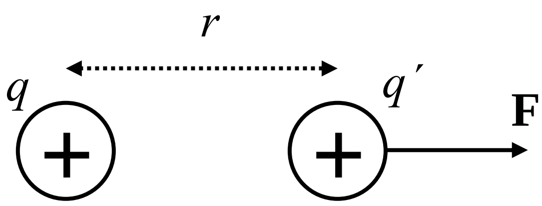
So far

- Geometrical Optics
  – Reflection and refraction from planar and spherical interfaces
  – Imaging condition in the paraxial approximation
  – Apertures & stops
  – Aberrations (violations of the imaging condition due to terms of order higher than paraxial or due to dispersion)
- Limits of validity of geometrical optics: features of interest are much bigger than the wavelength λ
  – Problem: point objects/images are *smaller* than λ!!!
  – So light focusing at a single point is an artifact of our approximations
  – To understand light behavior at scales $\sim \lambda$ we need to take into account the *wave* nature of light.

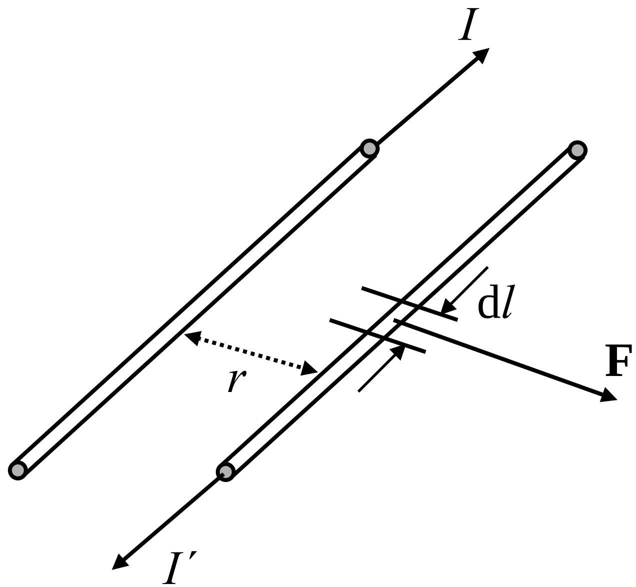
Step #1 towards wave optics: electro-dynamics

- Electromagnetic fields (definitions and properties) *in vacuo*
- Electromagnetic fields in matter
- Maxwell's equations
  - Integral form
  - Differential form
  - Energy flux and the Poynting vector
- The electromagnetic wave equation

Electric and magnetic forces



free charges



Coulomb force

$$|\mathbf{F}| = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

(dielectric) permitivity
of free space

Magnetic force

$$\frac{\mathrm{d}|\mathbf{F}|}{\mathrm{d}l} = \mu_0 \frac{II'}{2\pi r}$$

(magnetic) permeability
of free space

Note the units…

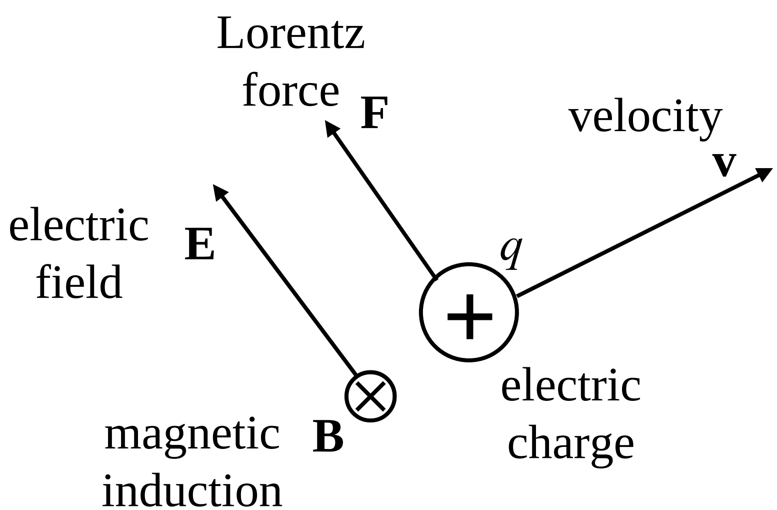
$$\begin{pmatrix}\text{Electric}\\ \text{force}\end{pmatrix} = \frac{1}{\varepsilon_0}\left(\frac{\text{Charge}}{\text{Distance}}\right)^2 \qquad |\mathbf{F}| = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

$$\begin{pmatrix}\text{Magnetic}\\ \text{force}\end{pmatrix} = \mu_0\left(\frac{\text{Charge}}{\text{Time}}\right)^2 \qquad \frac{\mathrm{d}|\mathbf{F}|}{\mathrm{d}l} = \mu_0\frac{II'}{2\pi r}$$

$$\Rightarrow (\varepsilon_0\mu_0)^{1/2} \equiv \left(\frac{\text{Distance}}{\text{Time}}\right) \equiv (\text{Speed})$$

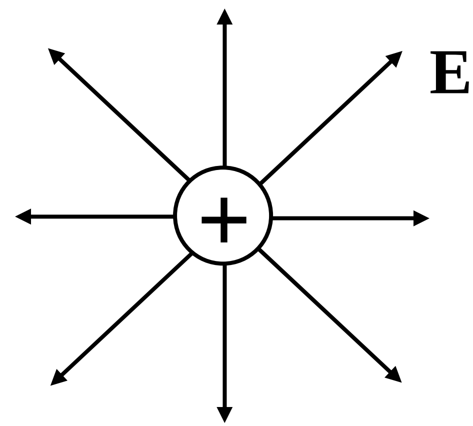
Electric and magnetic fields

Observation

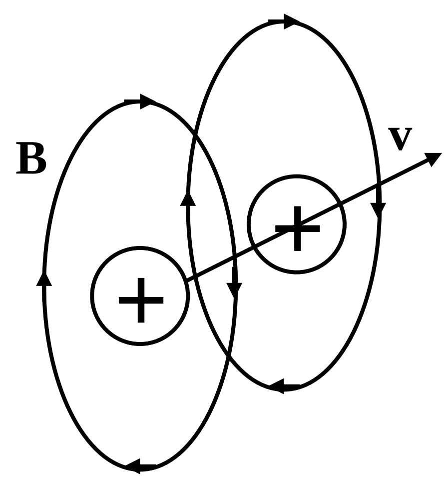


$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Generation

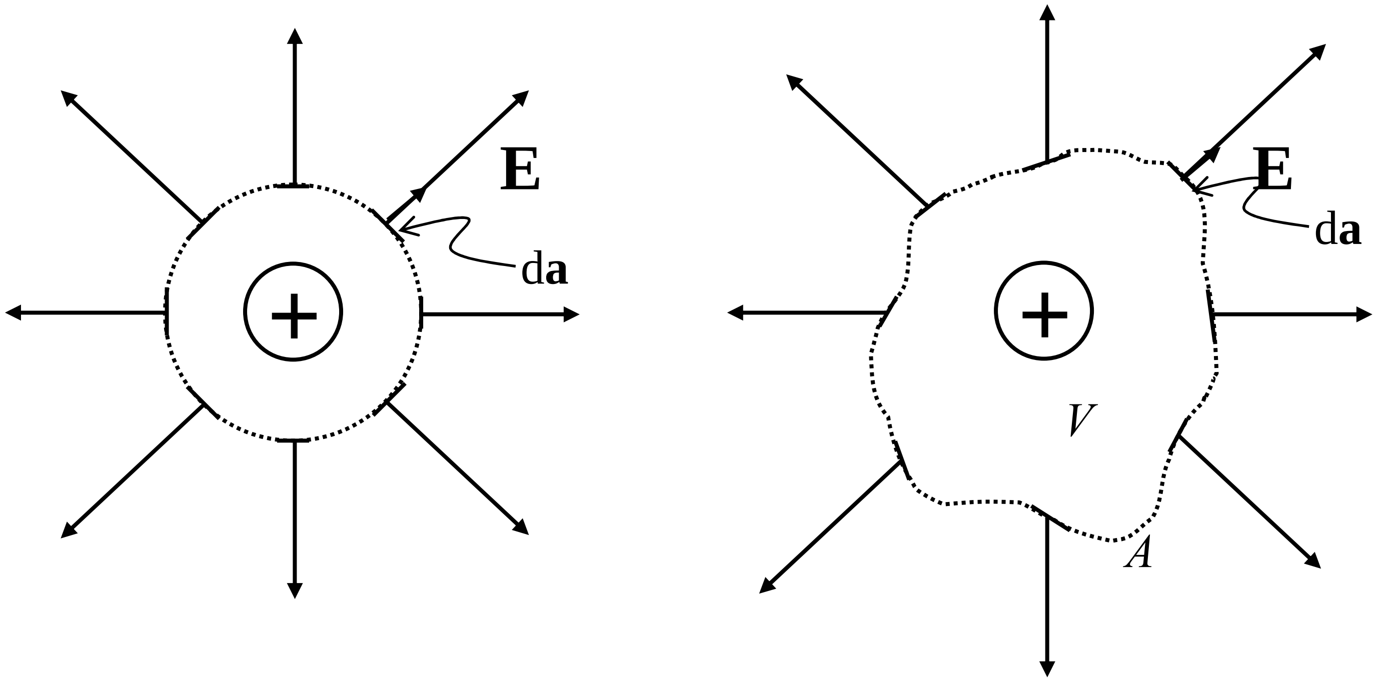


static charge:
$\Rightarrow$
electric field



electric current
(moving charges):
$\Rightarrow$
magnetic field

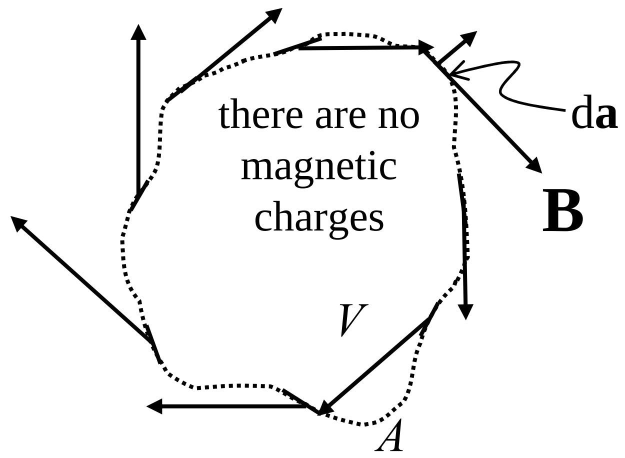
Gauss Law: electric fields



$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0} \iiint_V \rho \, \mathrm{d}V \quad \overset{\text{Gauss theorem}}{\Longleftrightarrow} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

