Today’s summary

- Polarization
- Energy / Poynting’s vector
- Reflection and refraction at a dielectric interface:
  - wave approach to derive Snell’s law
  - reflection and transmission coefficients
  - total internal reflection (TIR) revisited

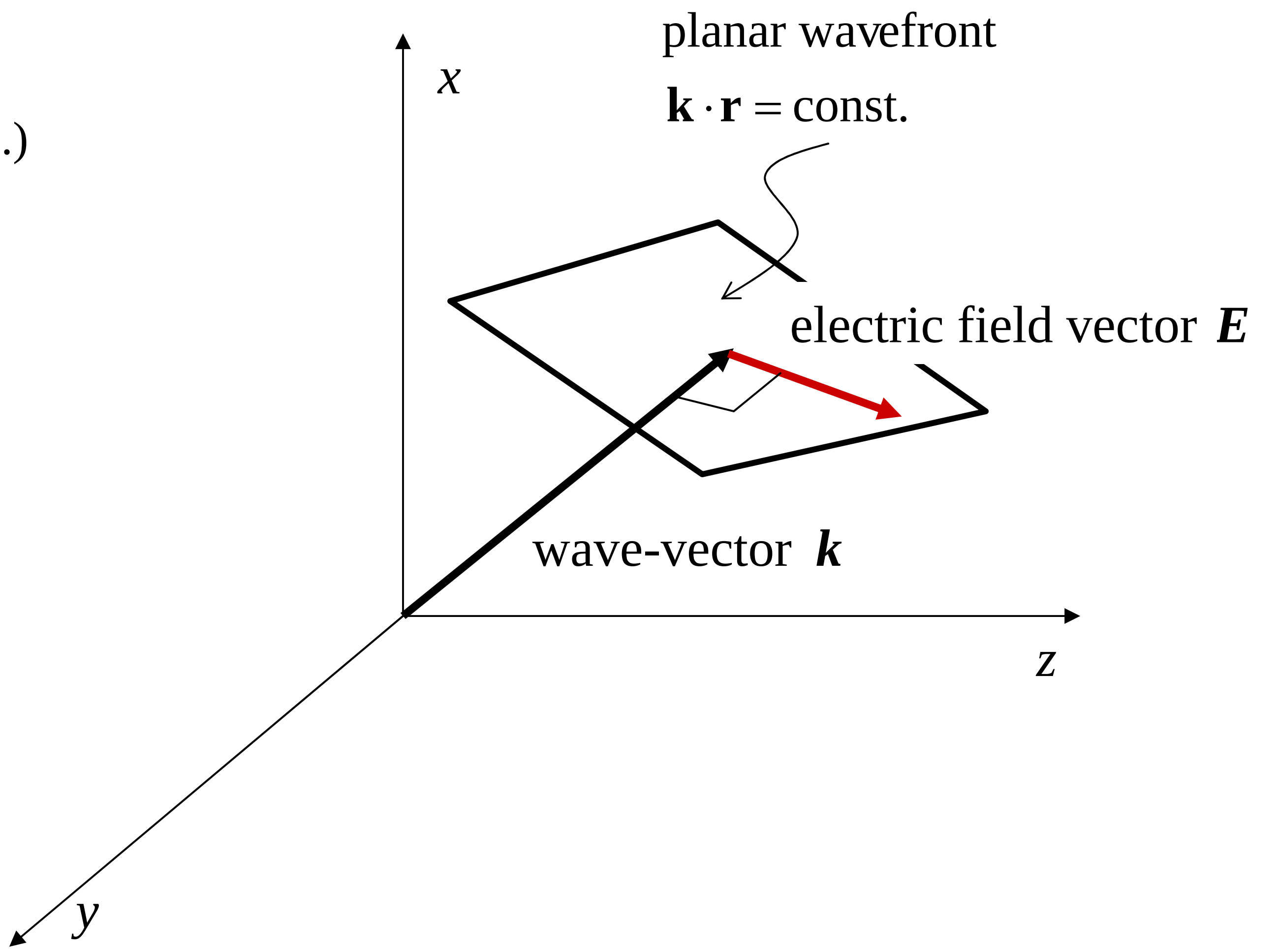
Polarization

Propagation and polarization

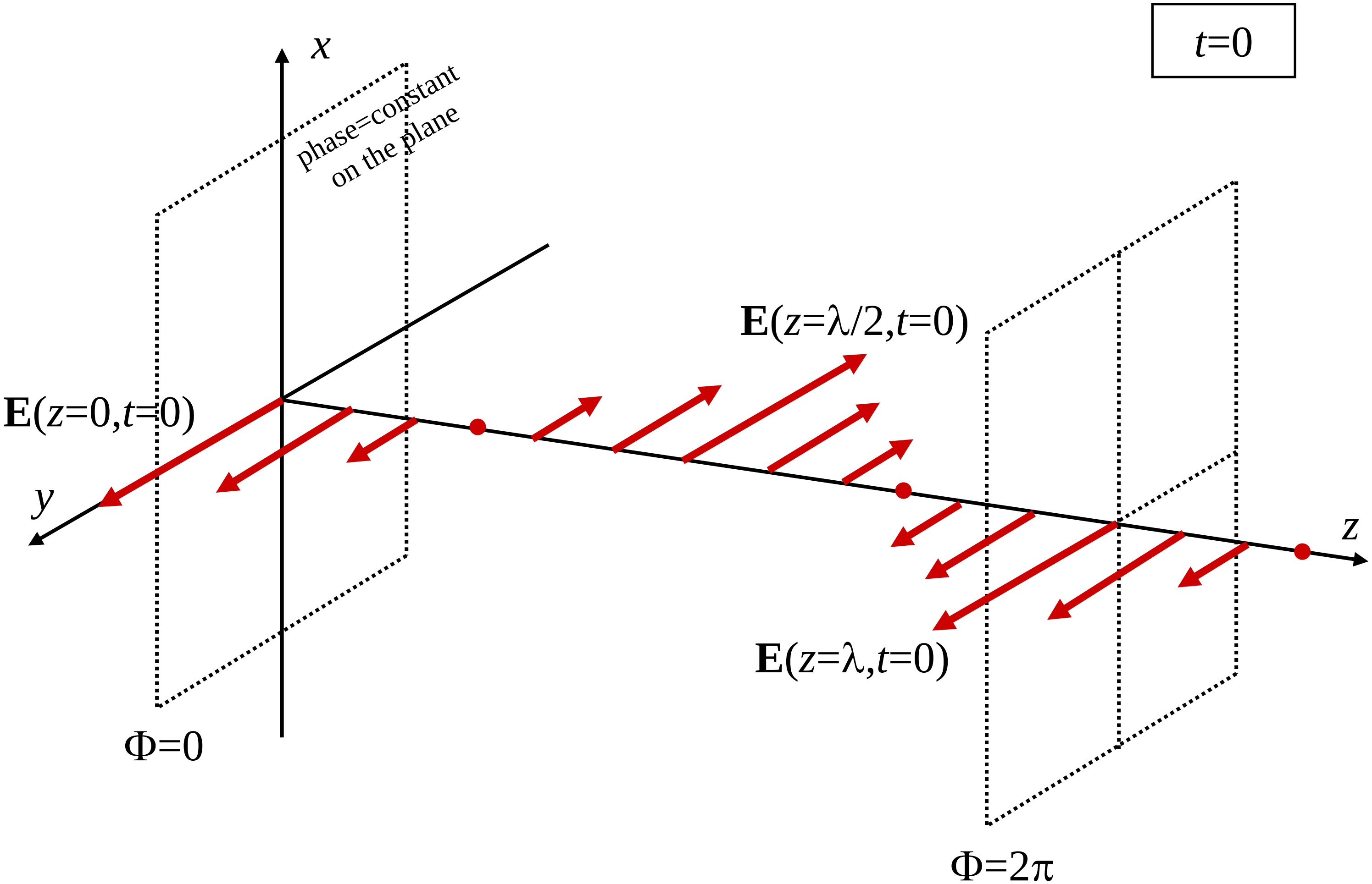
In isotropic media
(e.g. free space,
amorphous glass, etc.)

$\mathbf{k} \cdot \mathbf{E} = 0$
i.e. $\mathbf{k} \perp \mathbf{E}$

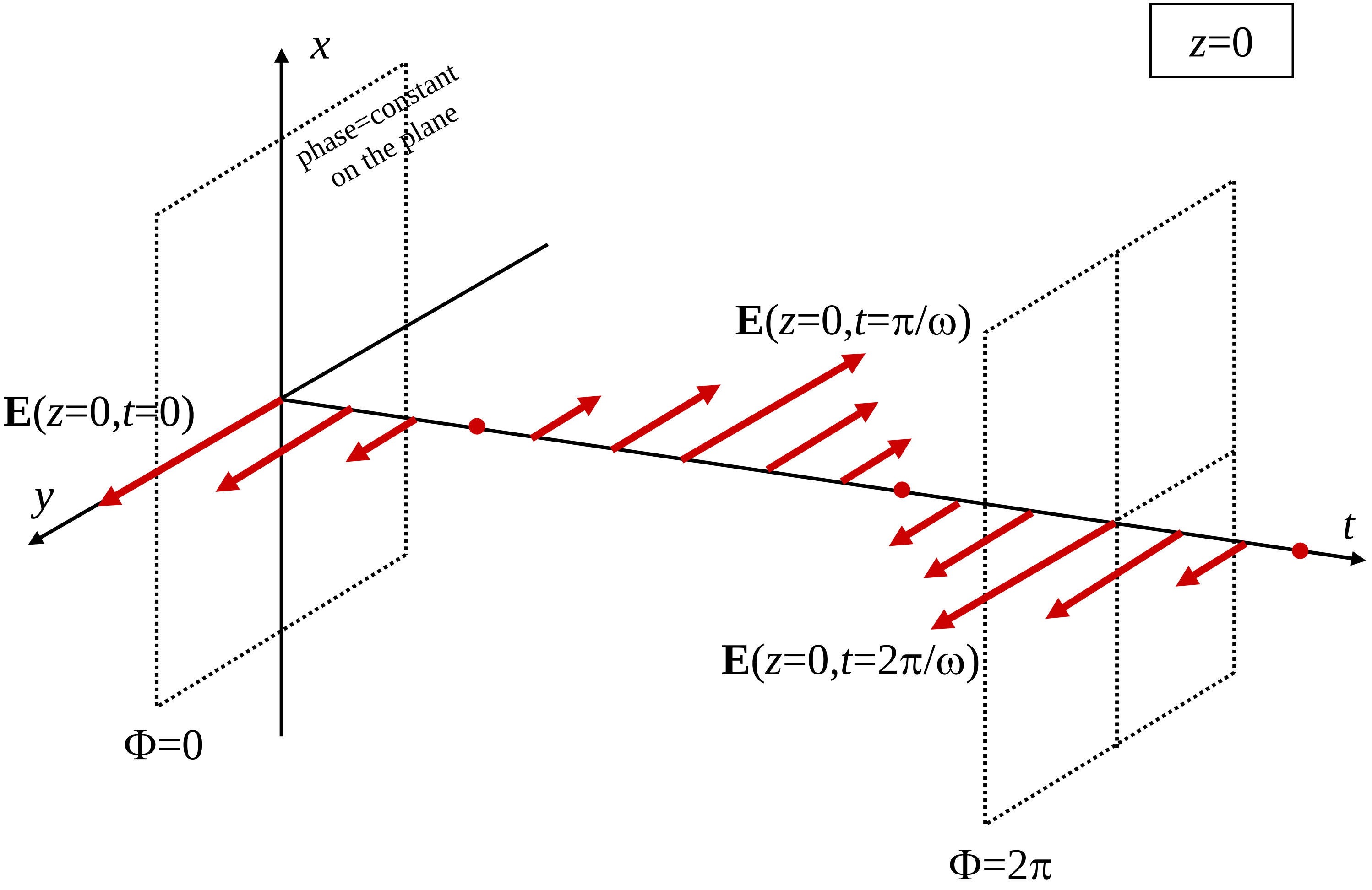
More generally,
$\mathbf{k} \cdot \mathbf{D} = 0$
(reminder: in
anisotropic media,
e.g. crystals, one
could have
$\mathbf{E}$ not parallel to $\mathbf{D}$)



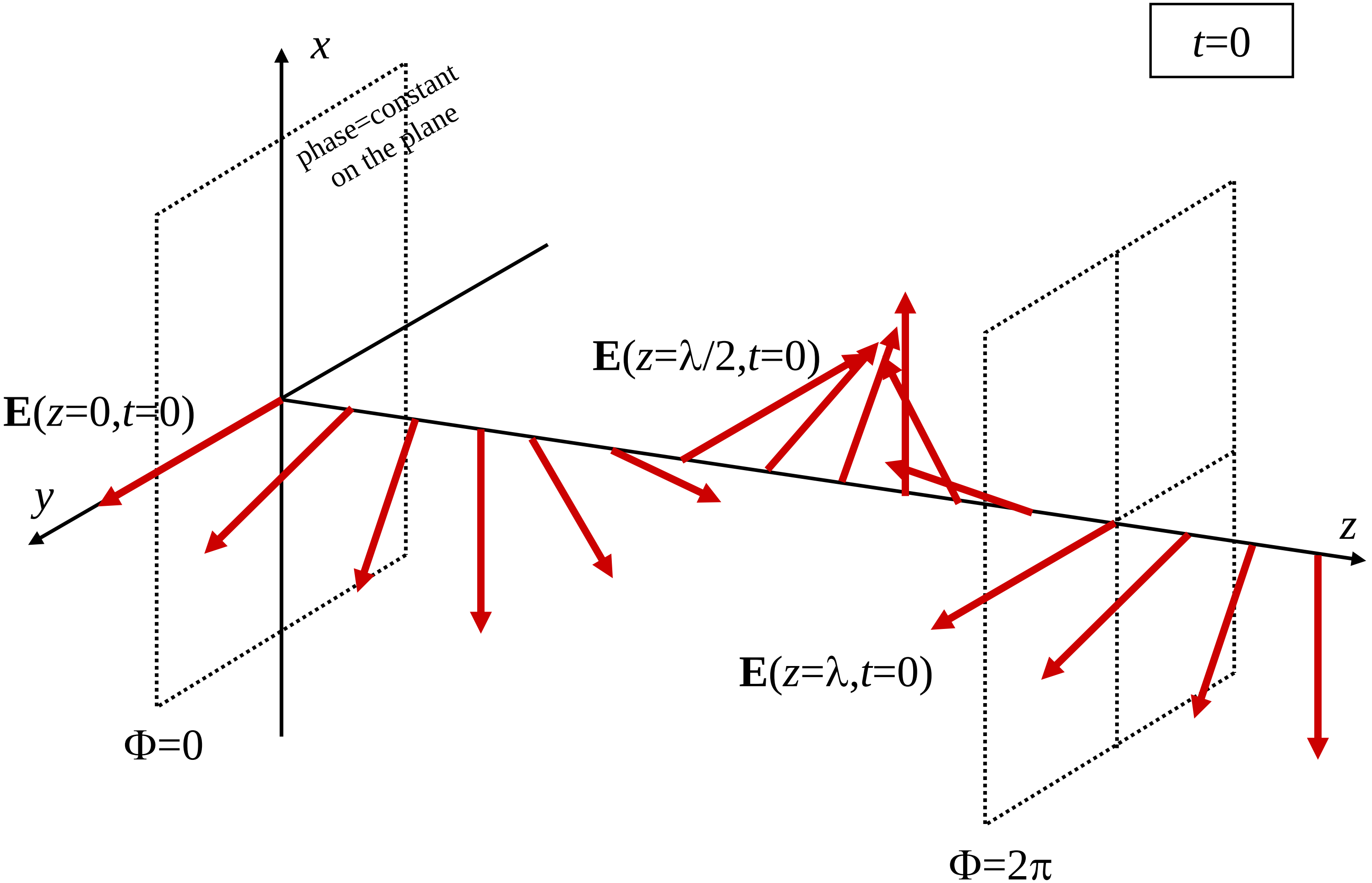
Linear polarization (frozen time)



