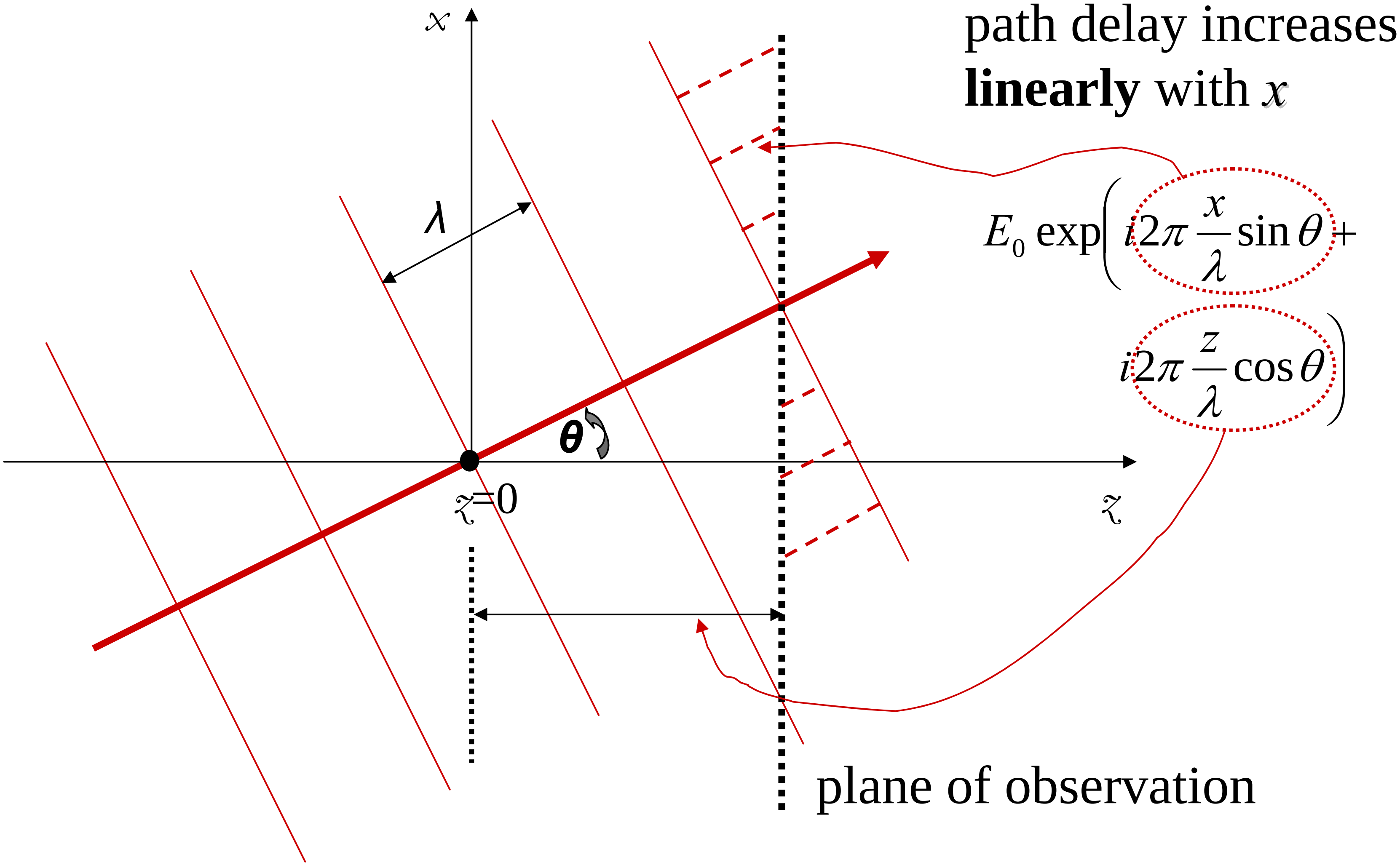