charge density

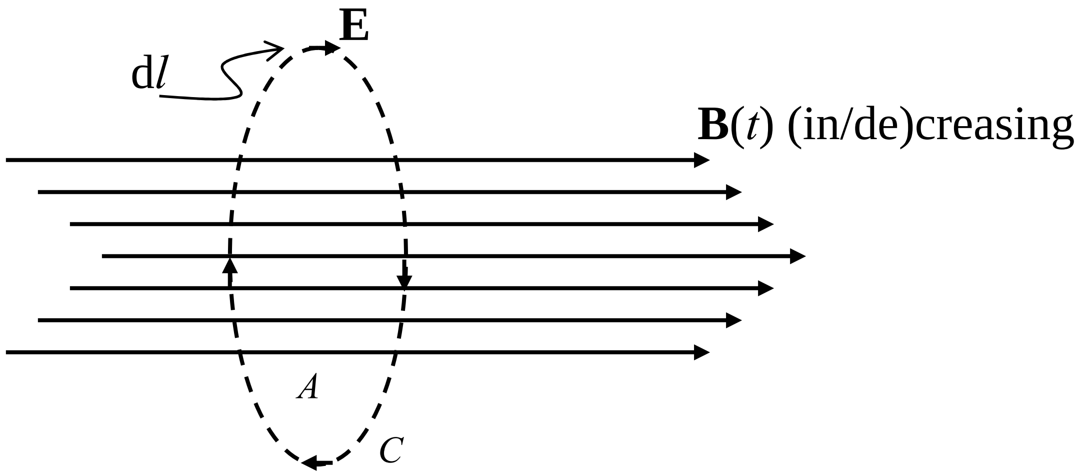
Gauss Law: magnetic fields



$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0 \quad \overset{\text{Gauss theorem}}{\Longleftrightarrow} \quad \nabla \cdot \mathbf{B} = 0$$

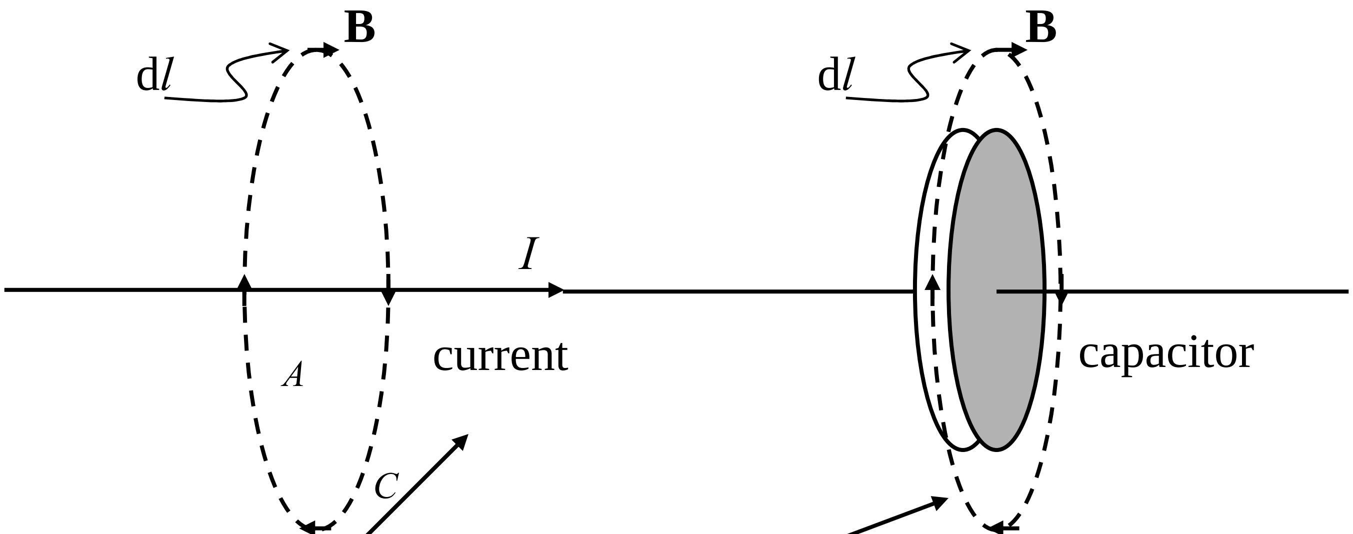
"magnetic charge" density

Faraday's Law: electromotive force



$$\oint_C \mathbf{E}\cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot \mathrm{d}\mathbf{a} \quad \xleftrightarrow{\text{Stokes theorem}} \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law: magnetic induction



$$\oint_C \mathbf{B}\cdot \mathrm{d}l = \mu_0\left(\iint_A \mathbf{J}\cdot \mathrm{d}\mathbf{a} + \underbrace{\iint_A \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\cdot \mathrm{d}\mathbf{a}}\right) \quad \xLeftrightarrow{\text{Stokes theorem}} \quad \nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

current density

Maxwell's extension,
Displacement current

Maxwell's equations

(in vacuo)

$$\oiint_A \mathbf{E}\cdot \mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0}\iiint_V \rho \,\mathrm{d}V \quad \Longleftrightarrow \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss/electric

$$\oiint_A \mathbf{B}\cdot \mathrm{d}\mathbf{a} = 0 \quad \Longleftrightarrow \quad \nabla\cdot\mathbf{B} = 0$$

Gauss/magnetic

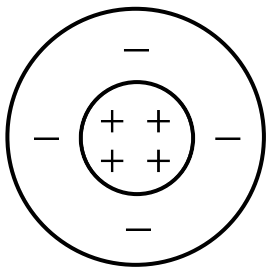
$$\oint_C \mathbf{E}\cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot\mathrm{d}\mathbf{a} \quad \Longleftrightarrow \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Faraday

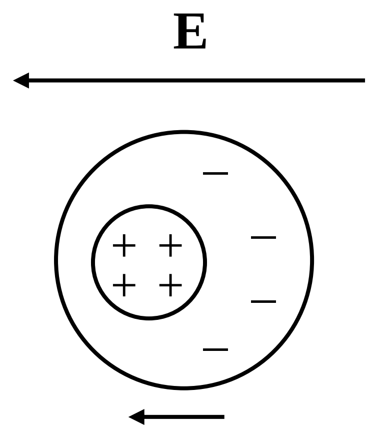
$$\oint_C \mathbf{B}\cdot \mathrm{d}l = \mu_0\iint_A \left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\mathrm{d}\mathbf{a} \quad \Longleftrightarrow \quad \nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$$

Ampere-Maxwell

Electric fields in dielectric media



atom in equilibrium



atom under electric field:

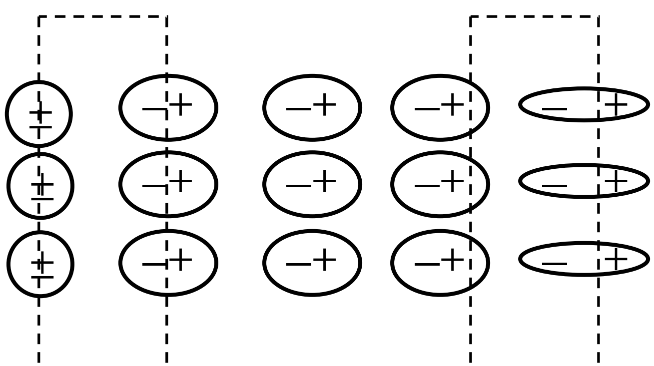
- charge neutrality is preserved
- spatial distribution of charges becomes assymetric

$$\mathbf{p} = q_+\mathbf{r}_+ - q_-\mathbf{r}_-$$

Dipole moment

$$\mathbf{P} = \sum \mathbf{p}$$

Polarization



Spatially variant polarization induces *local* charge imbalances (bound charges)

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$$

Electric displacement

Gauss Law: $$\nabla\cdot\mathbf{E}=\frac{1}{\varepsilon_0}\rho_{\text{total}}=\frac{1}{\varepsilon_0}\left(\rho_{\text{free}}+\rho_{\text{bound}}\right)$$

$$=\frac{1}{\varepsilon_0}\left(\rho_{\text{free}}-\nabla\cdot\mathbf{P}\right)$$

$$\nabla\cdot\underbrace{\left(\varepsilon_0\mathbf{E}+\mathbf{P}\right)}=\rho_{\text{free}}$$

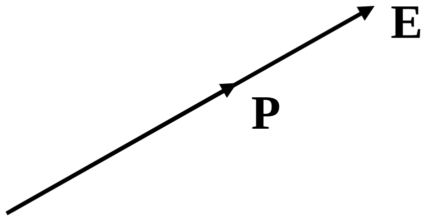
Electric displacement field: $\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$ $\Longrightarrow$ $\nabla\cdot\mathbf{D}=\rho_{\text{free}}$

Linear, isotropic polarizability: $\mathbf{P}=\varepsilon_0\chi\mathbf{E}$

$$\Longrightarrow\ \mathbf{D}=\varepsilon_0\left(1+\chi\right)\mathbf{E}\equiv\varepsilon\mathbf{E}$$

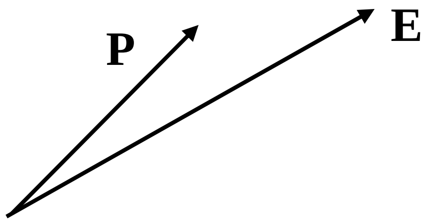
General cases of polarization

Linear, isotropic polarizability: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$



Linear, anisotropic polarizability:

$$\mathbf{P} = \varepsilon_0 \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \mathbf{E}$$



Nonlinear, isotropic polarizability: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E} + \varepsilon_0 \chi^{(2)} \mathbf{EE} + \ldots$

etc.