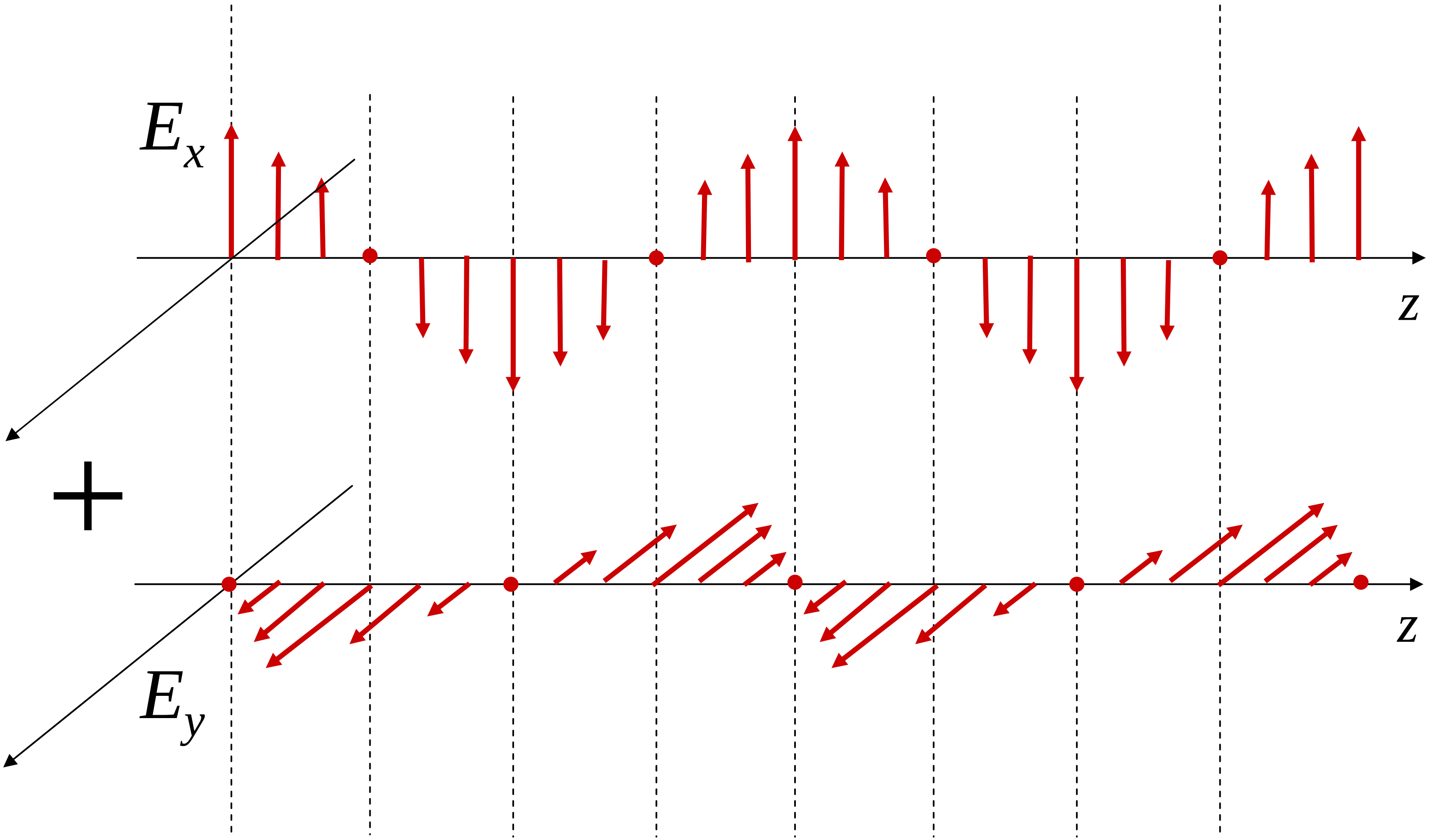
Linear polarization (fixed space)



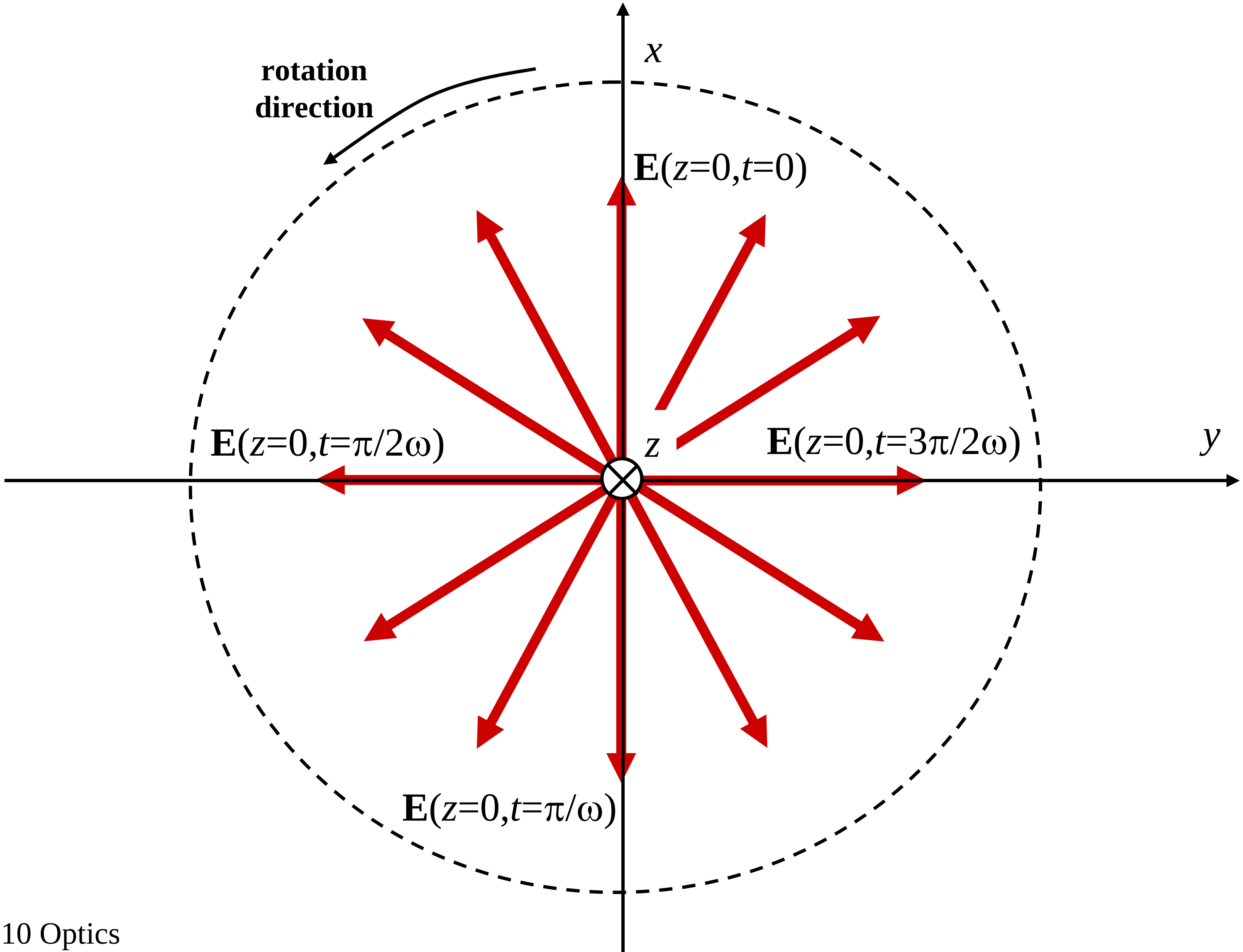
Circular polarization (frozen time)



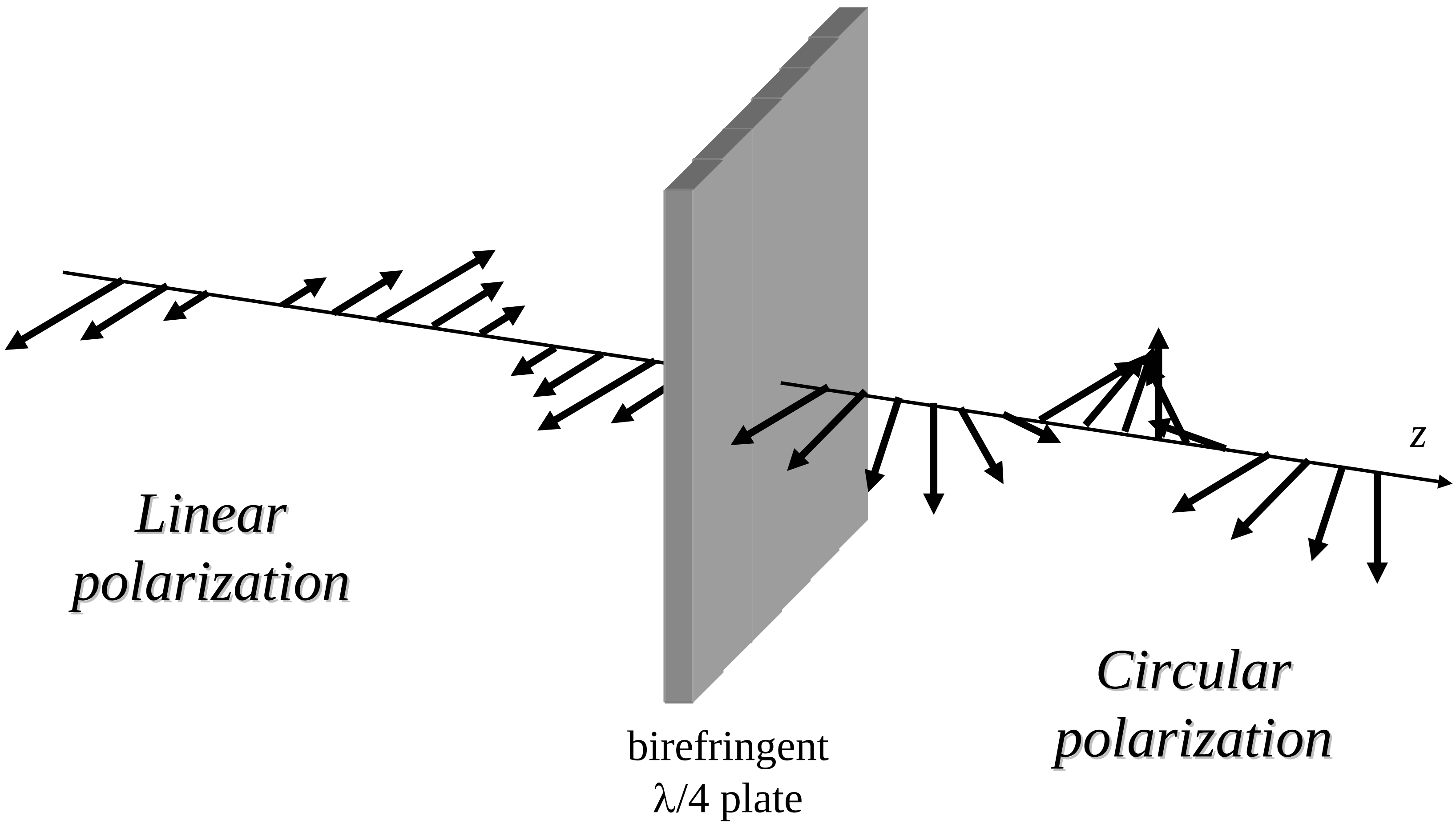
Circular polarization: linear components



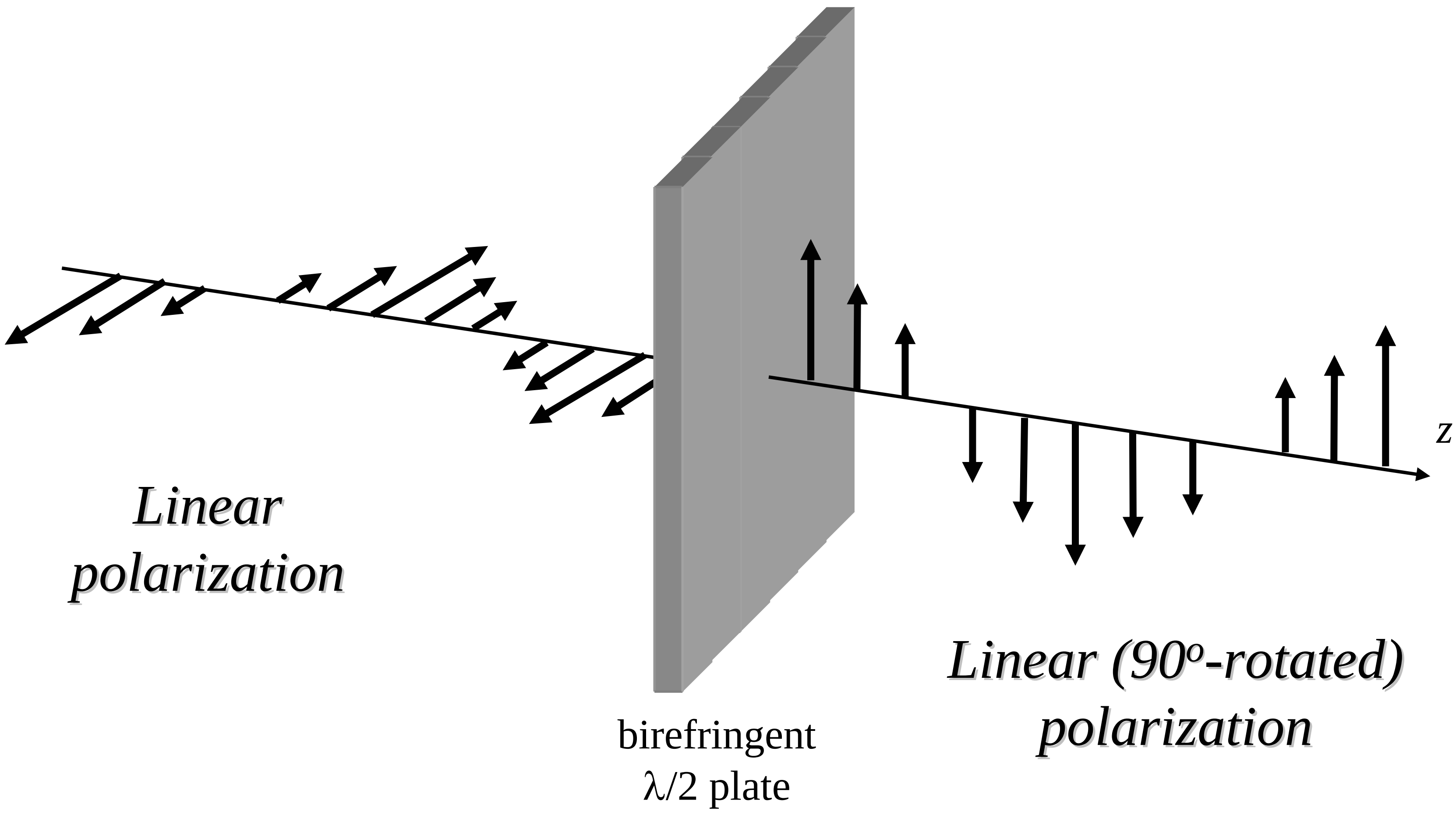
Circular polarization (fixed space)



