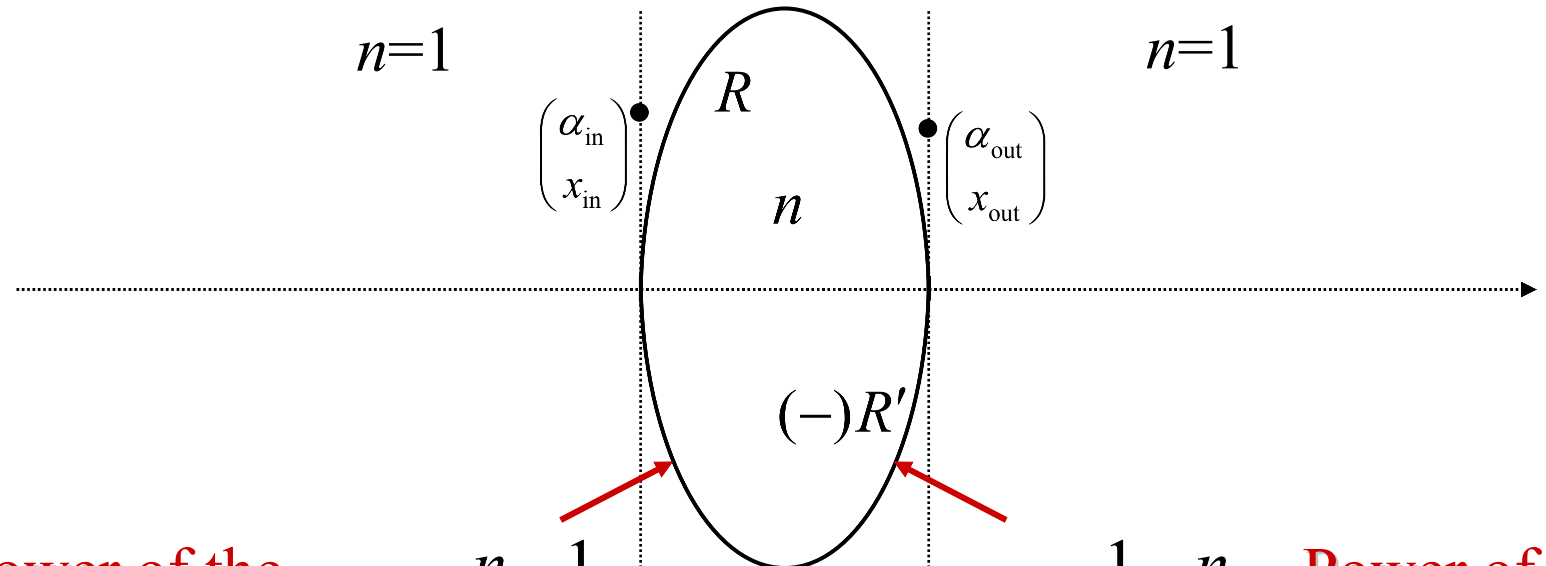
Thin lens in air



$$\begin{pmatrix}\alpha_{\text{out}}\\ x_{\text{out}}\end{pmatrix}=\begin{pmatrix}1 & -\dfrac{1-n}{R'}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -\dfrac{n-1}{R}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{in}}\\ x_{\text{in}}\end{pmatrix}$$

$$\begin{pmatrix}\alpha_{\text{out}}\\ x_{\text{out}}\end{pmatrix}=\begin{pmatrix}1 & -(n-1)\left(\dfrac{1}{R}-\dfrac{1}{R'}\right)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{in}}\\ x_{\text{in}}\end{pmatrix}$$

Thin lens in air



$$\begin{pmatrix}\alpha_{in}\\ x_{in}\end{pmatrix} \qquad \begin{pmatrix}\alpha_{out}\\ x_{out}\end{pmatrix}$$