Constitutive relationships

E: electric field

D: electric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

polarization

B: magnetic induction

H: magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

magnetization

Maxwell's equations

(in matter)

$$\oint\!\!\!\oint_A \mathbf{D}\cdot \mathrm{d}\mathbf{a} = \iiint_V \rho_{\text{free}}\,\mathrm{d}V \quad\Longleftrightarrow\quad \nabla\cdot\mathbf{D} = \rho_{\text{free}} \qquad \text{Gauss/electric}$$

$$\oint\!\!\!\oint_A \mathbf{B}\cdot \mathrm{d}\mathbf{a} = 0 \quad\Longleftrightarrow\quad \nabla\cdot\mathbf{B} = 0 \qquad \text{Gauss/magnetic}$$

$$\oint_C \mathbf{E}\cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot\mathrm{d}\mathbf{a} \quad\Longleftrightarrow\quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \text{Faraday}$$

$$\oint_C \mathbf{H}\cdot \mathrm{d}l = \mu_0\iint_A\left(\mathbf{J}_{\text{free}} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathrm{d}\mathbf{a} \quad\Longleftrightarrow\quad \nabla\times\mathbf{H} = \mu_0\left(\mathbf{J}_{\text{free}} + \frac{\partial\mathbf{D}}{\partial t}\right)$$

Ampere-Maxwell

Maxwell's equations $\Rightarrow$ wave equation

(in linear, anisotropic, non-magnetic matter, no free charges/currents)

$$\left.\begin{aligned}
&\nabla\cdot(\varepsilon\mathbf{E})=0\\
&\nabla\cdot\mathbf{B}=0\\
&\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\\
&\nabla\times\mathbf{B}=\mu_0\frac{\partial(\varepsilon\mathbf{E})}{\partial t}
\end{aligned}\right\}
\xrightarrow{\text{matter spatially and temporally invariant}}
\left\{\begin{aligned}
&\nabla\cdot\mathbf{E}=0\\
&\nabla\cdot\mathbf{B}=0\\
&\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\\
&\nabla\times\mathbf{B}=\mu_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}
\end{aligned}\right.$$

$$\left.\begin{aligned}
&\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\Rightarrow\nabla\times(\nabla\times\mathbf{E})=-\frac{\partial(\nabla\times\mathbf{B})}{\partial t}=-\mu_0\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}\\
&\nabla\times(\nabla\times\mathbf{E})=\nabla\underbrace{(\nabla\cdot\mathbf{E})}_{=0}-\nabla\cdot\nabla\mathbf{E}
\end{aligned}\right\}\Rightarrow
\boxed{\nabla^2\mathbf{E}-\mu_0\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0}$$

electromagnetic wave equation

Maxwell's equations $\Rightarrow$ wave equation

<u>(in linear, anisotropic, non-magnetic matter, no free charges/currents)</u>

$$\left.\begin{aligned}
\nabla\cdot\mathbf{E}&=0\\
\nabla\cdot\mathbf{B}&=0\\
\nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t}\\
\nabla\times\mathbf{B}&=\mu_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}
\end{aligned}\right\}\Longrightarrow\quad \boxed{\nabla^2\mathbf{E}-\mu_0\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0}$$

$$\mu_0\varepsilon\equiv\frac{1}{c^2}\qquad \boxed{\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0}$$

wave velocity

Light velocity and refractive index

$$\mu_0 \varepsilon_0 \equiv \frac{1}{c_{\text{vacuum}}^2}$$

c_{vacuum}: speed of light in vacuum

$$\varepsilon = (1 + \chi)\varepsilon_0 \equiv n^2 \varepsilon_0$$

n: index of refraction

$$\mu_0 \varepsilon = \frac{n^2}{c_{\text{vacuum}}^2} \equiv \frac{1}{c^2}$$

$c \equiv c_{\text{vacuum}}/n$: speed of light in medium of refr. index n

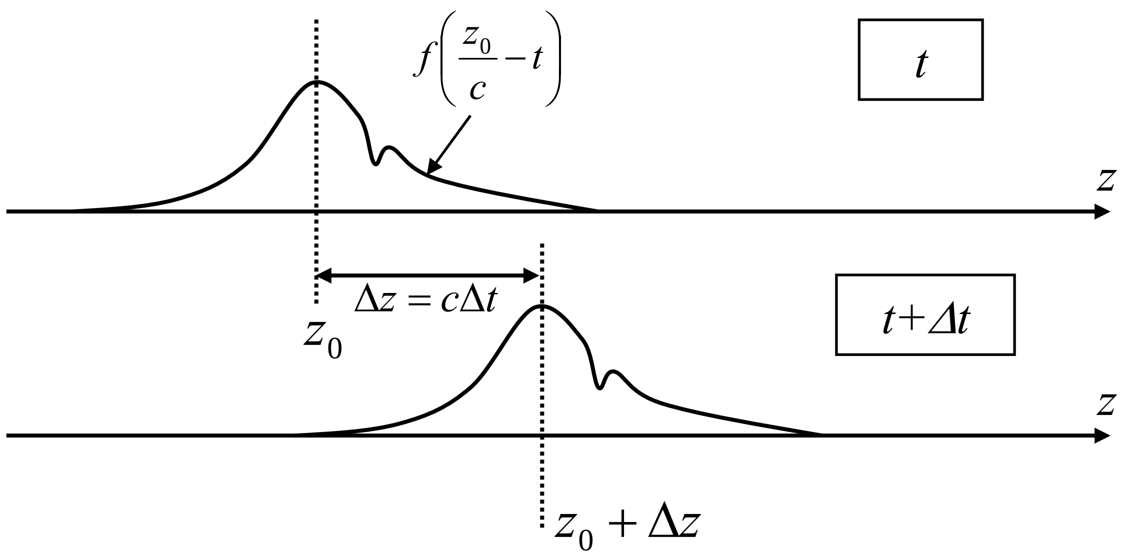
Simplified (1D, scalar) wave equation

$$\boxed{\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0}$$

- E is a scalar quantity (e.g. the component E_y of an electric field **E**)
- the geometry is symmetric in x, y $\Rightarrow$ the x, y derivatives are zero

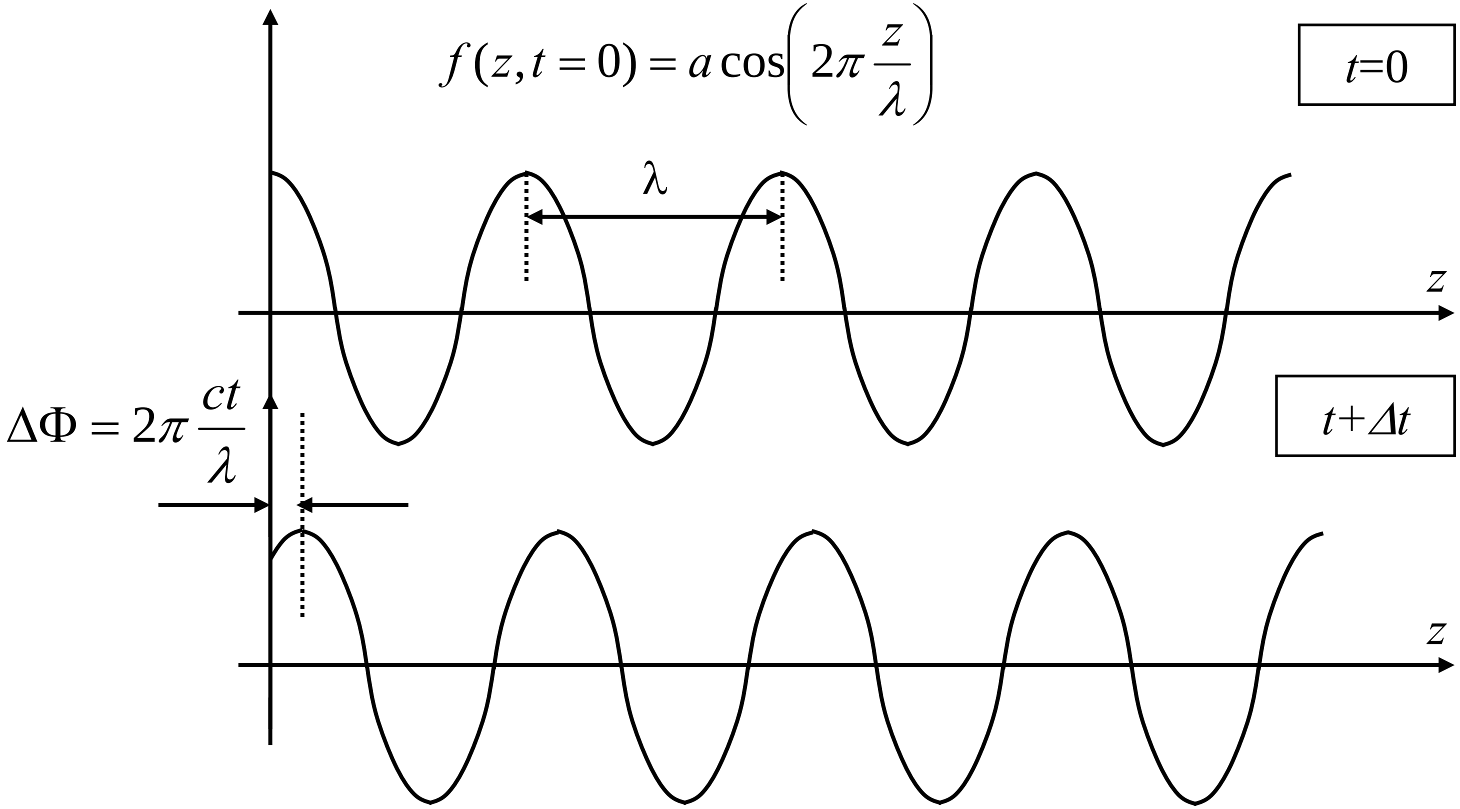
Solution:

$$E(z,t) = f\left(\frac{z}{c} - t\right) + g\left(\frac{z}{c} + t\right)$$



Special case: harmonic solution

$$f(z, t=0) = a\cos\left(2\pi\frac{z}{\lambda}\right)$$



$$f(z,t) = a\cos\underbrace{\left(2\pi\frac{(z-ct)}{\lambda}\right)}_{\Phi} \equiv a\cos\left(2\pi\nu\left(\frac{z}{c}-t\right)\right)$$

$\boxed{c = \lambda\nu}$
dispersion relation

Complex representation of waves

$$f(z,t) = A\cos\left(\frac{2\pi}{\lambda}z - 2\pi\nu t - \phi\right)$$

angular frequency

$$f(z,t) = A\cos(kz - \omega t - \phi) \qquad k = \frac{2\pi}{\lambda}, \;\; \omega = 2\pi\nu$$

wave-number

$$f(z,t) = A\cos(kz - \omega t - \phi)$$

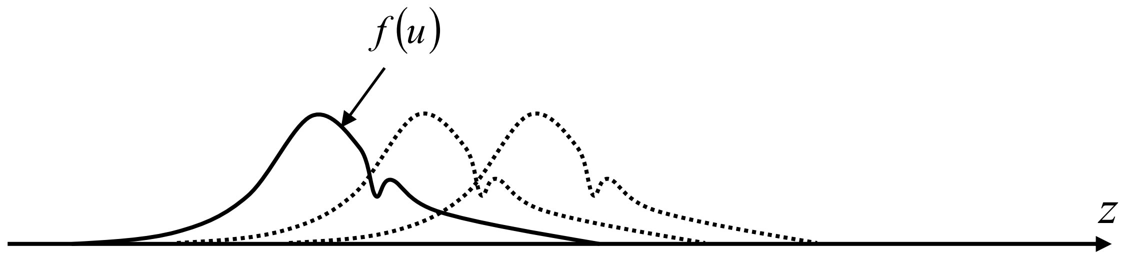
$$\hat{f}(z,t) = A\cos(kz - \omega t - \phi) + iA\sin(kz - \omega t - \phi)$$

i.e. $f(z,t) = \mathrm{Re}\left\{\hat{f}(z,t)\right\}$

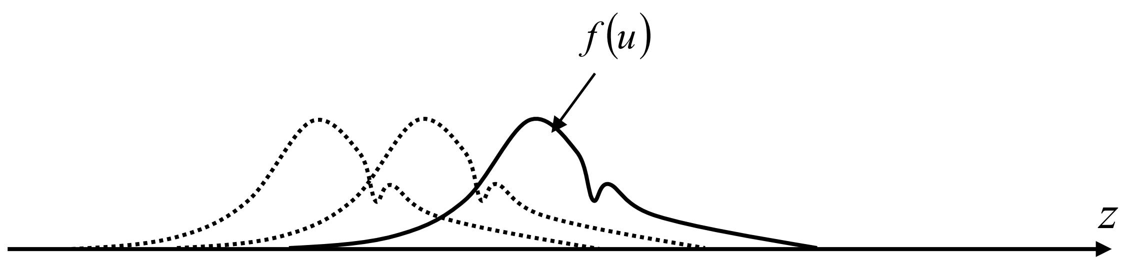
$\hat{f}(z,t)$ aka $\hat{f}(z,t) = A\,\mathrm{e}^{i(kz-\omega t-\phi)}$ complex representation

$A\,\mathrm{e}^{-i\phi}$ complex amplitude or "phasor"

Time reversal

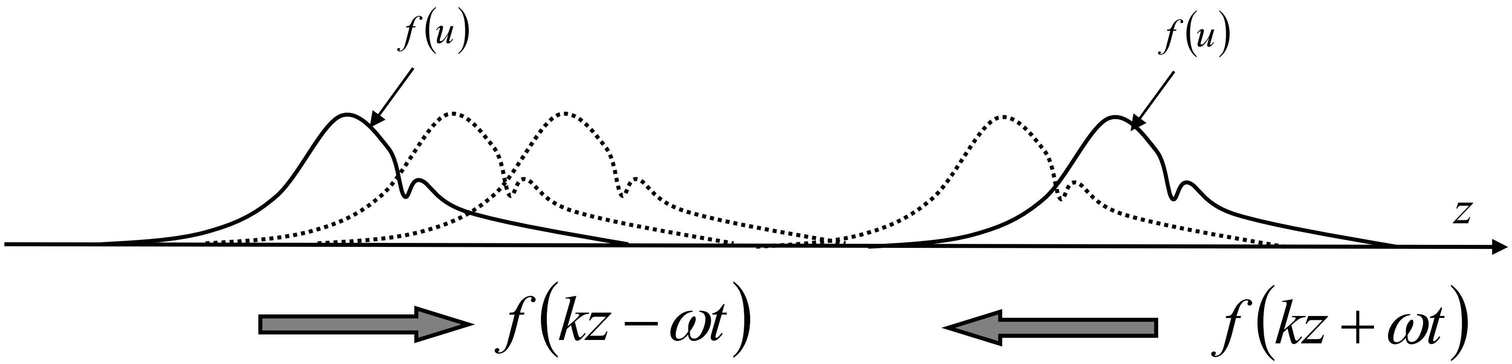


$$f\left(kz-\omega t\right)$$



$$f\left(kz+\omega t\right)$$

Superposition



$\Rightarrow f(kz-\omega t)+f(kz+\omega t)$ is also a solution

More generally, $f(kz-\omega t)+g(kz+\omega t)$ is a solution

Even more generally,

$$\begin{aligned}&f_1(kz-\omega t)+f_2(kz-\omega t)+\ldots\\&+g_1(kz+\omega t)+g_2(kz+\omega t)+\ldots\end{aligned}$$ is a solution

LINEARITY

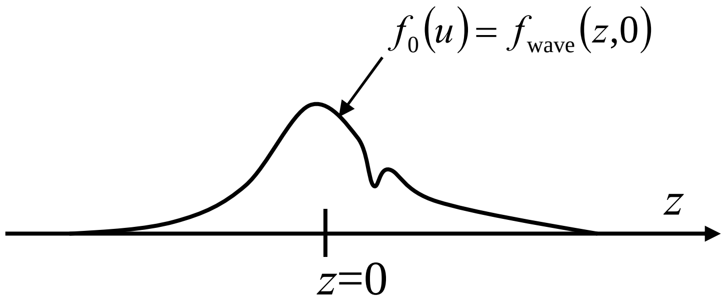
$\Updownarrow$

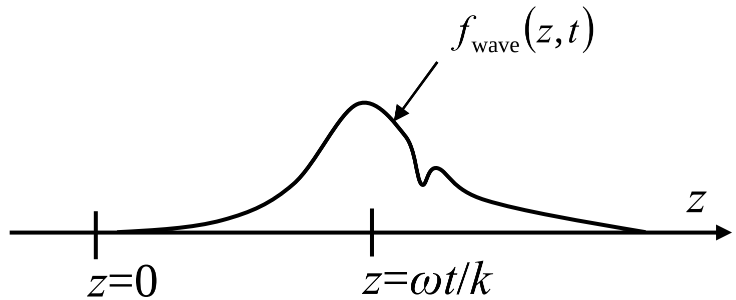
SUPERPOSITION

What is the solution to the wave equation?

- In general: the solution is an (arbitrary) superposition of propagating waves
- Usually, we have to impose
  - initial conditions (as in any differential equation)
  - boundary condition (as in most partial differential equations)

<u>Example:</u> initial value problem





$$f_{\text{wave}}(z,0) \equiv f_0(u) \Rightarrow f_{\text{wave}}(z,t) = f_0(kz - \omega t)$$

What is the solution to the wave equation?

- In general: the solution is an (arbitrary) superposition of propagating waves
- Usually, we have to impose
  - initial conditions (as in any differential equation)
  - boundary condition (as in most partial differential equations)

- Boundary conditions: we will not deal much with them in this class, but it is worth noting that physically they explain interesting phenomena such as waveguiding from the wave point of view (we saw already one explanation as TIR).