λ/4 plate

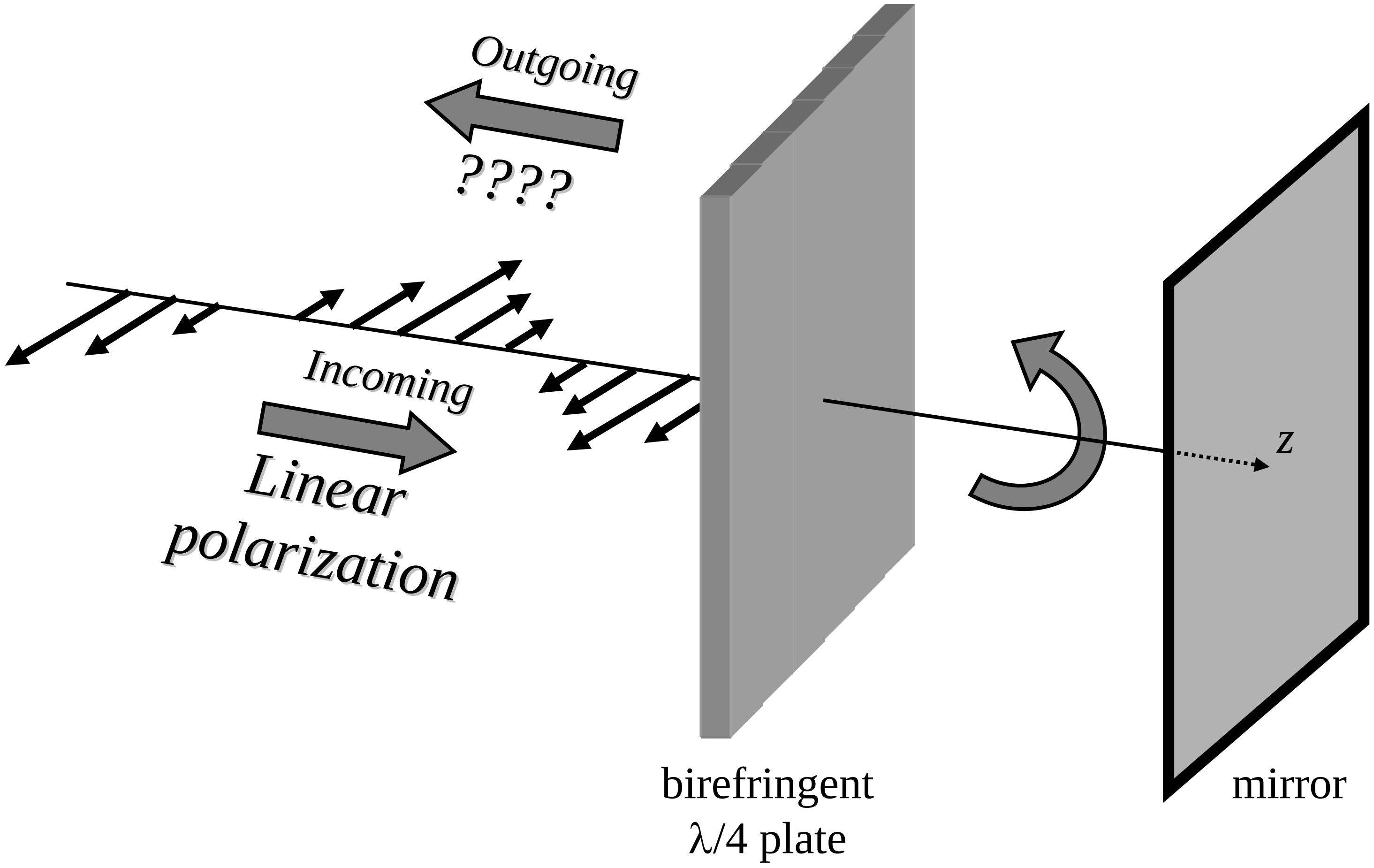

z
Linear polarization
birefringent
λ/4 plate
Circular polarization

λ/2 plate

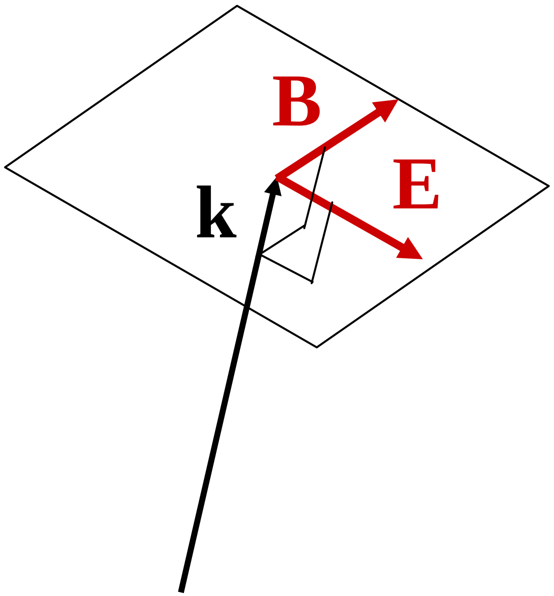


Linear polarization
z
birefringent
λ/2 plate
Linear (90°-rotated) polarization

Think about that



Relationship between E and B



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{where} \quad \mathbf{E} = \hat{\mathbf{x}} E_0 \, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\Rightarrow \nabla \times \equiv i\mathbf{k} \times \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -i\omega$$

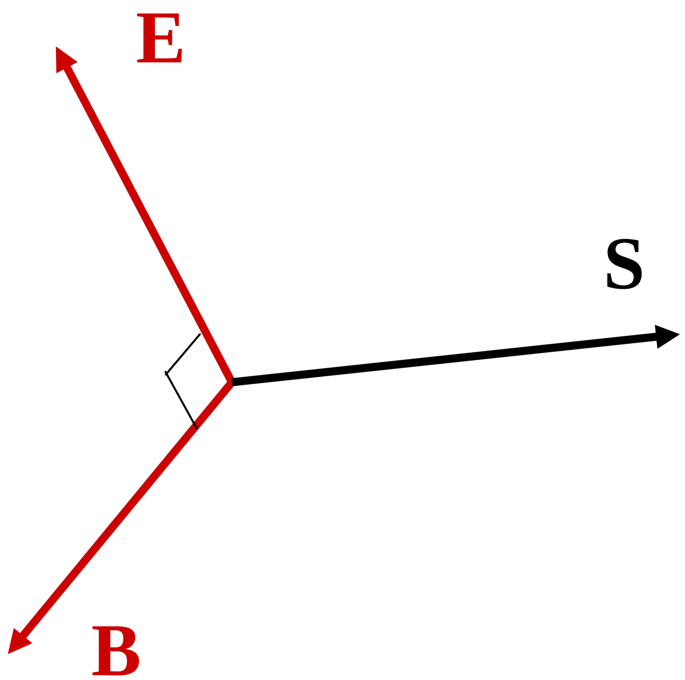
$$\Rightarrow \mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$$

Vectors **k**, **E**, **B** form a right-handed triad.

Note: free space or isotropic media only

Energy

The Poynting vector



$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = c^2\varepsilon_0\mathbf{E}\times\mathbf{B}$$

so in free space

$$\mathbf{S} \parallel \mathbf{k}$$

S has units of W/m^2
so it represents
energy flux (energy per
unit time & unit area)

Poynting vector and phasors (I)

$$\mathbf{S} = c^2 \varepsilon_0 \mathbf{E} \times \mathbf{B}$$

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} \Rightarrow \|\mathbf{B}\| = \frac{k}{\omega}\|\mathbf{E}\| = \frac{1}{c}\|\mathbf{E}\| \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$

For example, sinusoidal field propagating along z

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

Recall: for visible light, $\omega \sim 10^{14}\text{-}10^{15}$ Hz

Poynting vector and phasors (II)

Recall: for visible light, $\omega \sim 10^{14}$-10^{15}Hz

So any instrument will record the
average incident energy flux

$$\left\langle \left\| \mathbf{S} \right\| \right\rangle = \frac{1}{T} \int_{t}^{t+T} \left\| \mathbf{S} \right\| \mathrm{d}t$$

where T is the period ($T=\lambda/$c)

$\left\langle \left\| \mathbf{S} \right\| \right\rangle$ is called the *irradiance*, aka *intensity* of the optical field (units: W/m^2)

Poynting vector and phasors (III)

For example: sinusoidal electric field,

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos\left(kz - \omega t\right) \Rightarrow \left\|\mathbf{S}\right\| = c\varepsilon_0 E_0^2 \cos^2\left(kz - \omega t\right)$$

Then, at constant z:

$$\left\langle \cos^2(kz - \omega t) \right\rangle = \int_t^{t+T} \cos^2(kz - \omega t)\mathrm{d}t = \frac{1}{2}$$

$$\left\langle \left\|\mathbf{S}\right\| \right\rangle = \frac{1}{2} c\varepsilon_0 E_0^2$$

Poynting vector and phasors (IV)

Recall phasor representation:

$$f(z,t) = A\cos(kz - \omega t - \phi)$$

$$\hat{f}(z,t) = A\cos(kz - \omega t - \phi) + iA\sin(kz - \omega t - \phi)$$

$$\text{complex amplitude or "phasor":} \quad A\mathrm{e}^{-i\phi}$$

Can we use phasors to compute intensity?

Poynting vector and phasors (V)

Consider the superposition of *two* fields of the *same* frequency:

$$E_1(z,t) = E_{10}\cos(kz-\omega t)$$
$$E_2(z,t) = E_{20}\cos(kz-\omega t-\phi)$$
$$\langle\|\mathbf{S}\|\rangle = \frac{c\varepsilon_0}{T}\int_t^{t+T}(E_1+E_2)^2\,\mathrm{d}t = \ldots = \frac{c\varepsilon_0}{2}\left(E_{10}^2+E_{20}^2+2E_{10}E_{20}\cos\phi\right)$$

Now consider the two corresponding *phasors*:

$$E_{10}$$
$$E_{20}\,\mathrm{e}^{-i\phi}$$

and the quantity

$$\frac{c\varepsilon_0}{2}\left|E_{10}+E_{20}\,\mathrm{e}^{-i\phi}\right|^2 = \ldots = \frac{c\varepsilon_0}{2}\left(E_{10}^2+E_{20}^2+2E_{10}E_{20}\cos\phi\right)$$

Poynting vector and phasors (V)

Consider the superposition of *two* fields of the *same* frequency:

$$E_1(z,t) = E_{10}\cos(kz-\omega t)$$

$$E_2(z,t) = E_{20}\cos(kz-\omega t-\phi)$$

$$\langle\|\mathbf{S}\|\rangle = \frac{c\varepsilon_0}{T}\int_t^{t+T}(E_1+E_2)^2\,\mathrm{d}t = \ldots = \boxed{\frac{c\varepsilon_0}{2}\left(E_{10}^2+E_{20}^2+2E_{10}E_{20}\cos\phi\right)}$$

Now consider the two corresponding *phasors*:

$$E_{10}$$

$$E_{20}\,\mathrm{e}^{-i\phi}$$

and the quantity

$$\frac{c\varepsilon_0}{2}\left|E_{10}+E_{20}\,\mathrm{e}^{-i\phi}\right|^2 = \ldots = \boxed{\frac{c\varepsilon_0}{2}\left(E_{10}^2+E_{20}^2+2E_{10}E_{20}\cos\phi\right)}$$

= !!