Power of the first surface

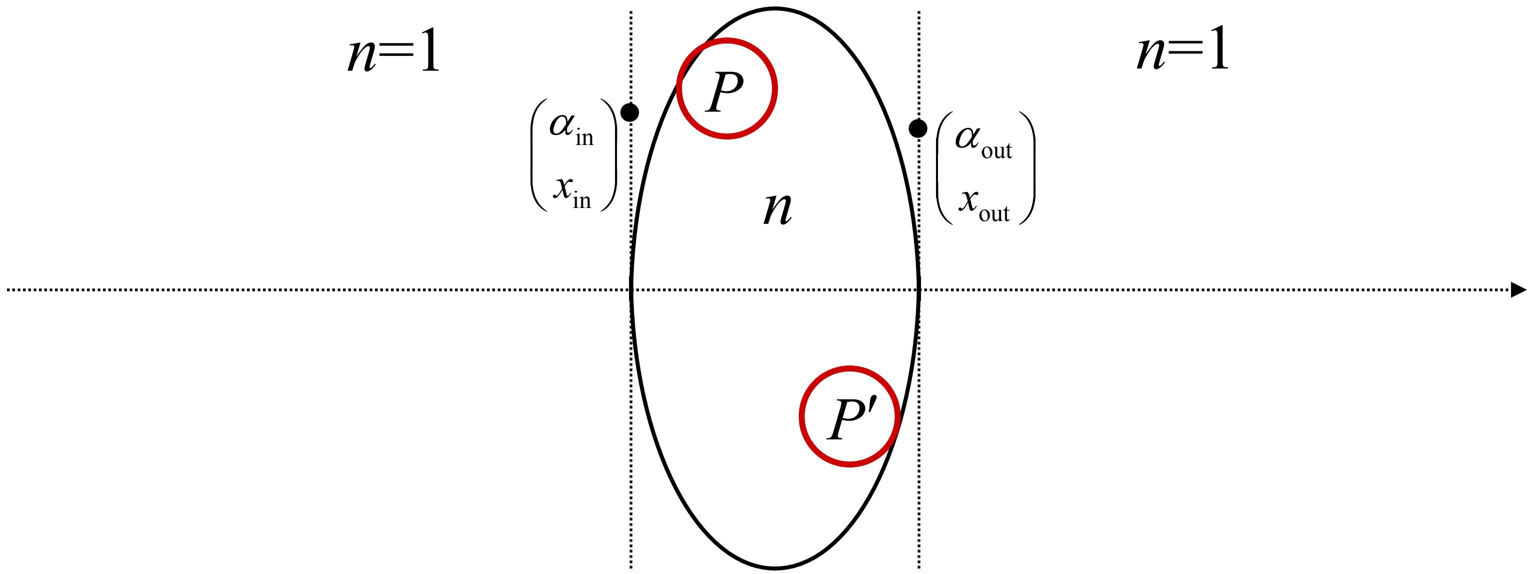
$$P=\frac{n-1}{R}$$

Power of the second surface

$$P'=\frac{1-n}{R'}$$

$$\begin{pmatrix}\alpha_{out}\\ x_{out}\end{pmatrix}=\begin{pmatrix}1 & -(n-1)\left(\frac{1}{R}-\frac{1}{R'}\right)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{in}\\ x_{in}\end{pmatrix}$$