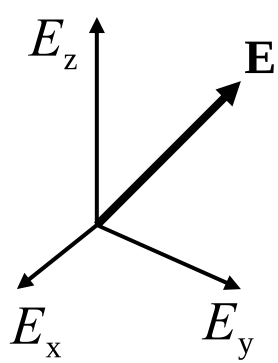
Elementary waves: plane, spherical

The EM vector wave equation

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

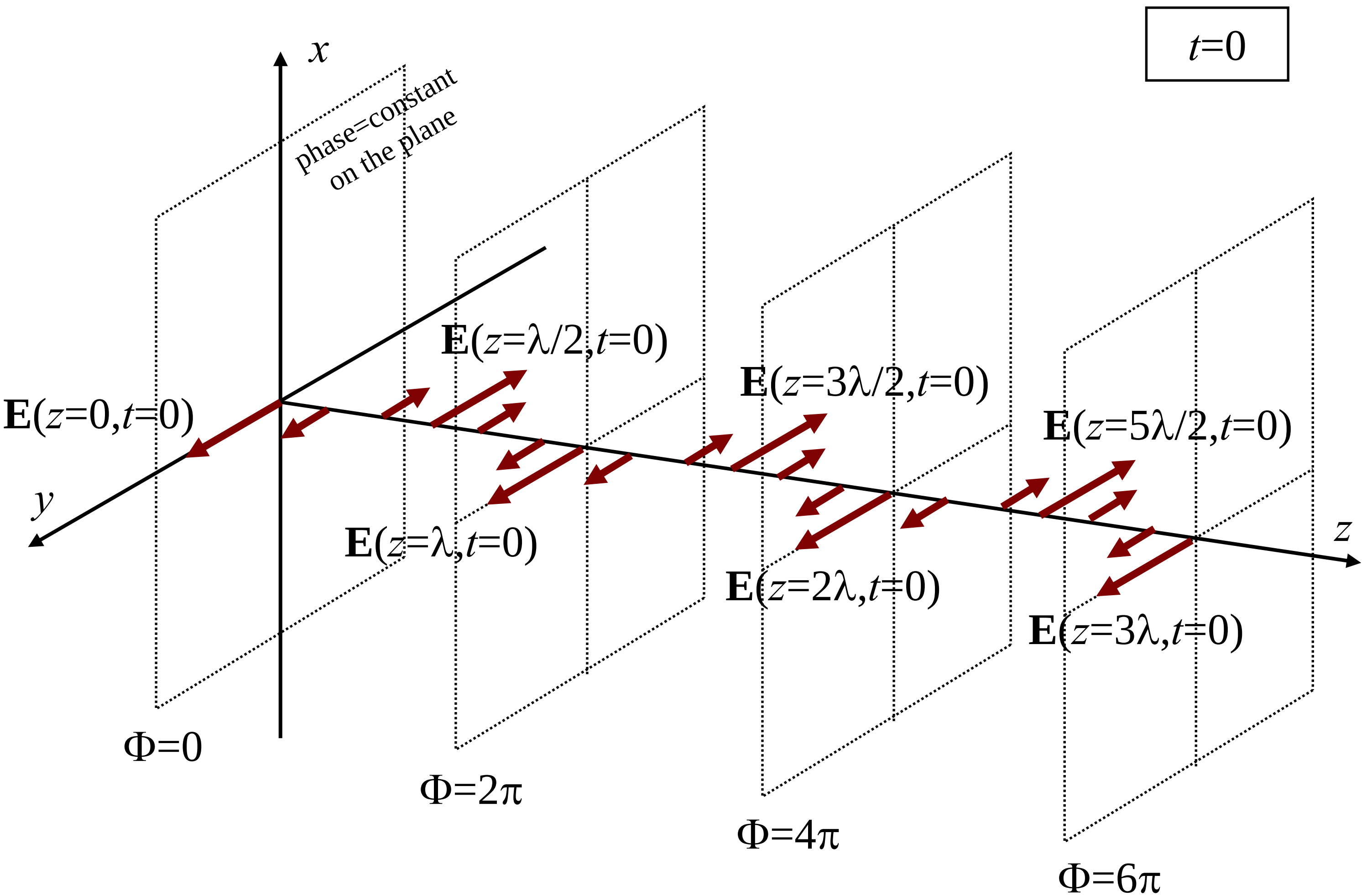
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{E} = E_{\mathrm{x}}\hat{\mathbf{x}} + E_{\mathrm{y}}\hat{\mathbf{y}} + E_{\mathrm{z}}\hat{\mathbf{z}}$$

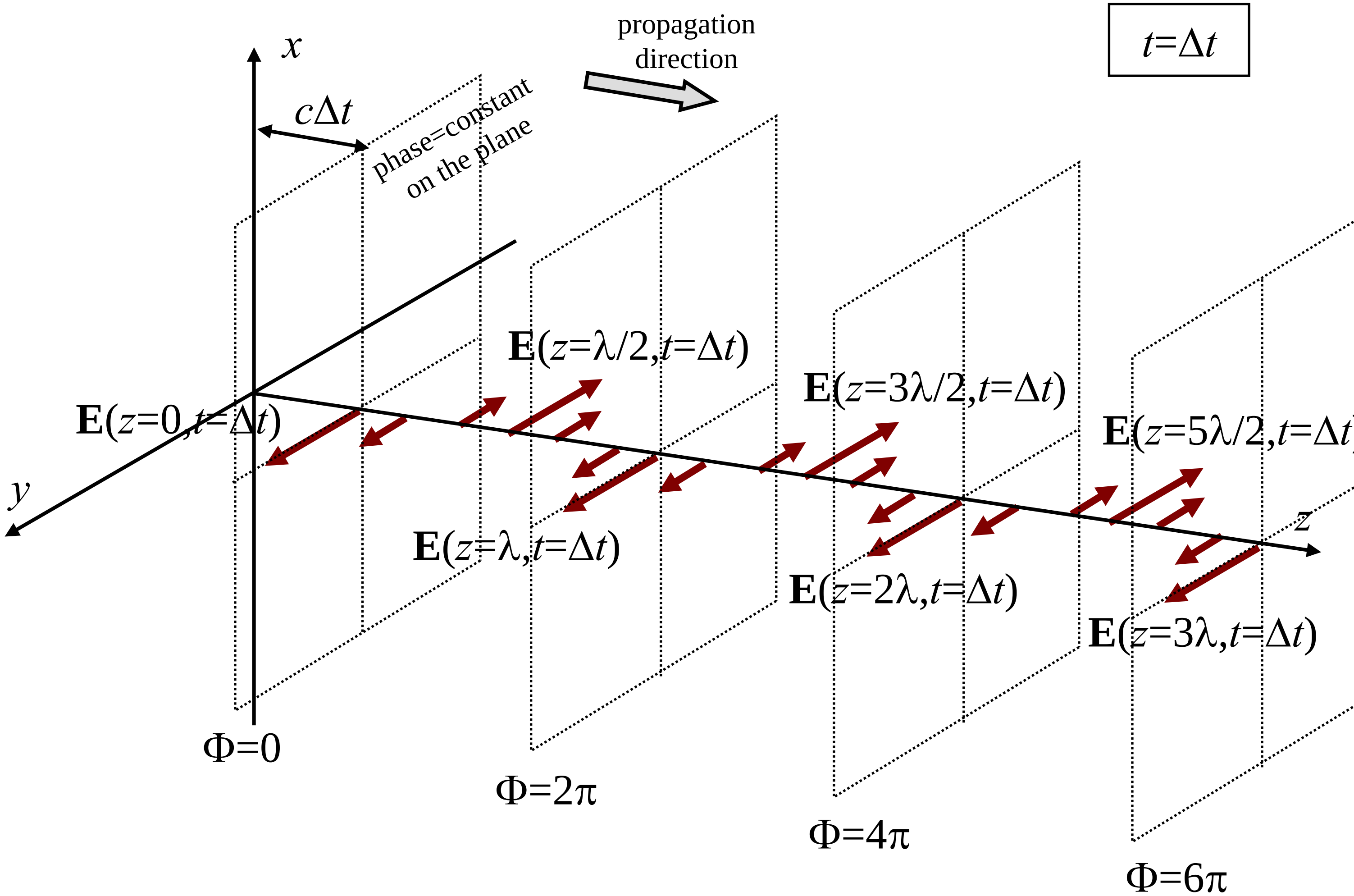


$$\left.\begin{aligned}
\frac{\partial^2 E_{\mathrm{x}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{x}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{x}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{x}}}{\partial t^2} &= 0\\
\frac{\partial^2 E_{\mathrm{y}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{y}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{y}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{y}}}{\partial t^2} &= 0\\
\frac{\partial^2 E_{\mathrm{z}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{z}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{z}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{z}}}{\partial t^2} &= 0
\end{aligned}\right\}$$

Harmonic solution in 3D: plane wave



Plane wave propagating



Complex representation of 3D waves

$$f(x,y,z,t) = A\cos\left(\frac{2\pi}{\lambda}\left(x\cos\alpha + y\cos\beta + z\cos\gamma\right) - \omega t - \phi_0\right)$$

$$f(x,y,z,t) = A\cos\left(k_x x + k_y y + k_z z - \omega t - \phi_0\right) \qquad k_x = \frac{2\pi}{\lambda}\cos\alpha, \text{ etc.}$$