Poynting vector and irradiance

Summary (free space or isotropic media)

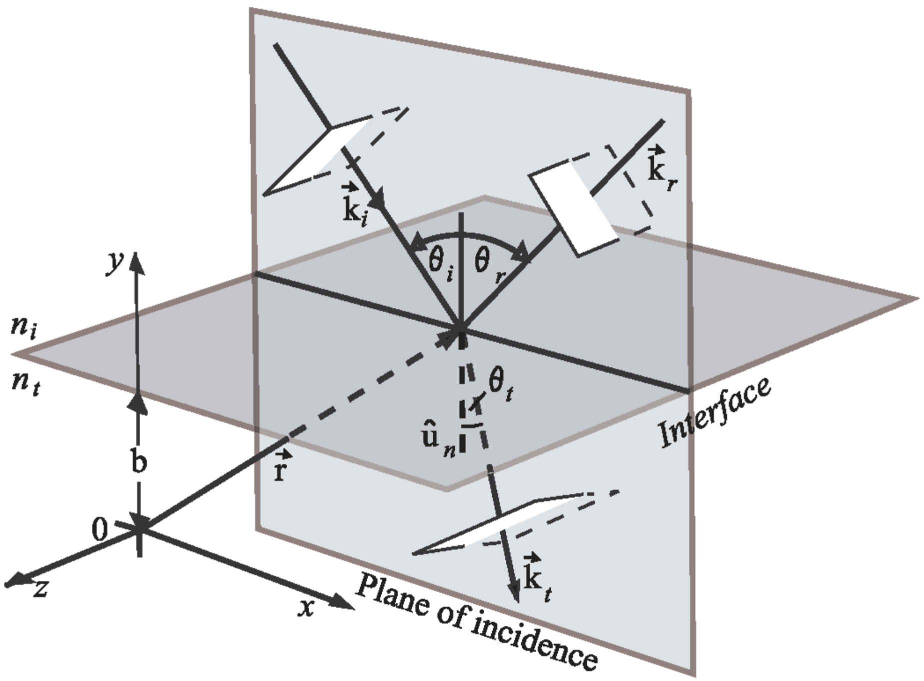
$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}; \quad \|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$ Poynting vector

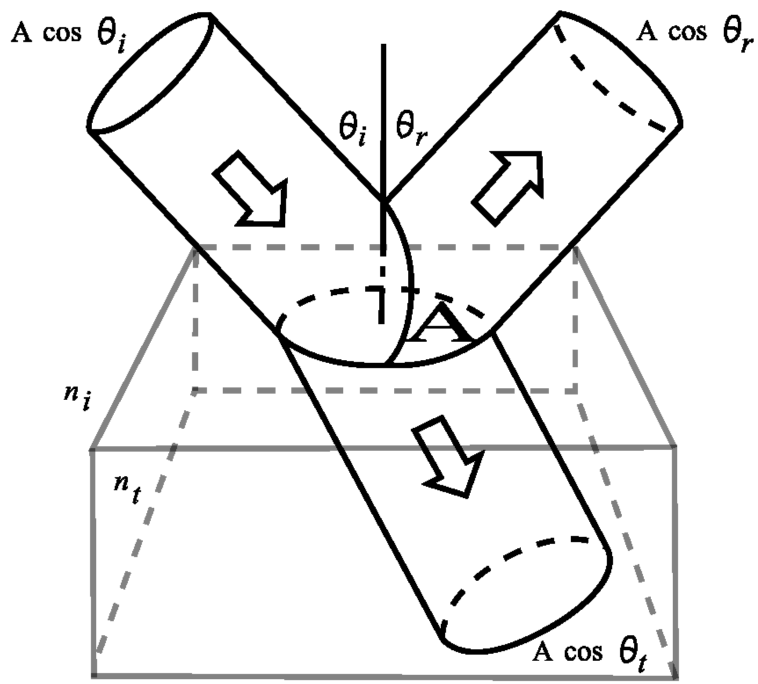
$$\langle\|\mathbf{S}\|\rangle = \frac{1}{T}\int_t^{t+T}\|\mathbf{S}\|\mathrm{d}t$$ Irradiance (or intensity)

$$\langle\|\mathbf{S}\|\rangle = \frac{c\varepsilon_0}{2}|\text{phasor}|^2 \quad \text{or} \quad \langle\|\mathbf{S}\|\rangle \propto |\text{phasor}|^2$$

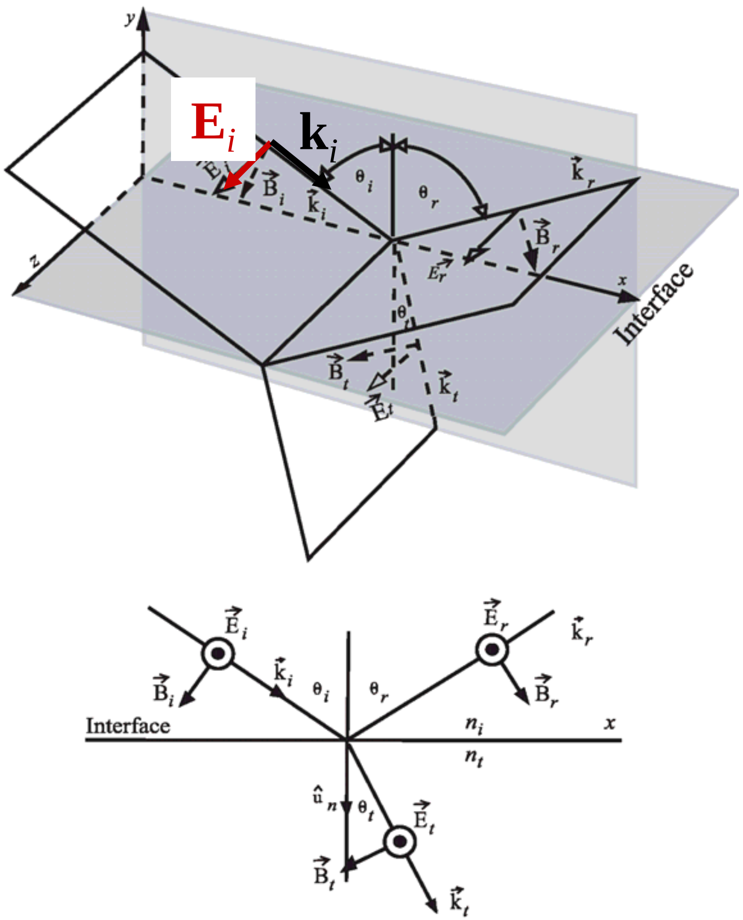
Reflection / Refraction
Fresnel coefficients

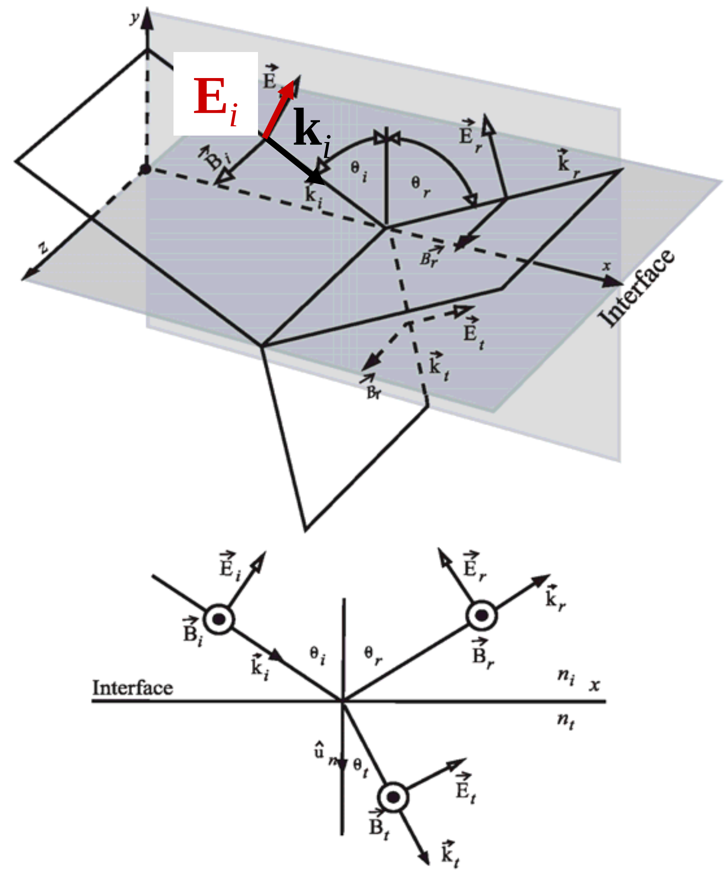
Reflection & transmission @ dielectric interface





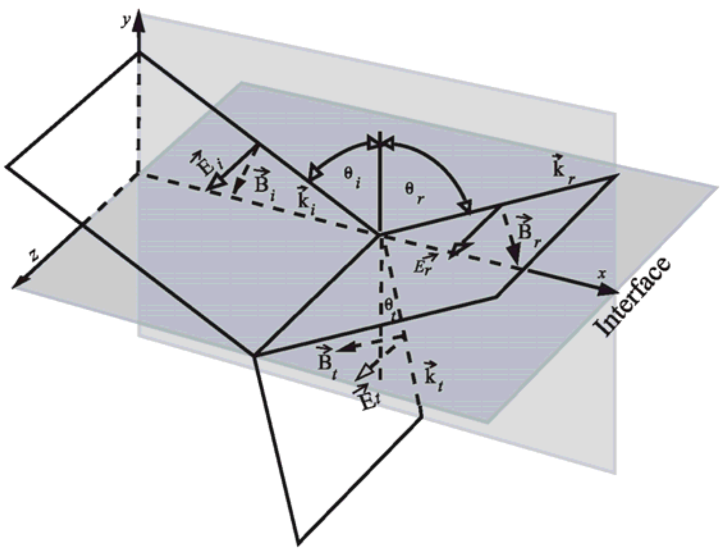
Reflection & transmission @ dielectric interface

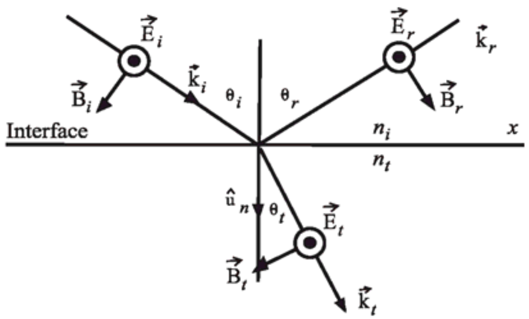




Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Incident electric field:

$$\mathbf{E}_i = \hat{\mathbf{z}} E_{0i} \exp\left[i\left(\mathbf{k}_i \cdot \mathbf{r} - \omega t\right)\right] = \\ = \hat{\mathbf{z}} E_{0i} \exp\left[i\left(k_i(-y\cos\theta_i + x\sin\theta_i) - \omega t\right)\right]$$

Reflected electric field:

$$\mathbf{E}_r = \hat{\mathbf{z}} E_{0r} \exp\left[i\left(\mathbf{k}_r \cdot \mathbf{r} - \omega t\right)\right] = \\ = \hat{\mathbf{z}} E_{0r} \exp\left[i\left(k_i(+y\cos\theta_r + x\sin\theta_r) - \omega t\right)\right]$$

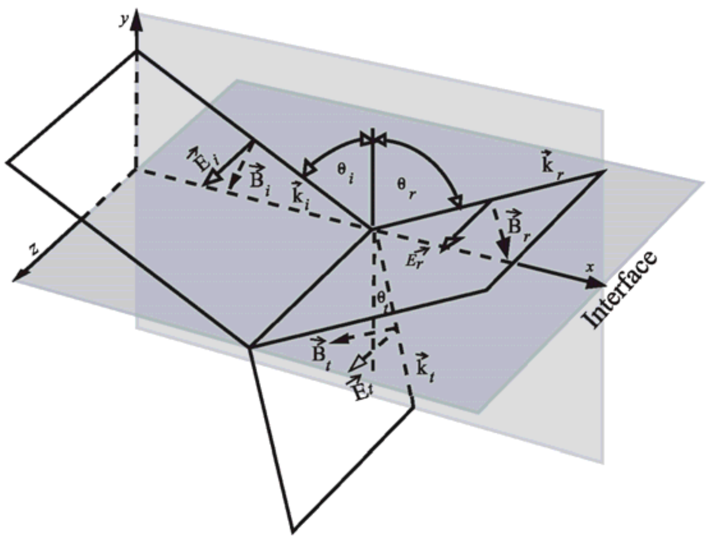
Transmitted electric field:

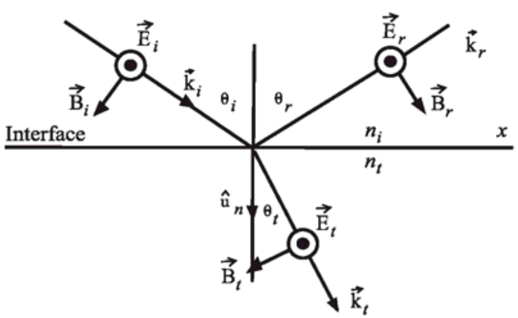
$$\mathbf{E}_t = \hat{\mathbf{z}} E_{0t} \exp\left[i\left(\mathbf{k}_t \cdot \mathbf{r} - \omega t\right)\right] = \\ = \hat{\mathbf{z}} E_{0t} \exp\left[i\left(k_t(-y\cos\theta_t + x\sin\theta_t) - \omega t\right)\right]$$

where: $k_i = \dfrac{2\pi n_i}{\lambda_{\text{vacuum}}}, \quad k_t = \dfrac{2\pi n_t}{\lambda_{\text{vacuum}}}$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Continuity of tangential electric field at the interface:

$$E_i(\text{tangential}) + E_r(\text{tangential}) = \\ = E_t(\text{tangential})$$

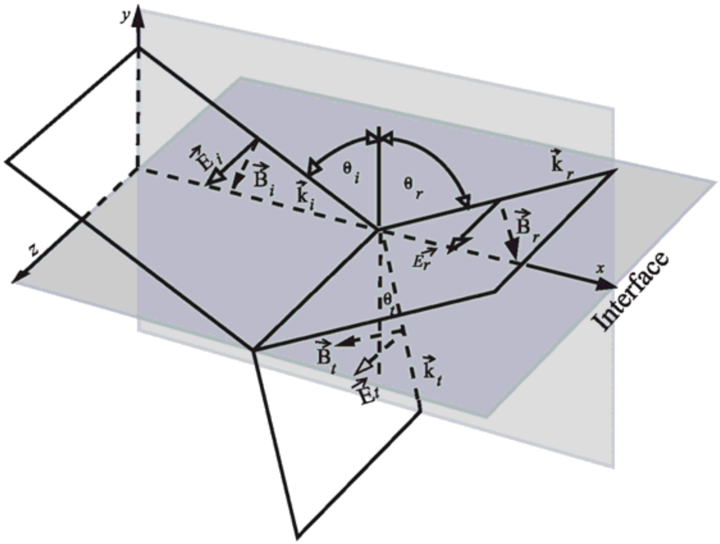
$$\Rightarrow E_{0i}\exp\left[i\left(k_i(-y\cos\theta_i + x\sin\theta_i) - \omega t\right)\right] + \\ E_{0r}\exp\left[i\left(k_i(+y\cos\theta_r + x\sin\theta_r) - \omega t\right)\right] = \\ E_{0t}\exp\left[i\left(k_t(-y\cos\theta_t + x\sin\theta_t) - \omega t\right)\right]$$

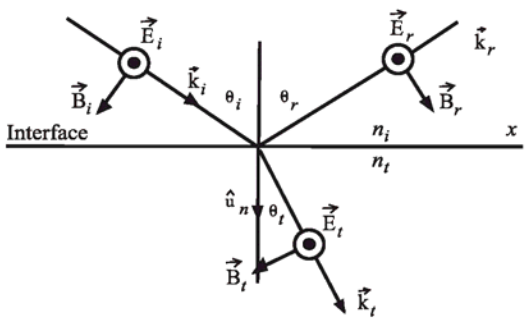
But at the interface y=0 so

$$E_{0i}\exp\left[i\left(k_i x\sin\theta_i - \omega t\right)\right] + \\ E_{0r}\exp\left[i\left(k_i x\sin\theta_r - \omega t\right)\right] = \\ E_{0t}\exp\left[i\left(k_t x\sin\theta_t - \omega t\right)\right]$$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Continuity of tangential electric field at the interface:

$$E_{0i}\exp\left[i\left(k_i x\sin\theta_i-\omega t\right)\right]+$$
$$E_{0r}\exp\left[i\left(k_i x\sin\theta_r-\omega t\right)\right]=$$
$$E_{0t}\exp\left[i\left(k_t x\sin\theta_t-\omega t\right)\right]$$