Thin lens in air

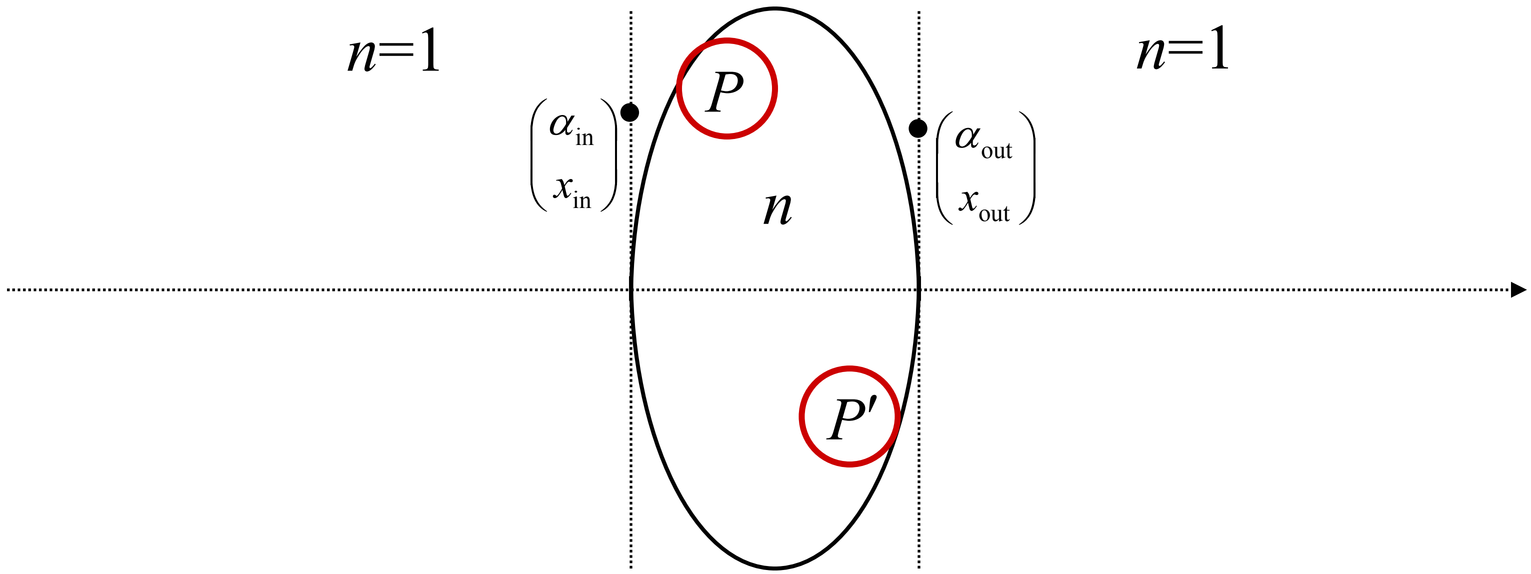


$$\begin{pmatrix}\alpha_{\text{out}}\\ x_{\text{out}}\end{pmatrix}=\begin{pmatrix}1 & -P_{\text{thin lens}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{in}}\\ x_{\text{in}}\end{pmatrix}$$

$$P_{\text{thin lens}}=P+P'=\frac{n-1}{R}+\frac{1-n}{R'}=\boxed{(n-1)\left(\frac{1}{R}-\frac{1}{R'}\right)}$$

Lens-maker's formula

Thin lens in air



$$\begin{pmatrix}\alpha_{out}\\ x_{out}\end{pmatrix}=\begin{pmatrix}1 & -P_{\text{thin lens}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{in}\\ x_{in}\end{pmatrix} \Rightarrow$$

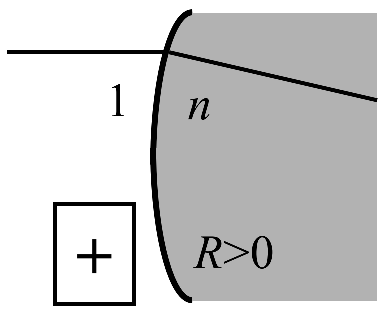
$$\alpha_{out}=\alpha_{in}-P_{\text{thin lens}}\,x_{in}$$

$$x_{out}=x_{in}$$

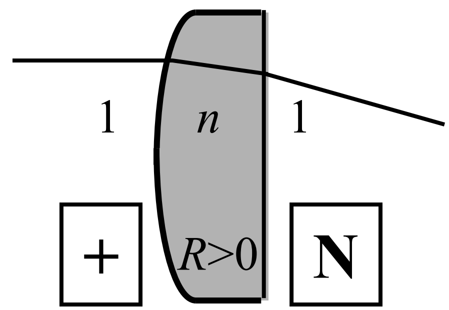
Ray bending is proportional to the distance from the axis

The power of surfaces

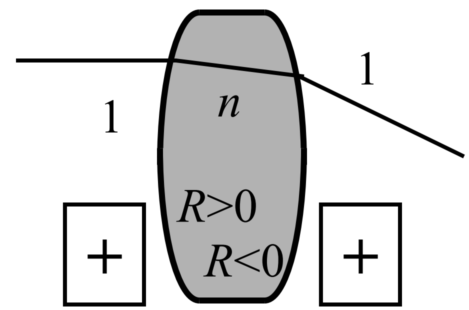
- Positive power bends rays “inwards”



Simple spherical
refractor (positive)

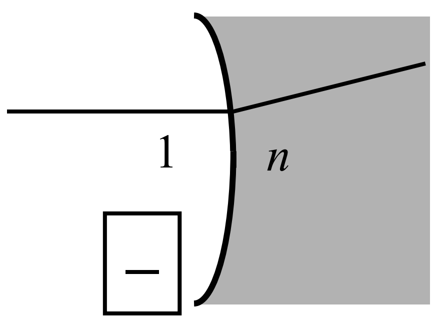


Plano-convex
lens

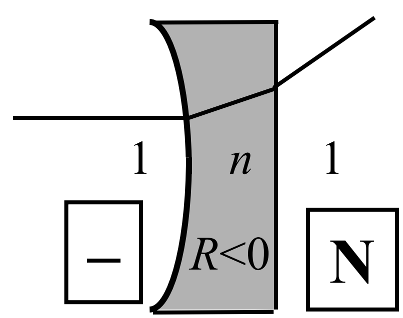


Bi-convex
lens

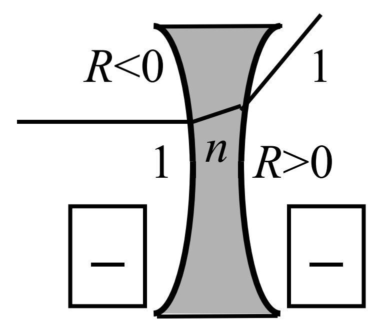
- Negative power bends rays “outwards”



Simple spherical
refractor (negative)



Plano-concave
lens



Bi-concave
lens