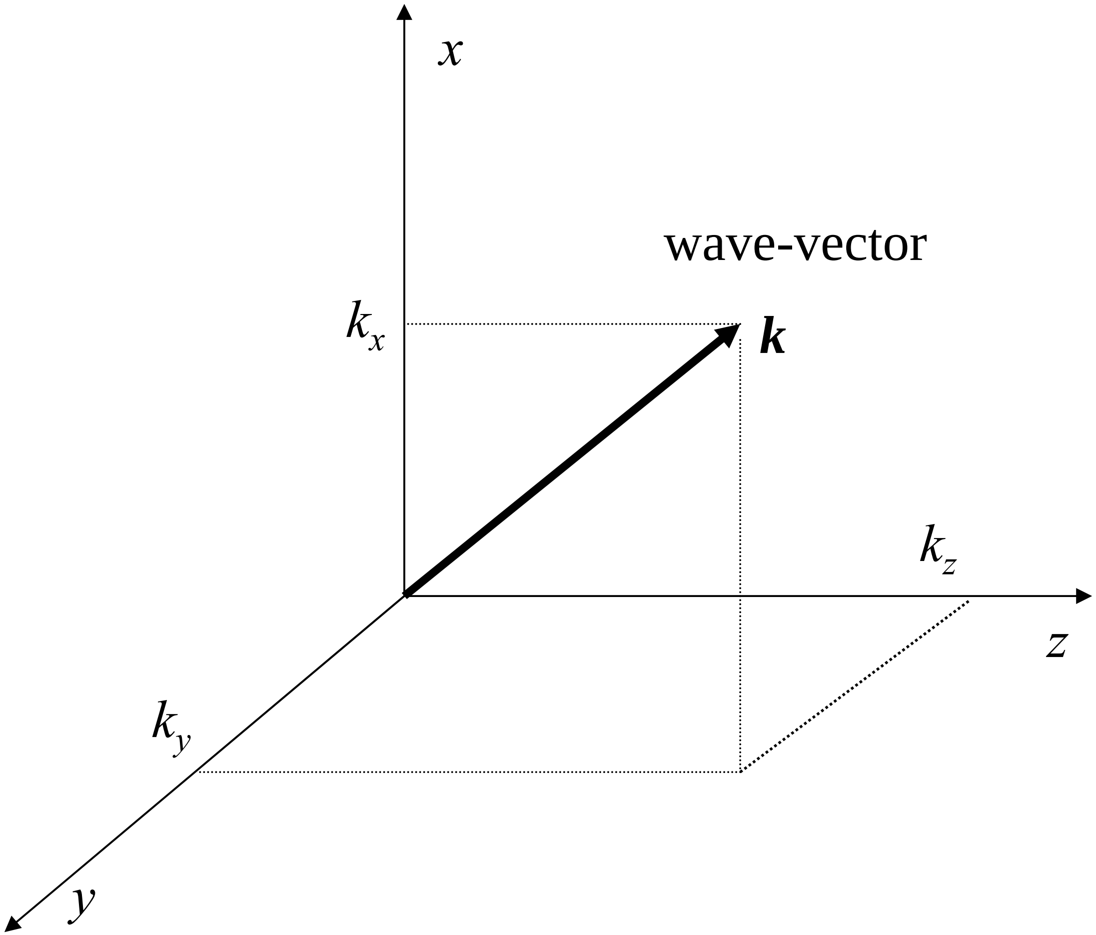
$$\hat{f}(x,y,z,t) = A\,\mathrm{e}^{i\left(k_x x + k_y y + k_z z - \omega t - \phi_0\right)} \quad \text{complex representation}$$

$$A\,\mathrm{e}^{-i\phi(x,y,z)} \quad \text{complex amplitude or "phasor"}$$

$$\text{where } \phi(x,y,z) \equiv k_x x + k_y y + k_z z - \phi_0$$

"Wavefront" : surface $\phi(x,y,z) = \text{const.}$

Plane wave



Plane wave

$$a(\mathbf{r}) = A_0\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

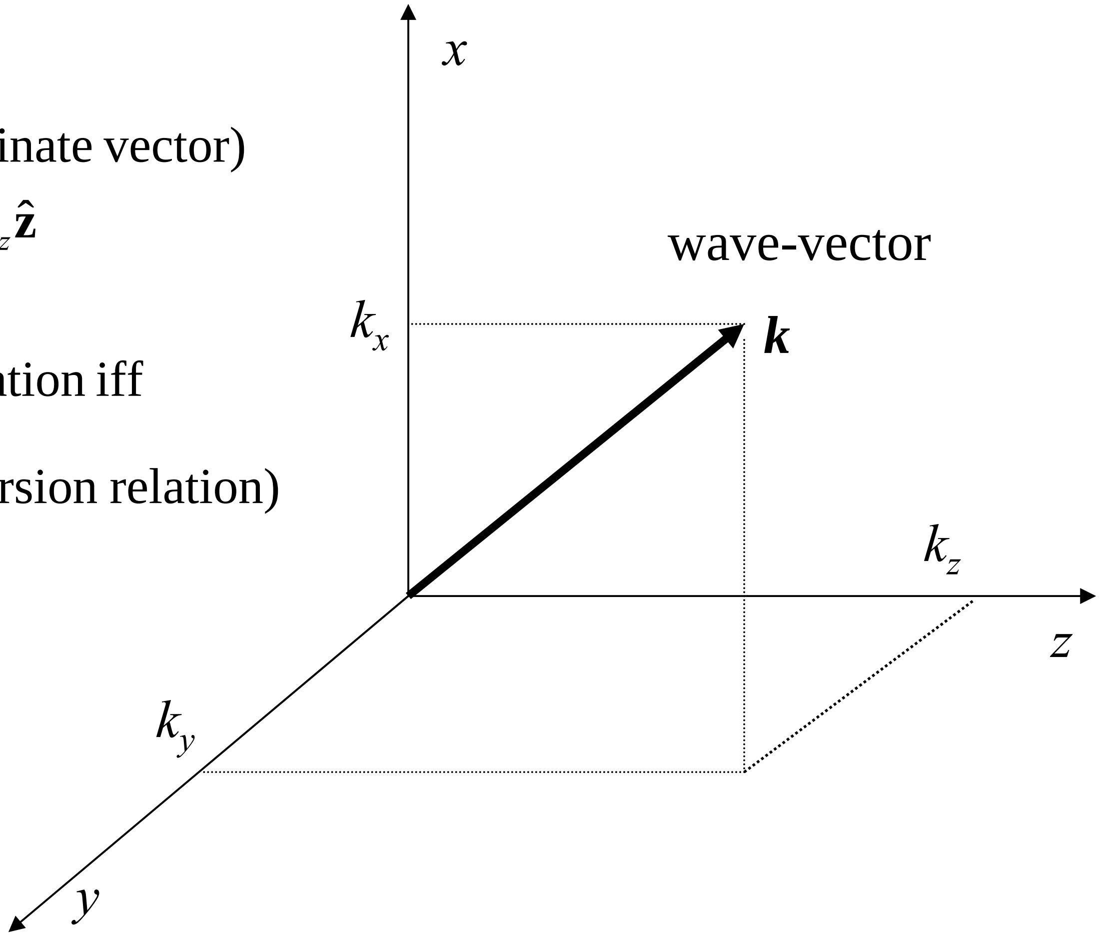
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

(Cartesian coordinate vector)

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$$

solves wave equation iff

$$|\mathbf{k}| = \frac{\omega}{c} \qquad \text{(dispersion relation)}$$



Plane wave

$$a(\mathbf{r}) = A_0\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

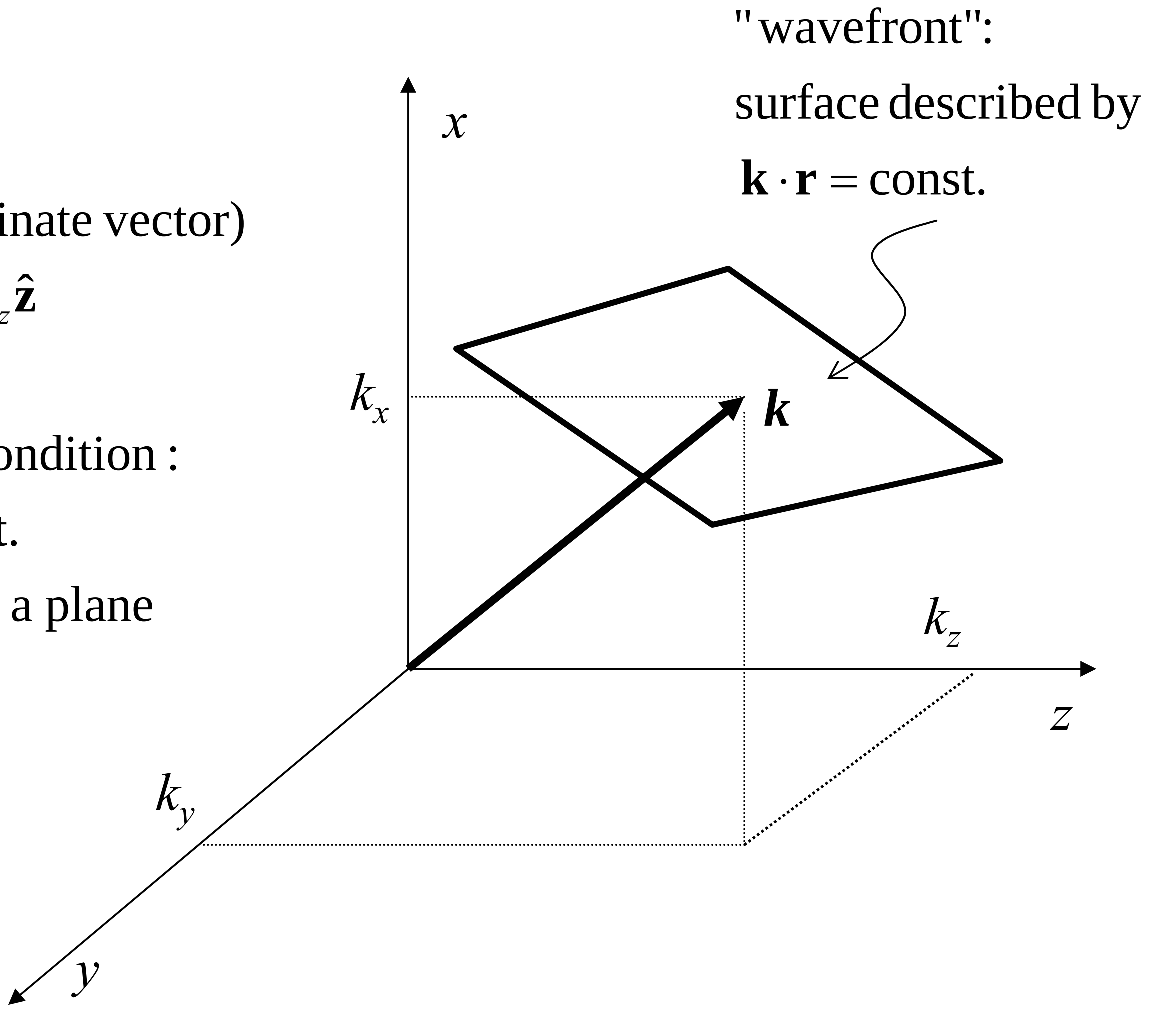
(Cartesian coordinate vector)