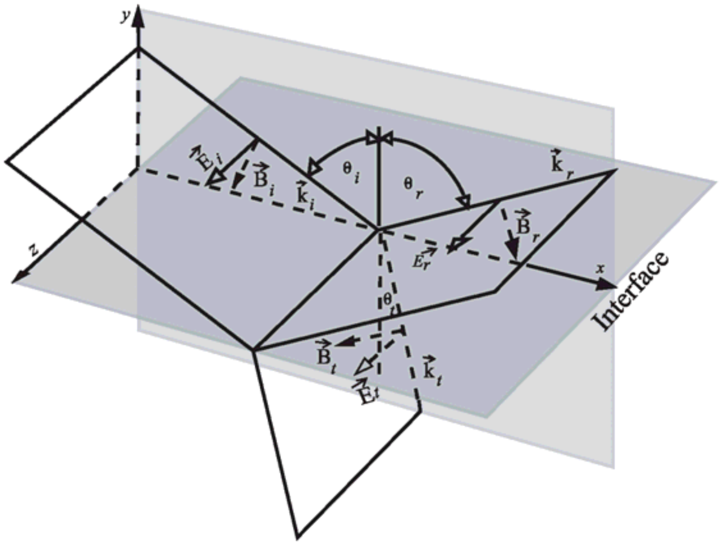
Since the exponents must be equal for all *x*, we obtain

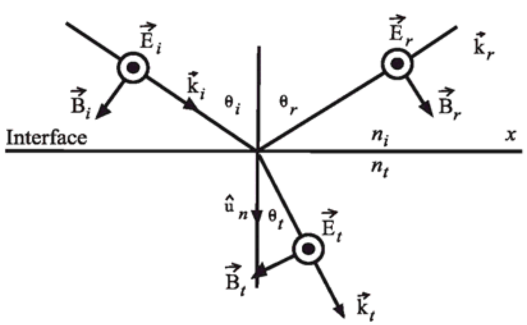
$$\theta_i=\theta_r \qquad \text{and}$$

$$k_i\sin\theta_i=k_t\sin\theta_t \Leftrightarrow \frac{2\pi n_i}{\lambda_{\text{vacuum}}}\sin\theta_i=\frac{2\pi n_t}{\lambda_{\text{vacuum}}}\sin\theta_t$$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Continuity of tangential electric field at the interface:

$$\theta_i = \theta_r$$ law of reflection

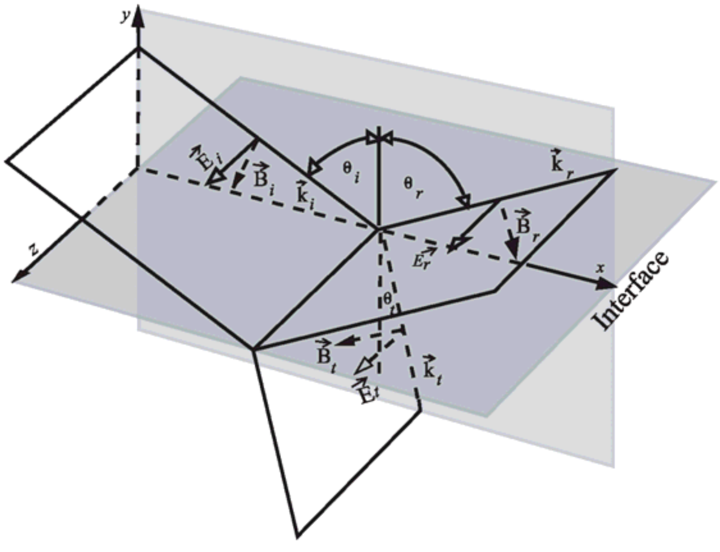
$$n_i \sin\theta_i = n_t \sin\theta_t$$

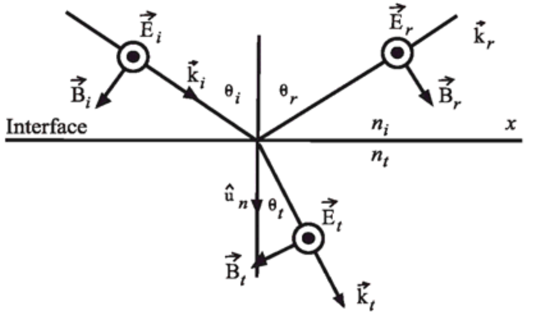
Snell's law of refraction

so wave description is equivalent to Fermat's principle!! ☺

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Incident electric field:

$$\mathbf{E}_i = \hat{\mathbf{z}} E_{0i} \exp\left[i\left(k_i(-y\cos\theta_i + x\sin\theta_i) - \omega t\right)\right]$$

Reflected electric field:

$$\mathbf{E}_r = \hat{\mathbf{z}} E_{0r} \exp\left[i\left(k_i(+y\cos\theta_r + x\sin\theta_r) - \omega t\right)\right]$$

Transmitted electric field:

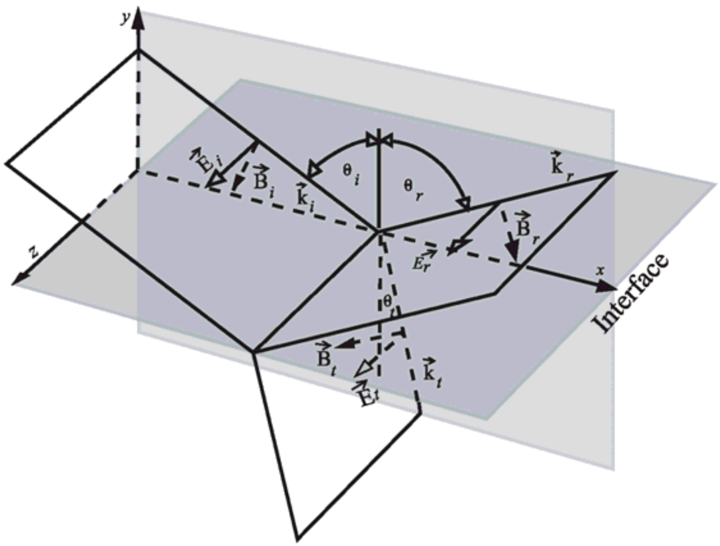
$$\mathbf{E}_t = \mathbf{z} E_{0t} \exp\left[i\left(k_t(-y\cos\theta_t + x\sin\theta_t) - \omega t\right)\right]$$

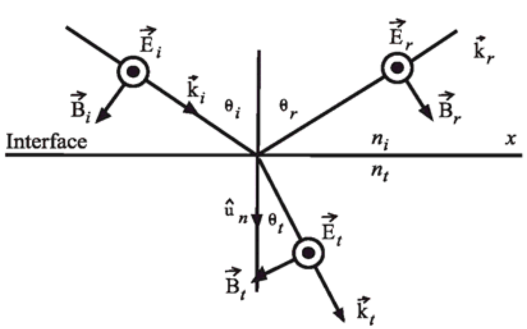
Need to calculate the reflected and transmitted amplitudes E_{0r}, E_{0t}

i.e. need *two* equations

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





Continuity of tangential electric field at the interface gives us one equation:

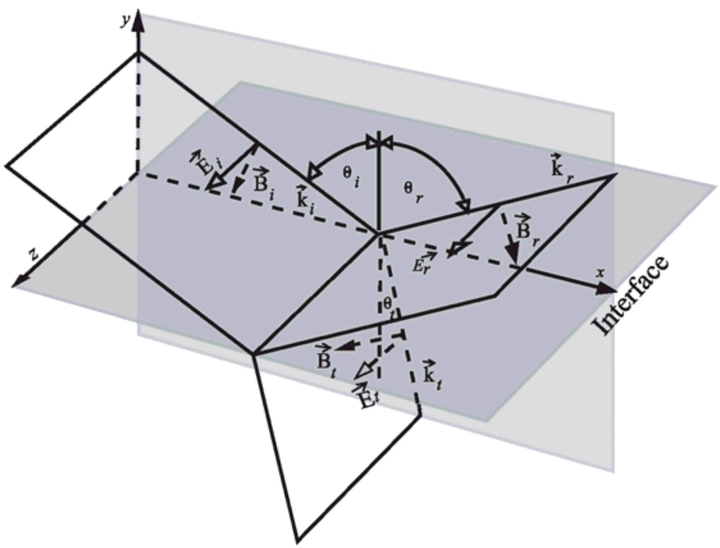
$$E_{0i}\exp\left[i\left(k_i x\sin\theta_i-\omega t\right)\right]+$$
$$E_{0r}\exp\left[i\left(k_i x\sin\theta_r-\omega t\right)\right]=$$
$$E_{0t}\exp\left[i\left(k_t x\sin\theta_t-\omega t\right)\right]$$

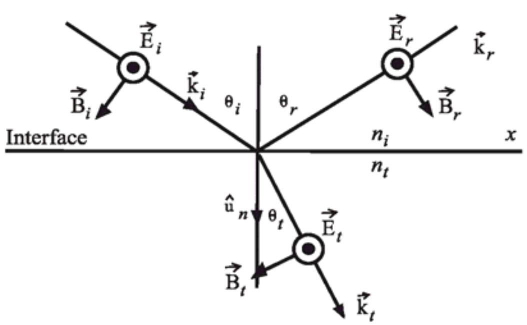
which after satisfying Snell's law becomes

$$E_{0i}+E_{0r}=E_{0t}$$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





The second equation comes from continuity of tangential magnetic field at the interface:

$$B_i(\text{tangential}) + B_r(\text{tangential}) =$$

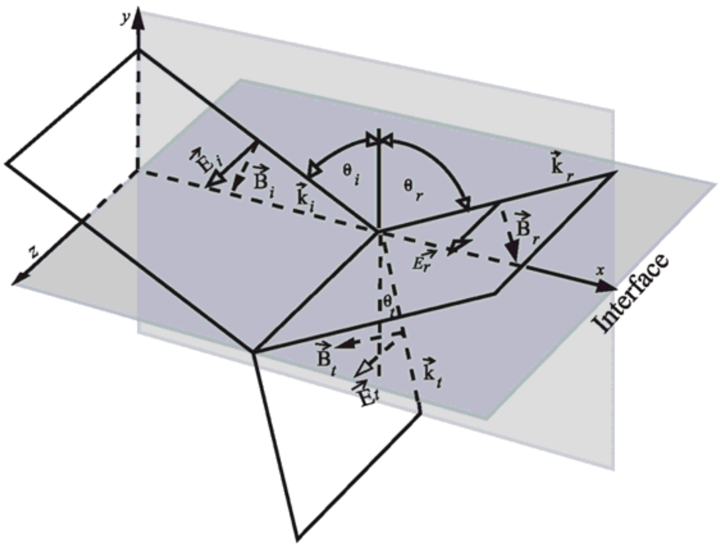
$$= B_t(\text{tangential})$$

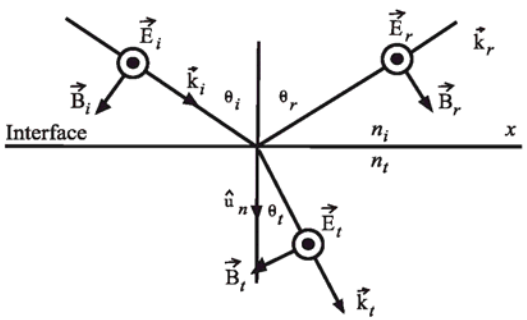
Recall $\mathbf{B} = \dfrac{1}{\omega}\mathbf{k}\times\mathbf{E}$

$$= \frac{1}{\omega}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ k\sin\theta & k\cos\theta & 0 \\ 0 & 0 & E_0 \end{vmatrix}$$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence





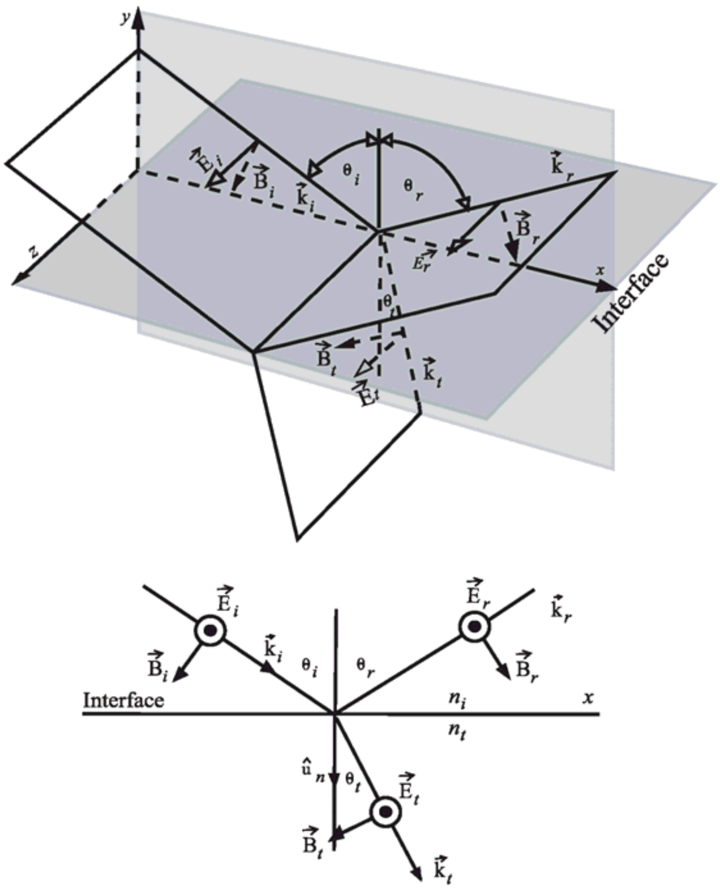
So continuity of tangential magnetic field B_x at the interface y=0 becomes:

$$k_i E_{0i} \cos\theta_i - k_i E_{0r} \cos\theta_r = k_t E_{0t} \cos\theta_t \Leftrightarrow$$

$$n_i E_{0i} \cos\theta_i - n_i E_{0r} \cos\theta_r = n_t E_{0t} \cos\theta_t$$

Reflection & transmission @ dielectric interface

I. Polarization normal to plane of incidence



Solving the 2×2 system of equations:

$$E_{0i} + E_{0r} = E_{0t}$$

$$n_i E_{0i} \cos\theta_i - n_i E_{0r} \cos\theta_r = n_t E_{0t} \cos\theta_t$$

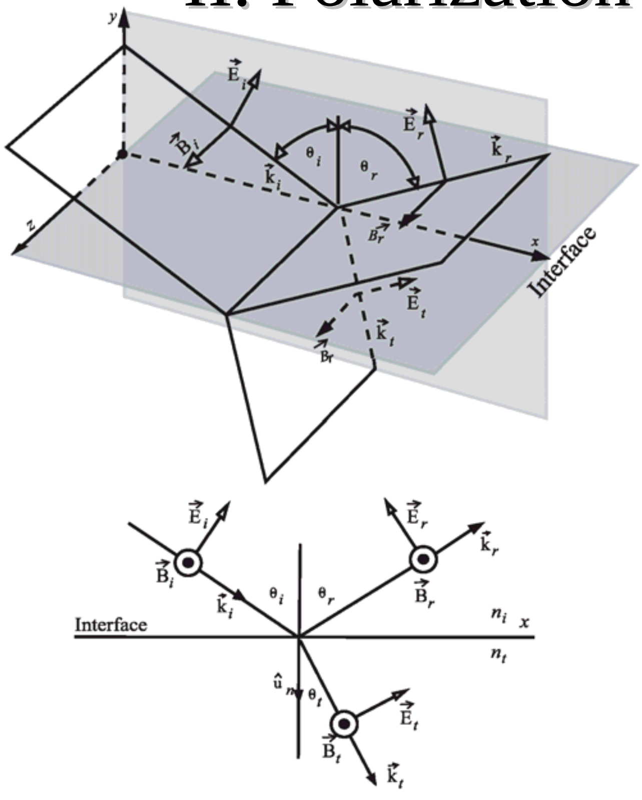
we finally obtain

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

Reflection & transmission @ dielectric interface

II. Polarization parallel to plane of incidence

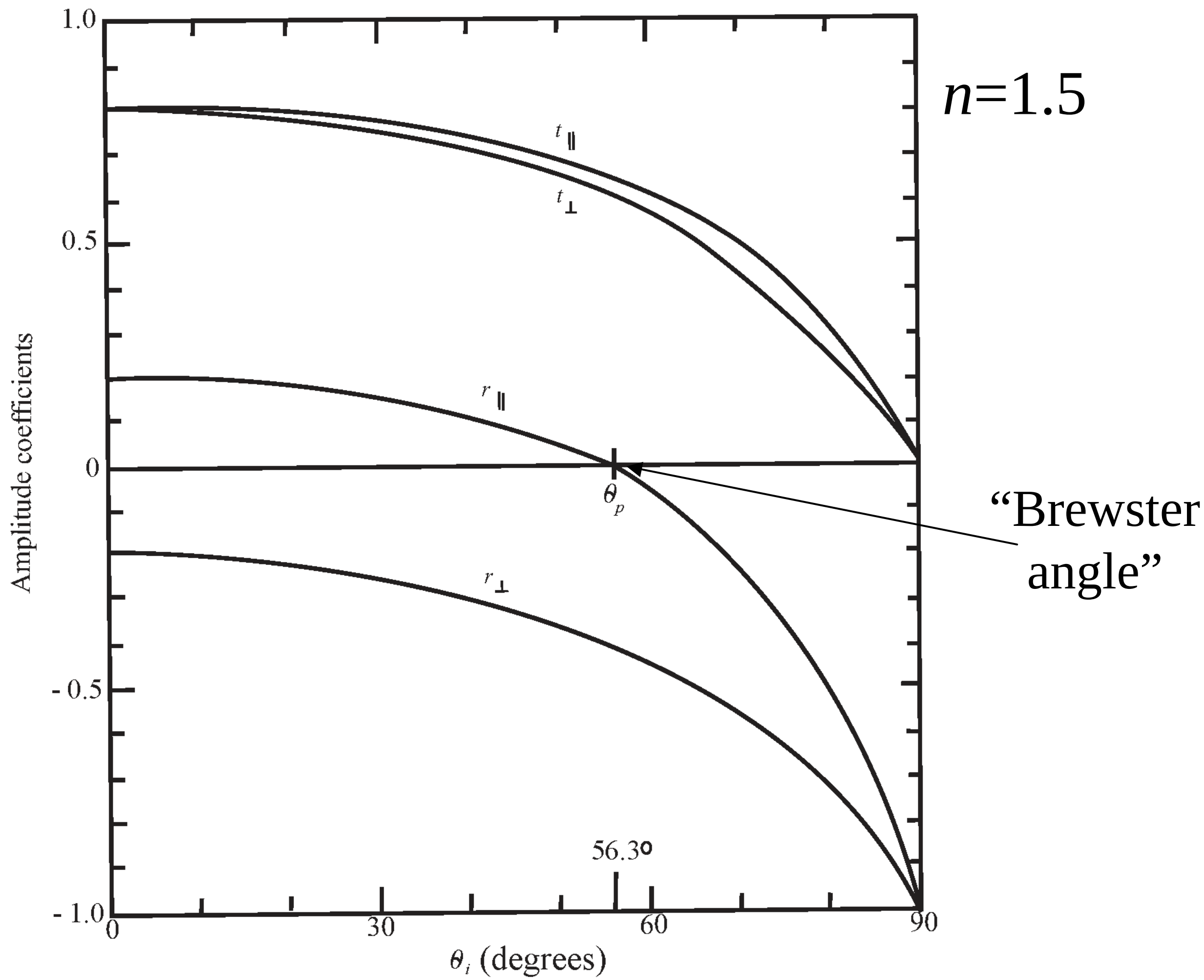


Following a similar procedure ...

$$r_{\|} = \left(\frac{E_{0r}}{E_{0i}} \right)_{\|} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t_{\|} = \left(\frac{E_{0t}}{E_{0i}} \right)_{\|} = \frac{2 n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Reflection & transmission @ dielectric interface

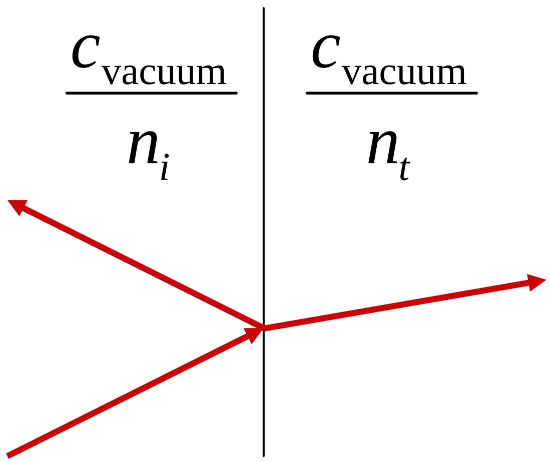


Reflection & transmission of *energy* @ dielectric interface

Recall Poynting vector definition:

$$\|\mathbf{S}\| = c\,\varepsilon_0 \|\mathbf{E}\|^2$$

different on the two sides of the interface

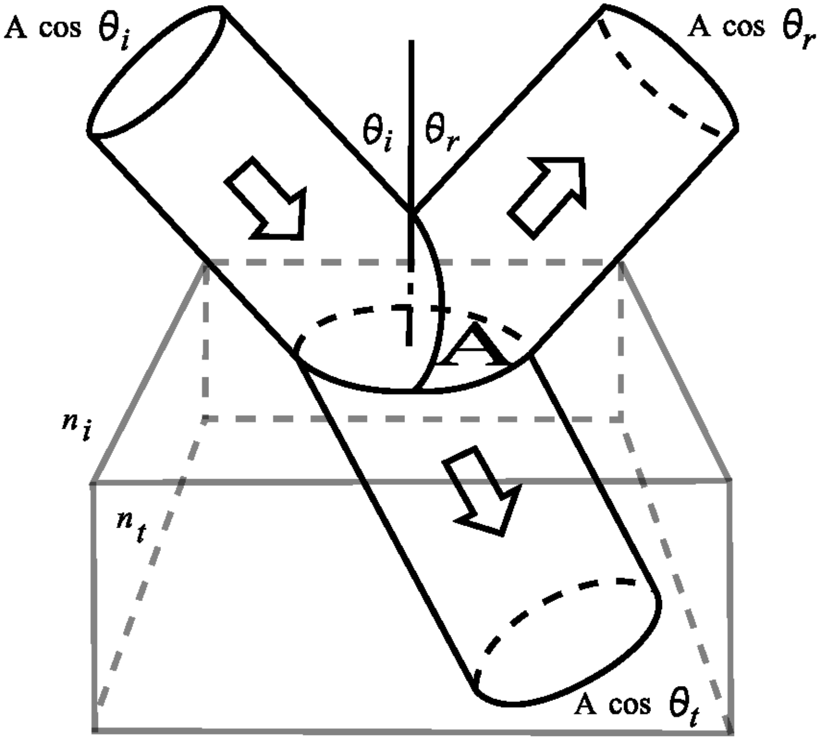


$$R = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

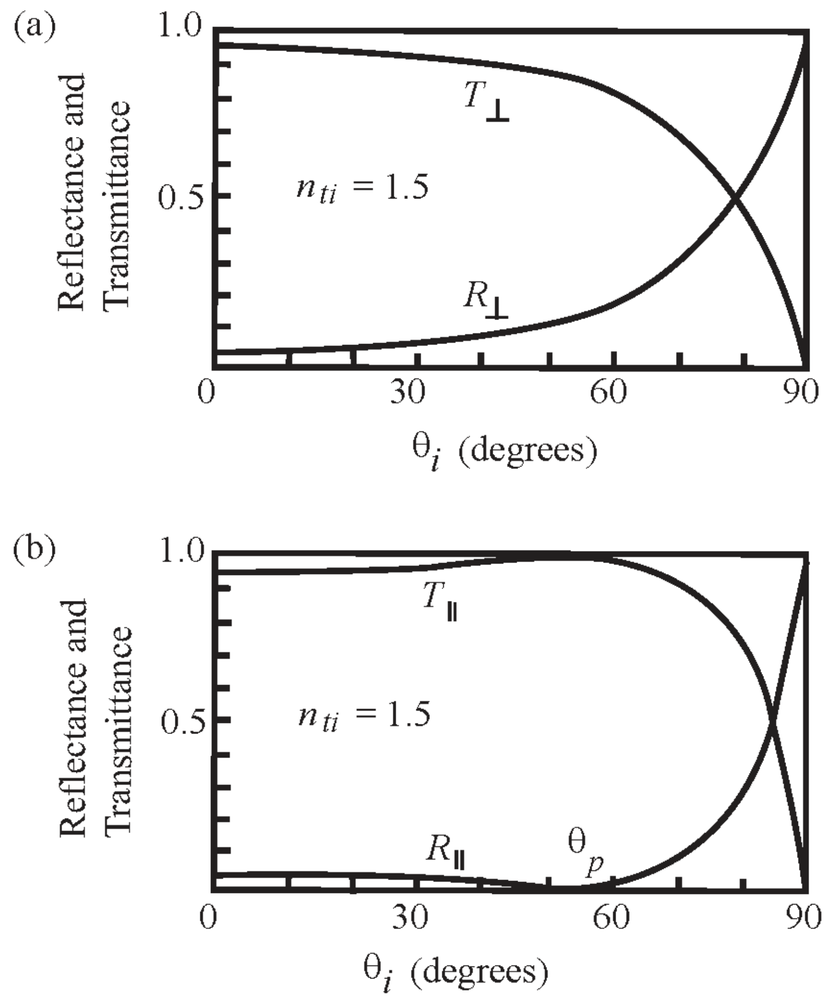
$$T = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left(\frac{E_{0t}}{E_{0i}}\right)^2 = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\, t^2$$

Energy conservation

$$R + T = 1, \text{ i.e. } \quad r^2 + \frac{n_t \cos \theta_t}{n_i \cos \theta_i} t^2 = 1$$



Reflection & transmission of *energy* @ dielectric interface



Normal incidence

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Note: independent of polarization

$\theta_i = 0$ and $\theta_t = 0$ →

$$r_\perp = r_\parallel = \frac{n_t - n_i}{n_t + n_i}$$

$$t_\perp = t_\parallel = \frac{2n_i}{n_t + n_i}$$

$$R_\perp = R_\parallel = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$$

$$T_\perp = T_\parallel = \frac{4n_t n_i}{(n_t + n_i)^2}$$

Brewster angle

Recall Snell's Law $$n_i \sin\theta_i = n_t \sin\theta_t$$

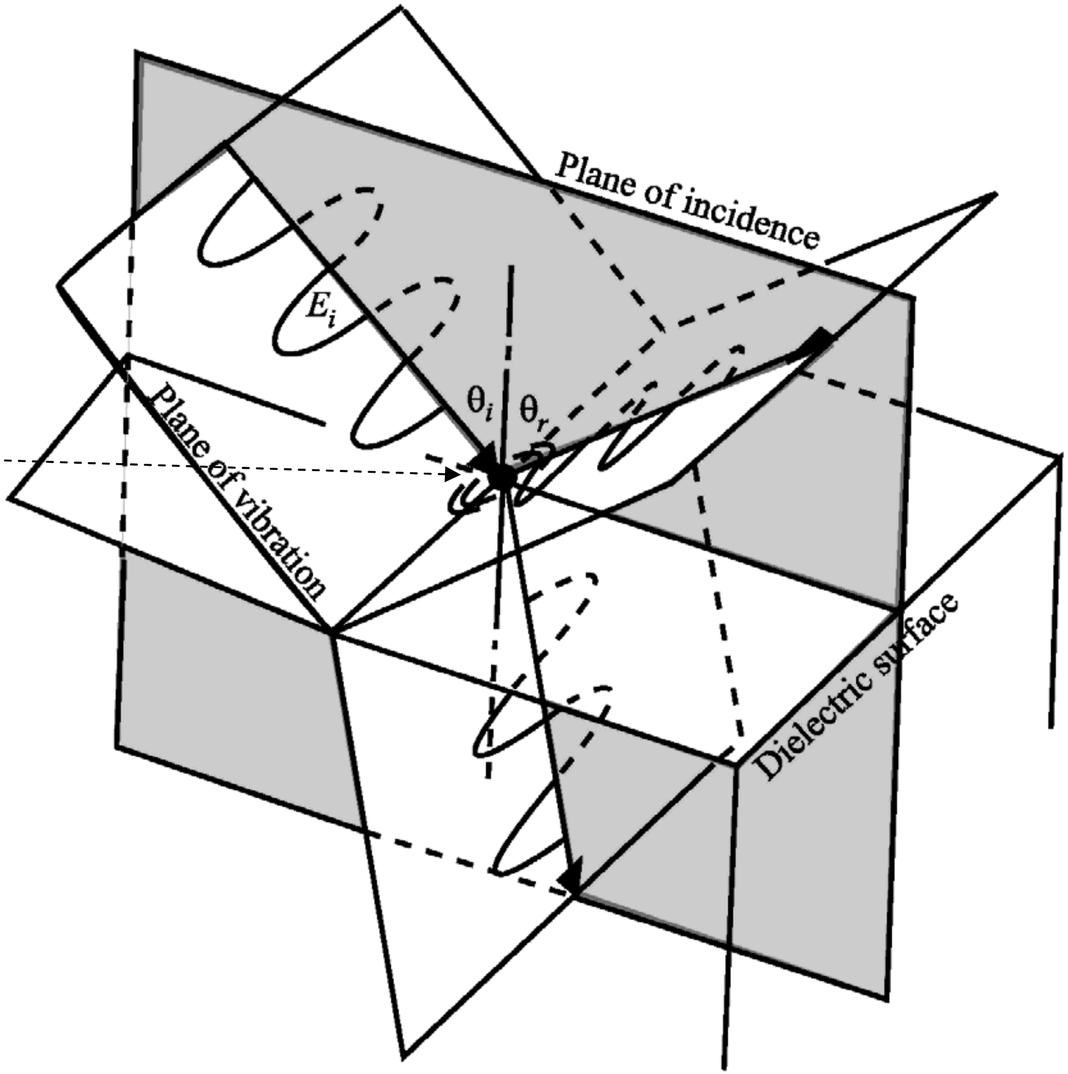
$$r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_t+\theta_i)}$$

$$r_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t} = +\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}$$

If $n_i \neq n_t$, $r_\perp \neq 0$ for all θ_i. When $\theta_\mathrm{i} - \theta_\mathrm{t} = \frac{\pi}{2}$ $(\tan\theta_i = \frac{n_t}{n_i})$, $r_\parallel = 0$.