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$$

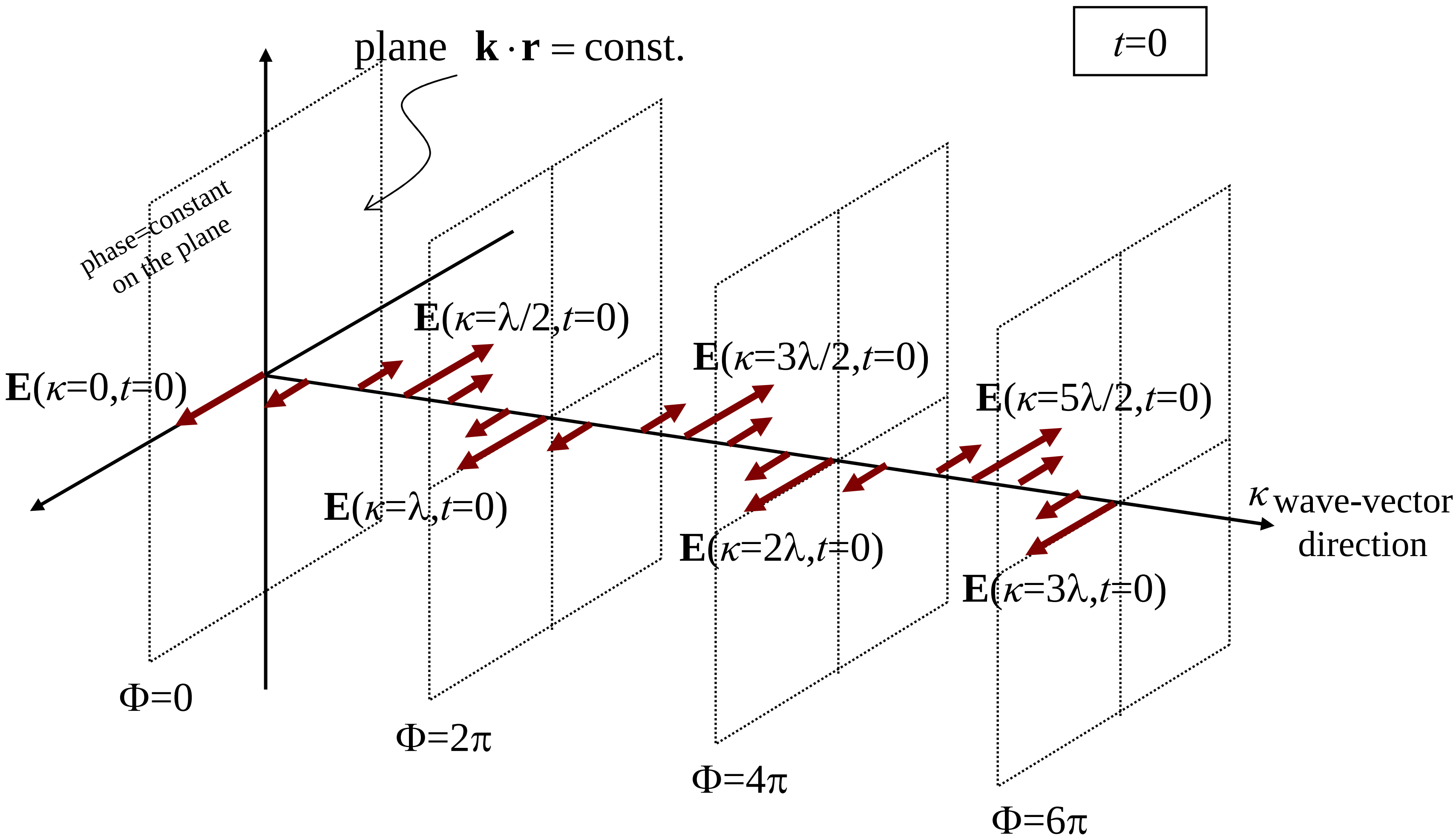
constant phase condition :