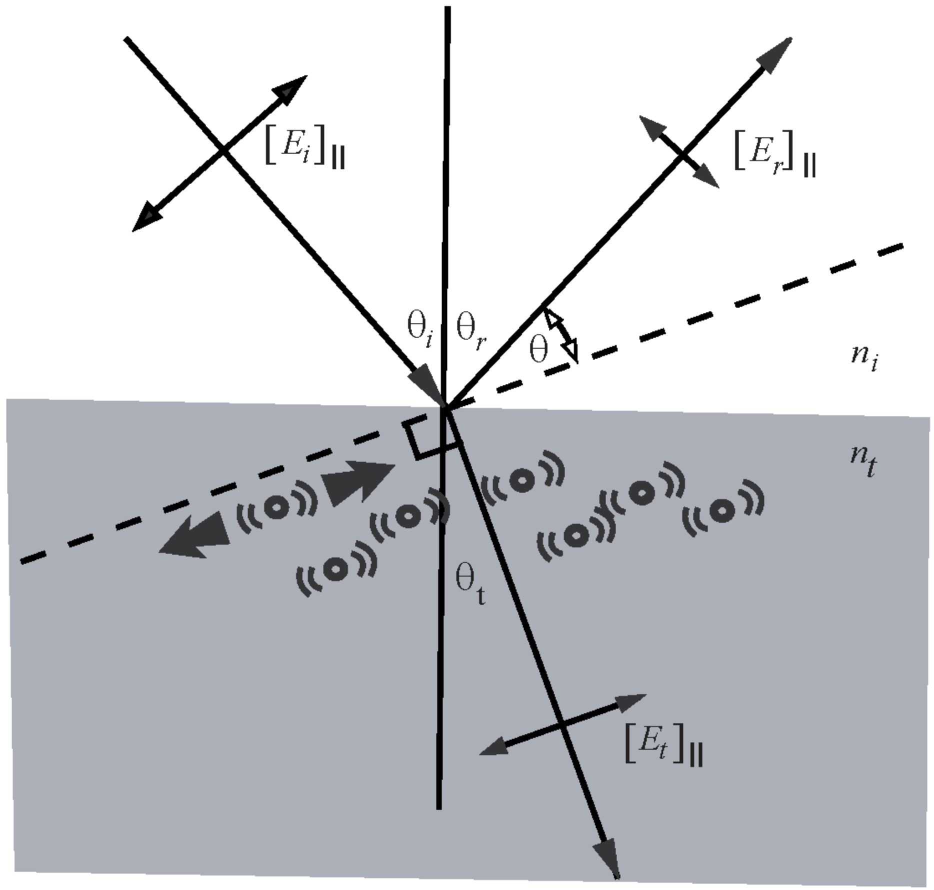
This angle is known as Brewster's angle. Under such circumstances, for an incoming unpolarized wave, only the component polarized normal to the incident plane will be reflected.

Why does Brewster happen?



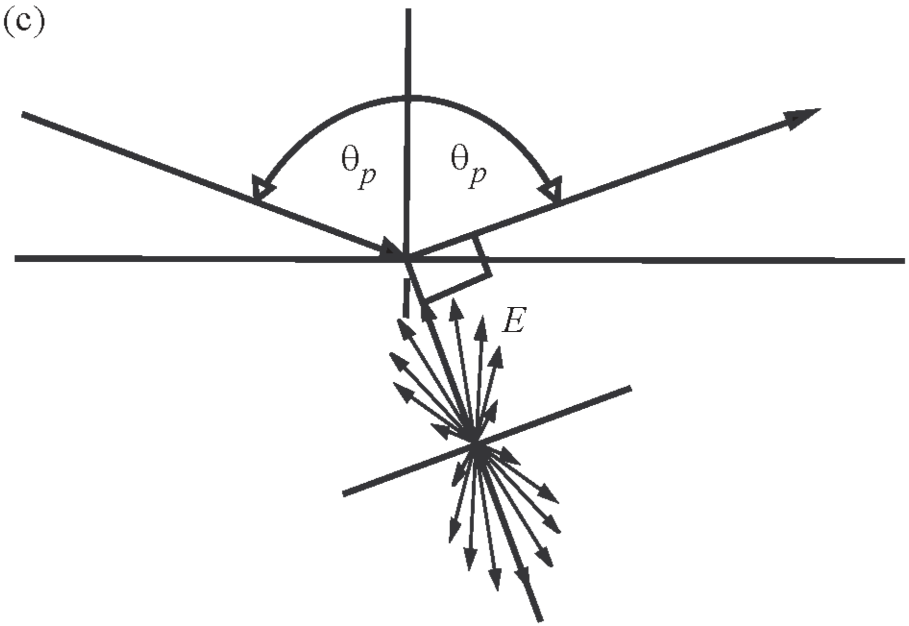
elemental dipole radiator excited by the incident field

Why does Brewster happen?



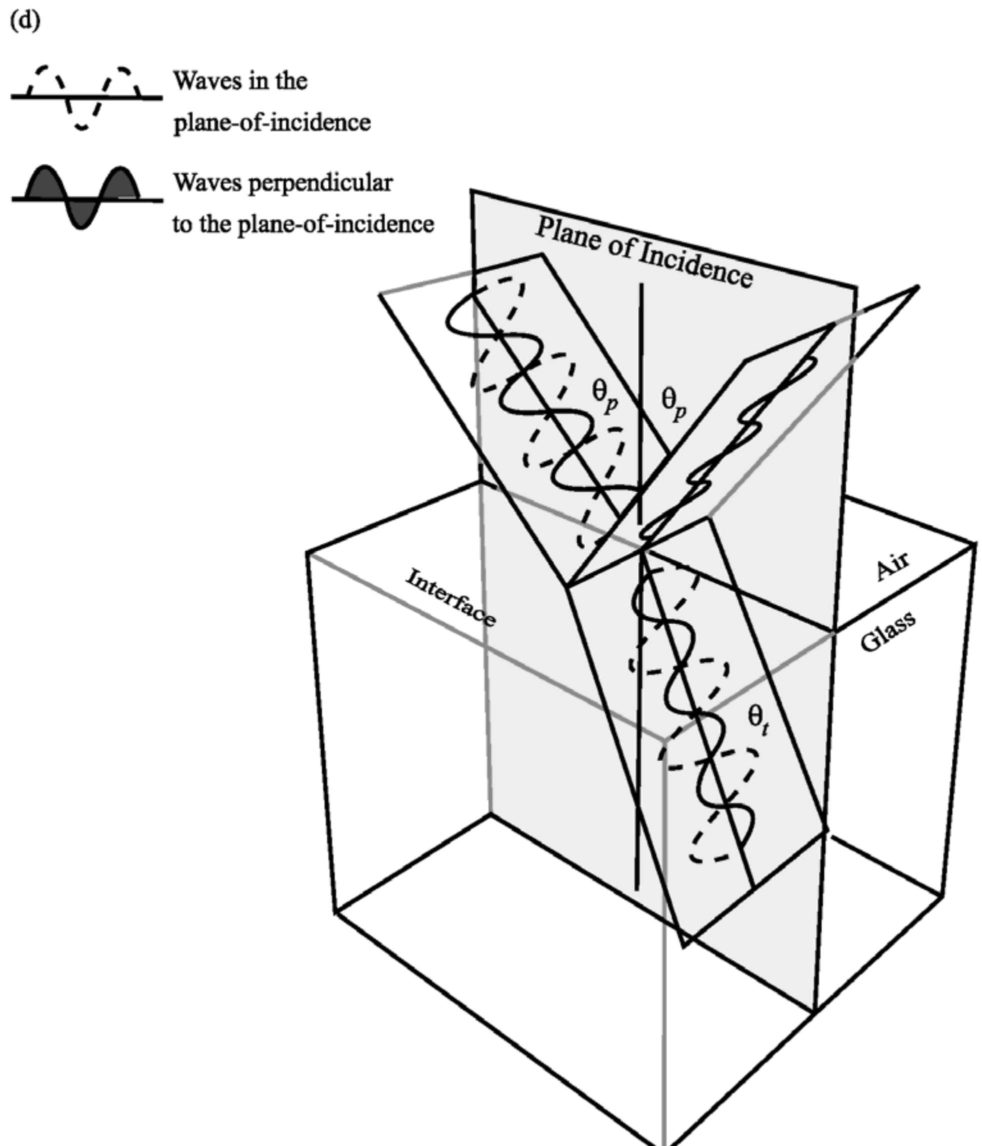
Why does Brewster happen?

(c)

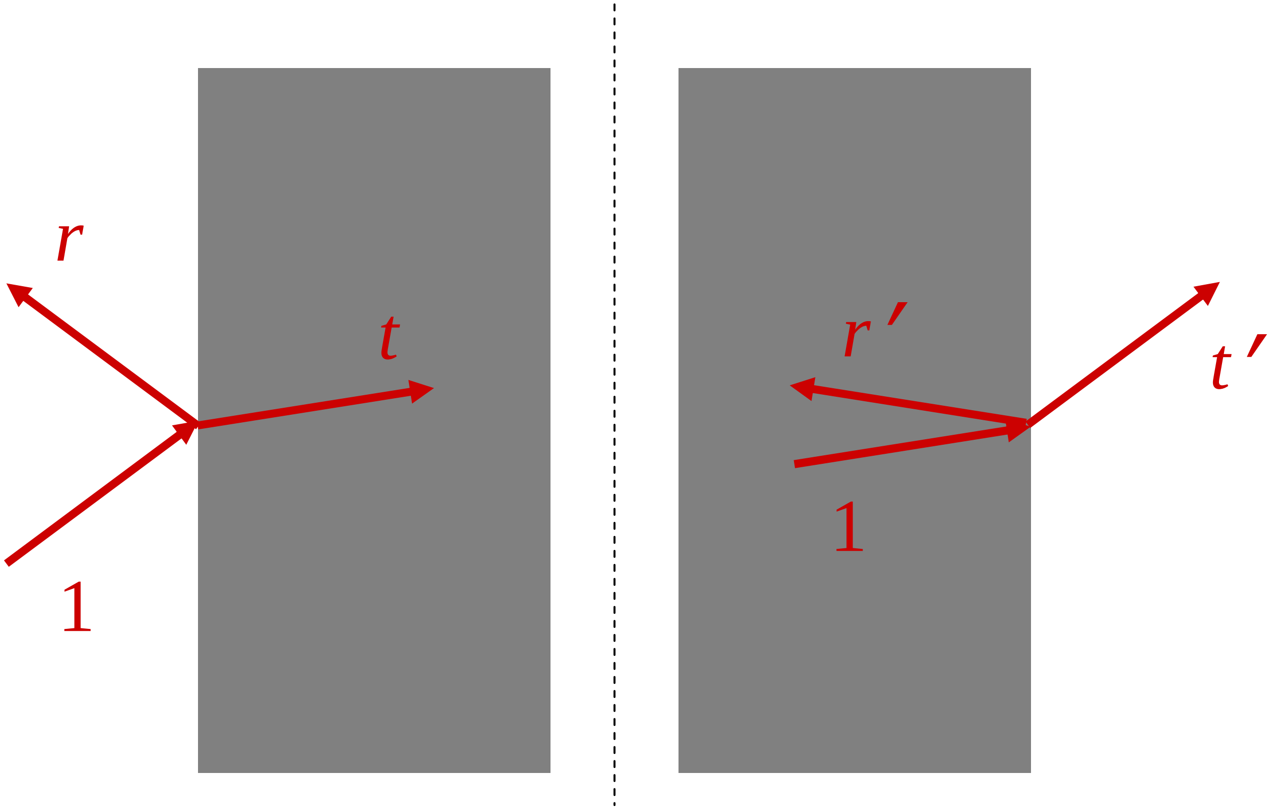


(d)

Why does Brewster happen?

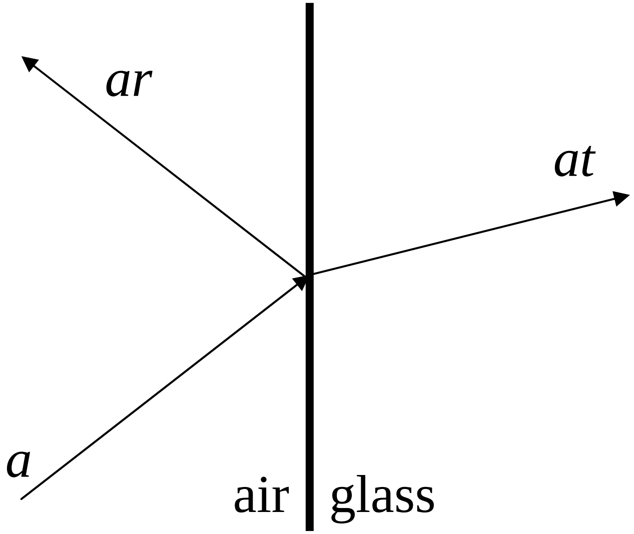


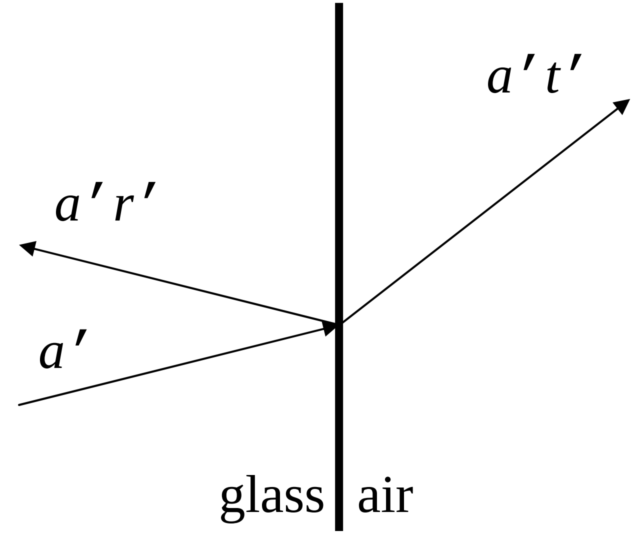
Turning the tables



Is there a relationship between *r*, *t* and *r′*, *t′* ?