$$\mathbf{k}\cdot\mathbf{r} - \omega t = \text{const.}$$

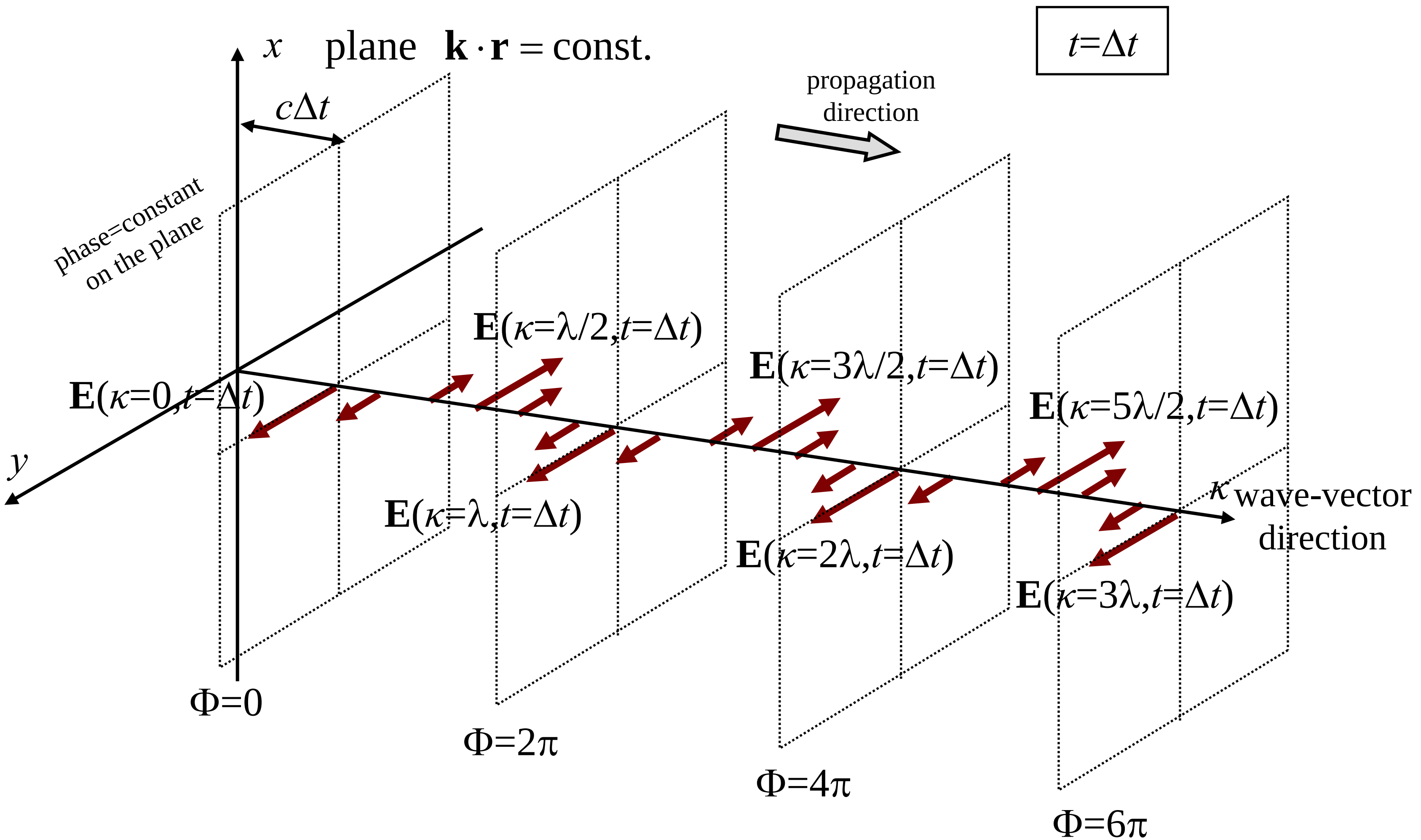
$\Rightarrow$ wave-front is a plane



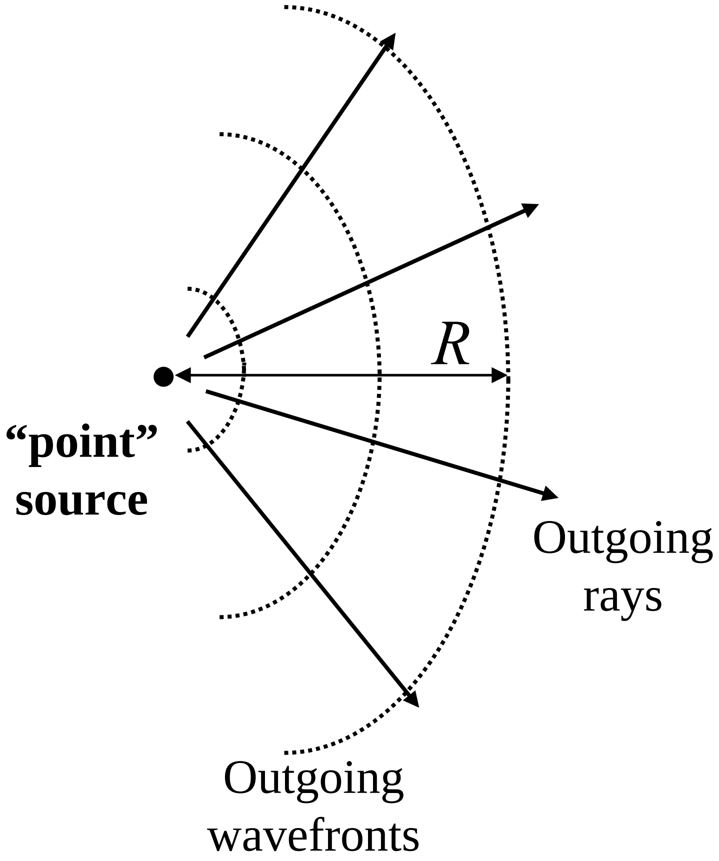
Plane wave propagating



Plane wave propagating



Spherical wave



equation of wavefront

$$kR - \omega t = \text{constant}$$

$$a(x, y, z, t) = A \frac{\cos\left(kR - \omega t + \pi / 2\right)}{R}$$

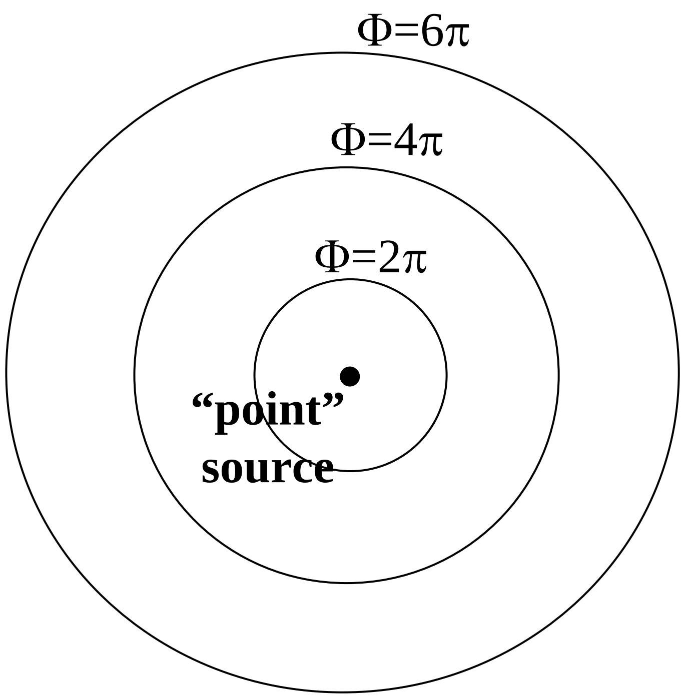
exponential notation

$$a(x, y, z, \cancel{t}) = A \frac{\exp\left\{ i\left(kR - \cancel{\omega t}\right) \right\}}{iR}$$

paraxial approximation

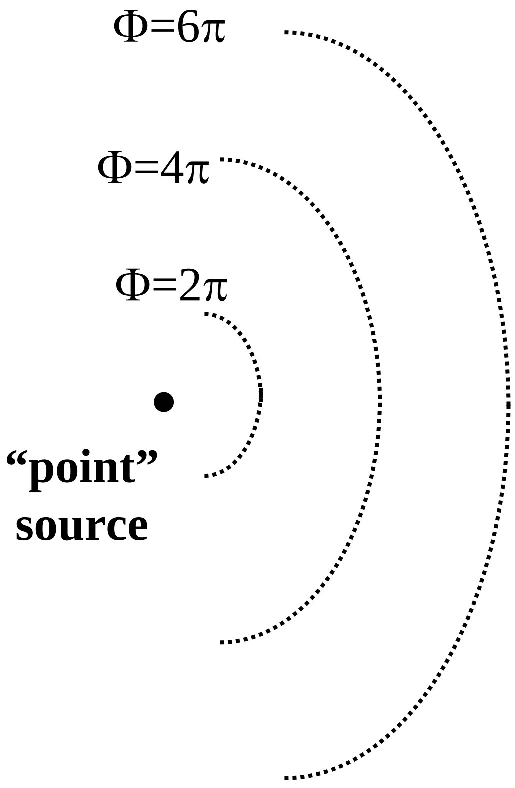
$$a(x, y, z) = \frac{A}{iR} \exp\left\{ i2\pi \frac{z}{\lambda} + i\pi \frac{x^2 + y^2}{\lambda z} \right\}$$

Spherical wave



spherical wavefronts

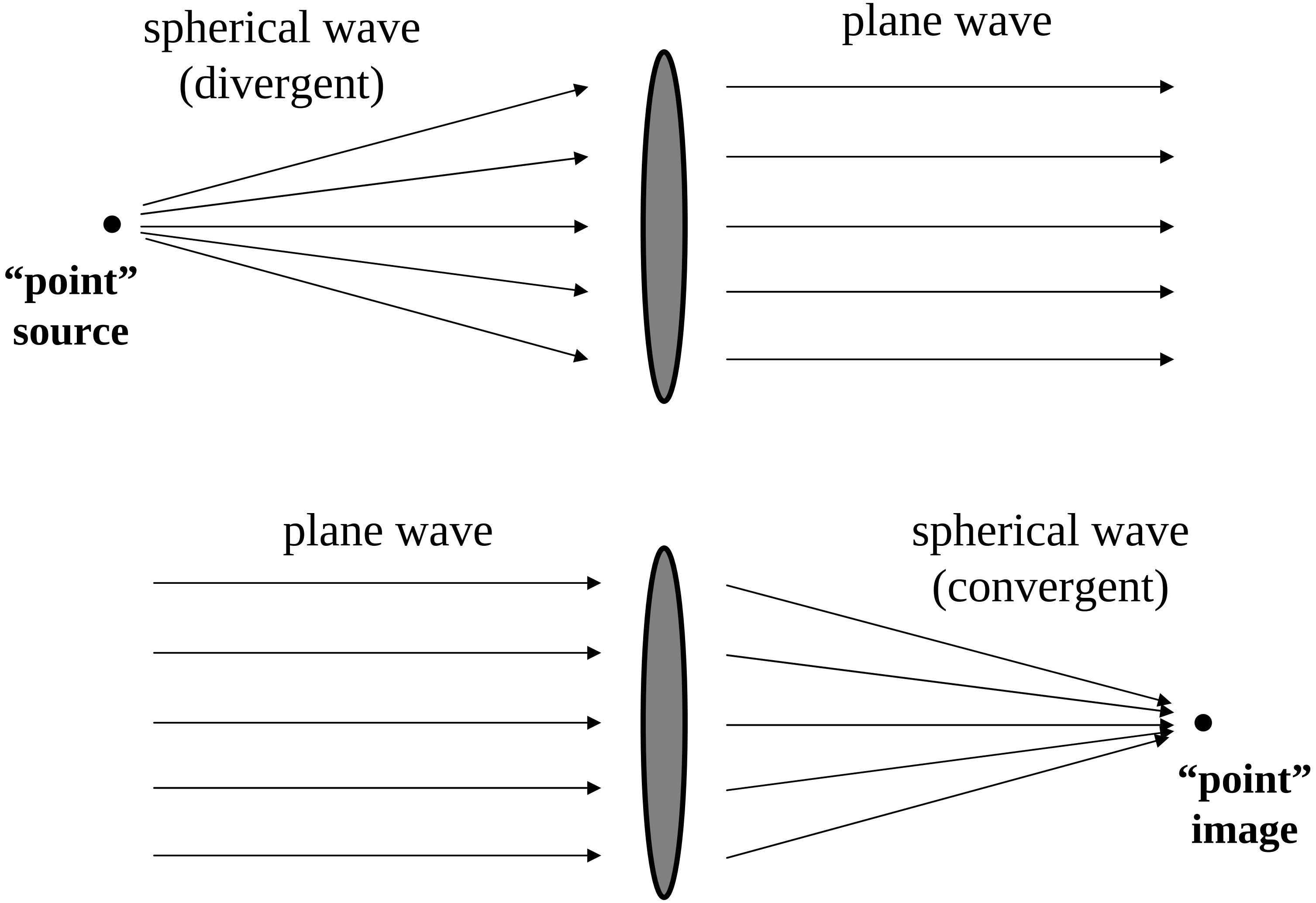
exact



parabolic wavefronts

paraxial approximation/
/Gaussian beams

The role of lenses



The role of lenses