Relation between *r*, *r′* and *t*, *t′*





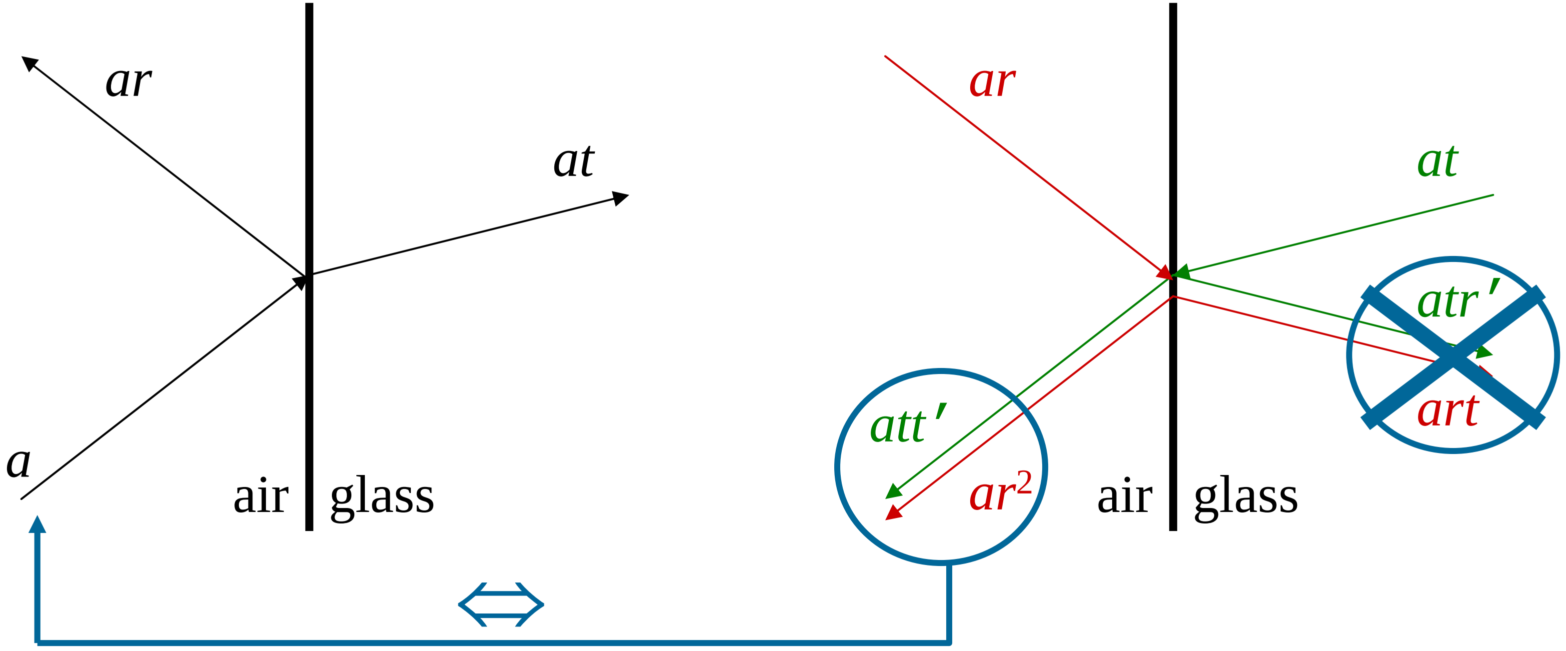
$$r' = -r$$

$$r^2 + tt' = 1$$

Stokes relationships

Proof: algebraic from the Fresnel coefficients or using the property of *preservation of the field properties upon time reversal*

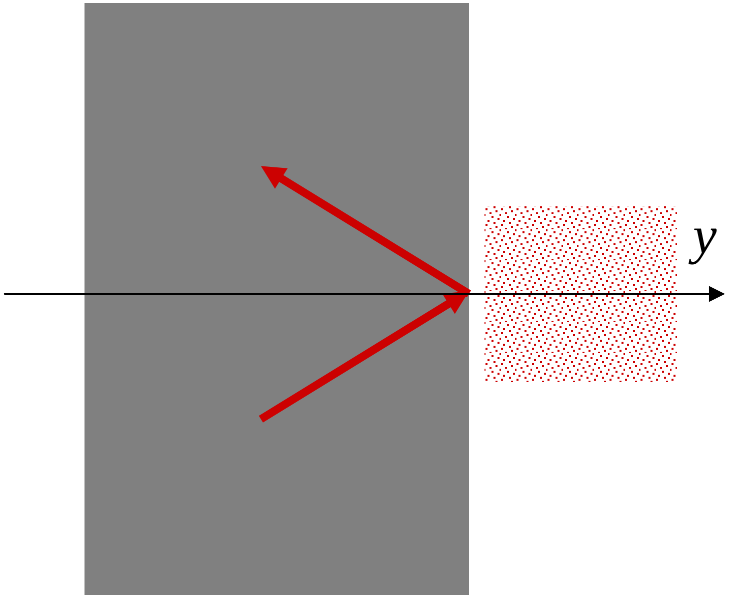
Proof using time reversal



$$a(r + r')t = 0 \Rightarrow r = -r'$$

$$a(r^2 + tt') = a \Rightarrow r^2 + tt' = 1$$

Total Internal Reflection



no energy transmitted

Happens when

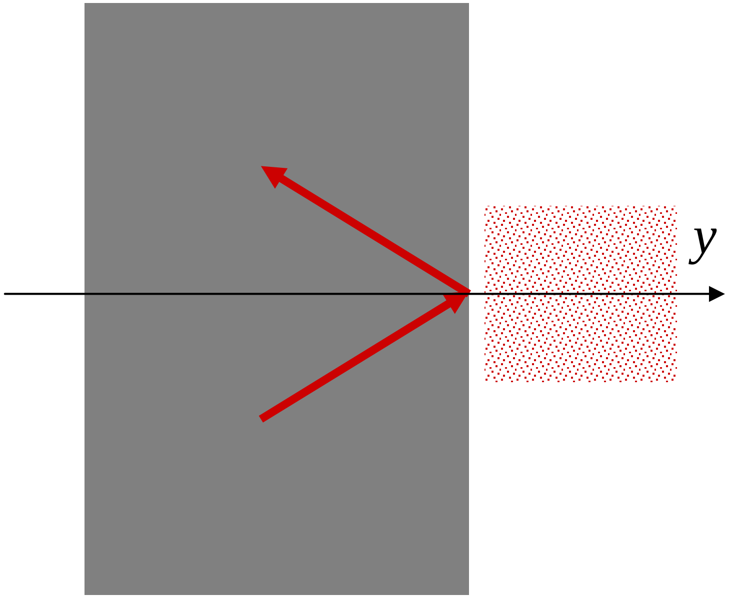
$$n_i \sin\theta_i > 1$$

Substitute into Snell's law

$$\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i > 1$$

ok if θ_t complex

Total Internal Reflection



no energy transmitted

Propagating component

$$E_t \propto \exp\left[ik_t y \cos\theta_t\right]$$

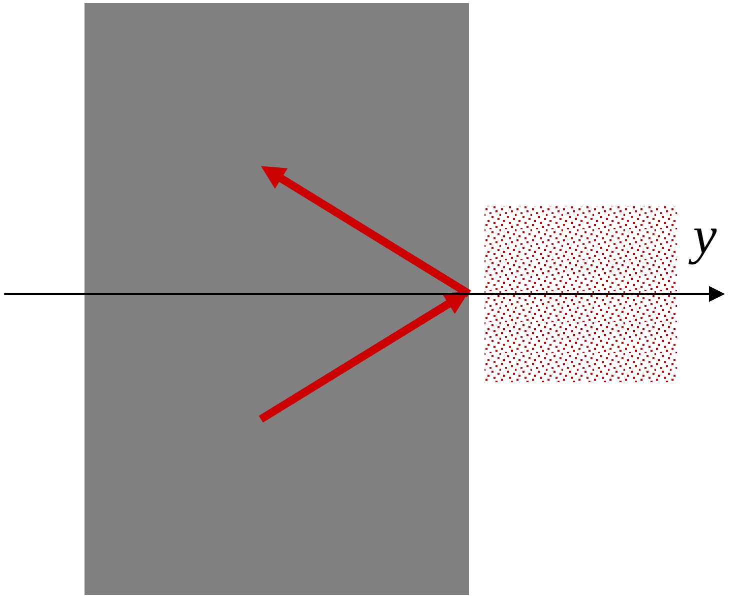
where

$$\cos\theta_t = \pm i\sqrt{\frac{n_i^2 \sin^2\theta_i}{n_t^2} - 1}$$

so

$$E_t \propto \exp\left[-k_t y\sqrt{\frac{n_i^2 \sin^2\theta_i}{n_t^2} - 1}\right]$$

Total Internal Reflection



$$E_t \propto \exp\left[-k_t y \sqrt{\frac{n_i^2 \sin^2\theta_i}{n_t^2} - 1}\right]$$

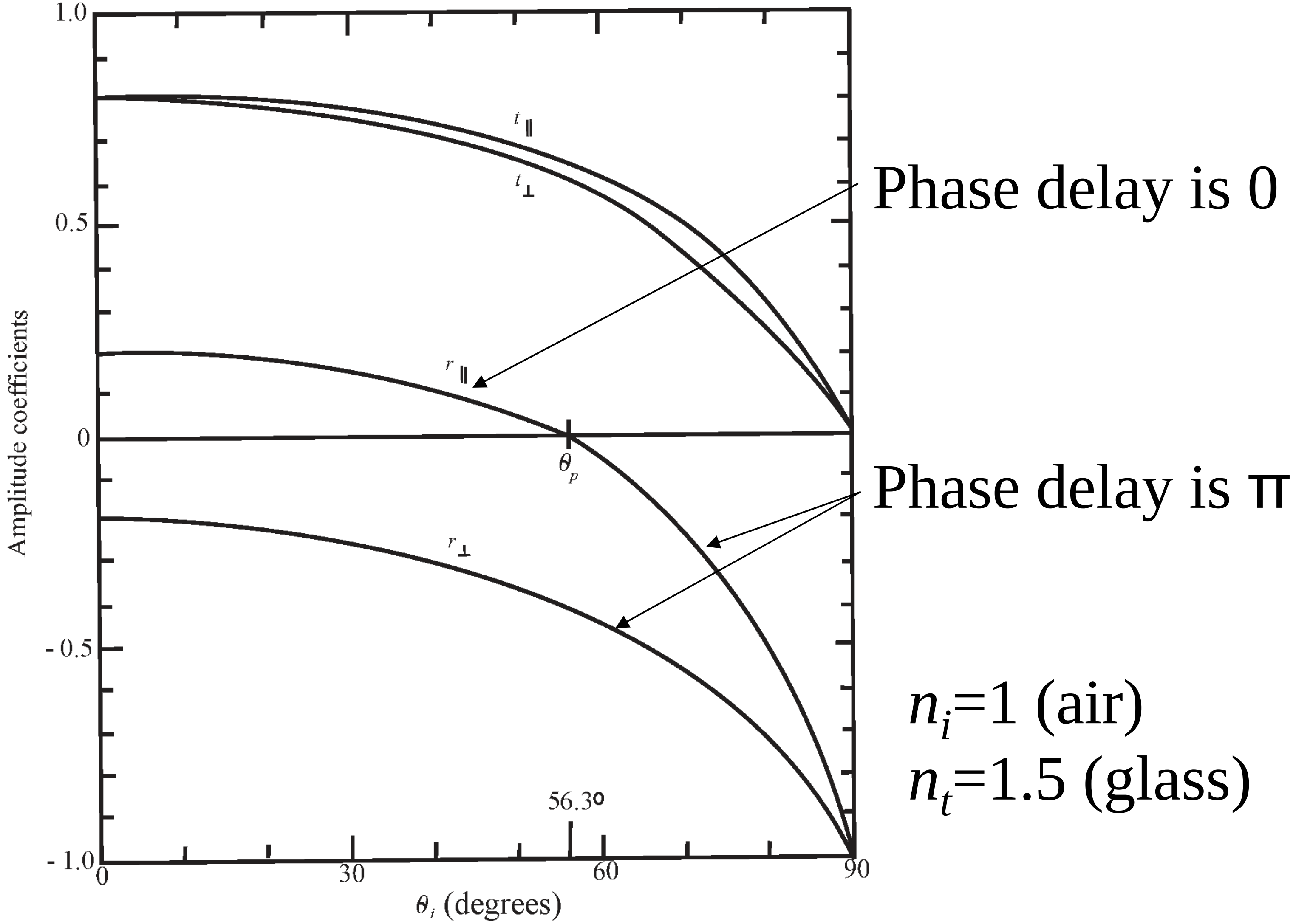
Pure exponential decay
≡ *evanescent* wave

It can be shown that:

$$\left\langle \left\| \mathbf{S}_t \right\| \right\rangle \approx 0$$

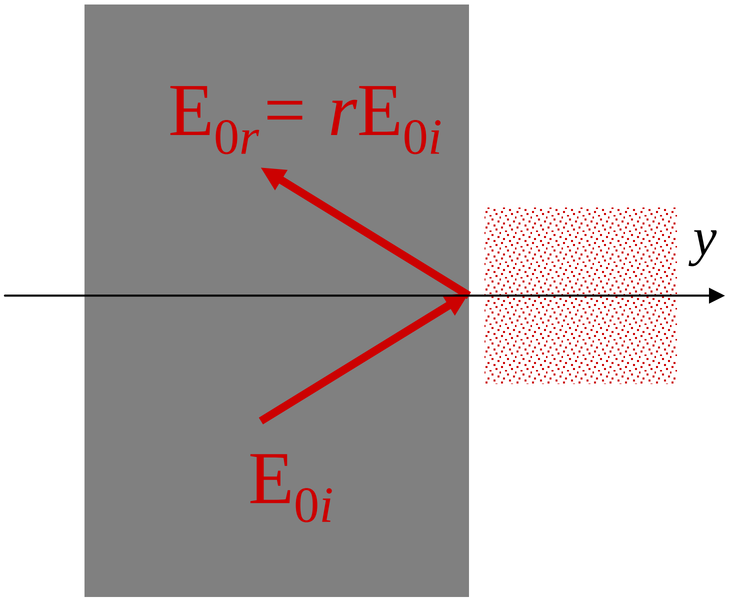
no energy transmitted

Phase delay upon reflection



n_i=1 (air)
n_t=1.5 (glass)

Phase delay upon TIR



$$r_{\parallel} = e^{i\delta_{\parallel}} \quad \text{and} \quad r_{\perp} = e^{i\delta_{\perp}}$$

where

$$\tan\frac{\delta_{\parallel}}{2} = -\frac{n_i\sqrt{n_i^2\sin^2\theta_i - n_t^2}}{n_t^2\cos\theta_i}$$

$$\tan\frac{\delta_{\perp}}{2} = -\frac{\sqrt{n_i^2\sin^2\theta_i - n_t^2}}{n_i\cos\theta_i}$$

Phase delay upon TIR