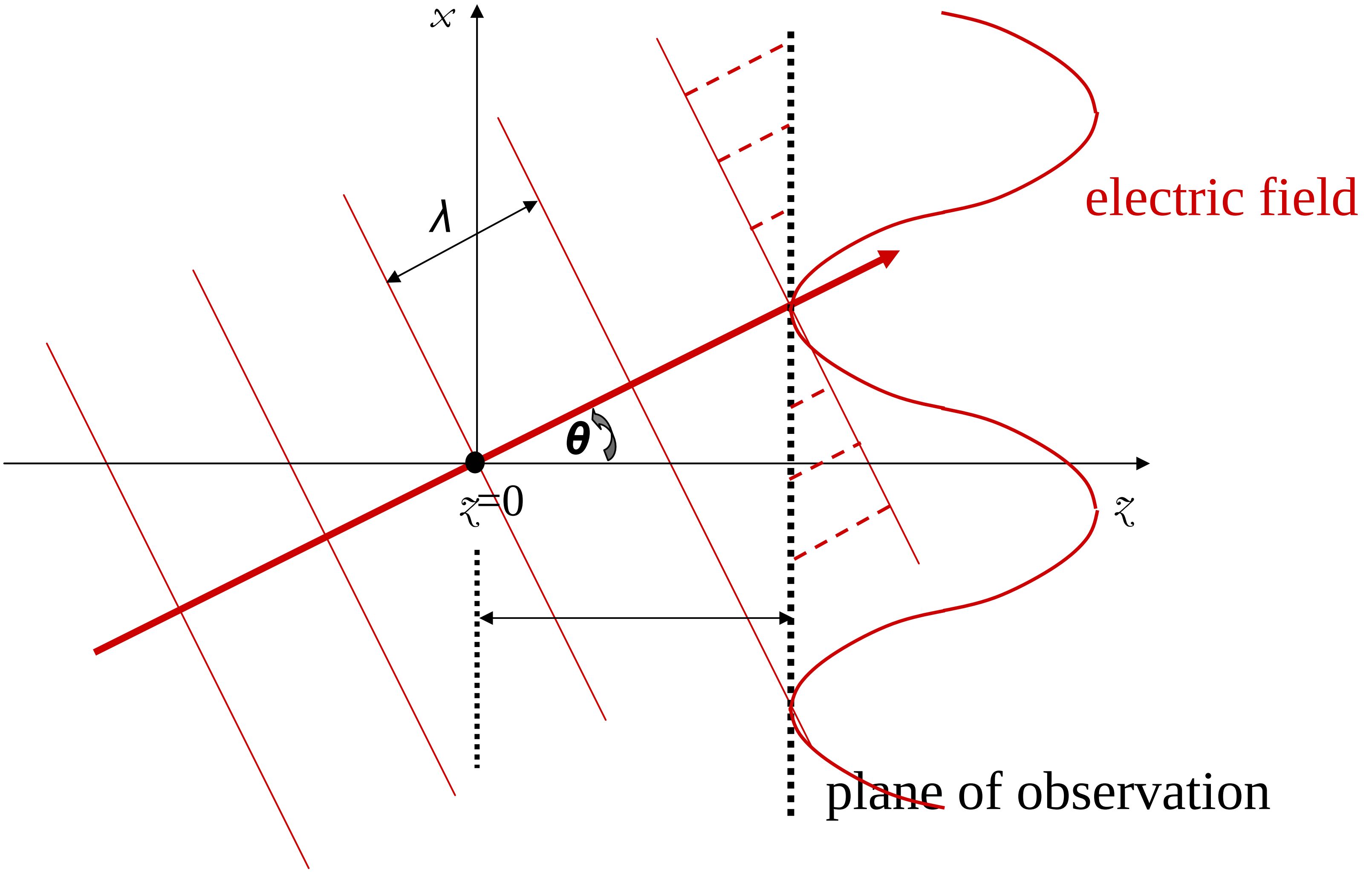
The spatial frequency domain

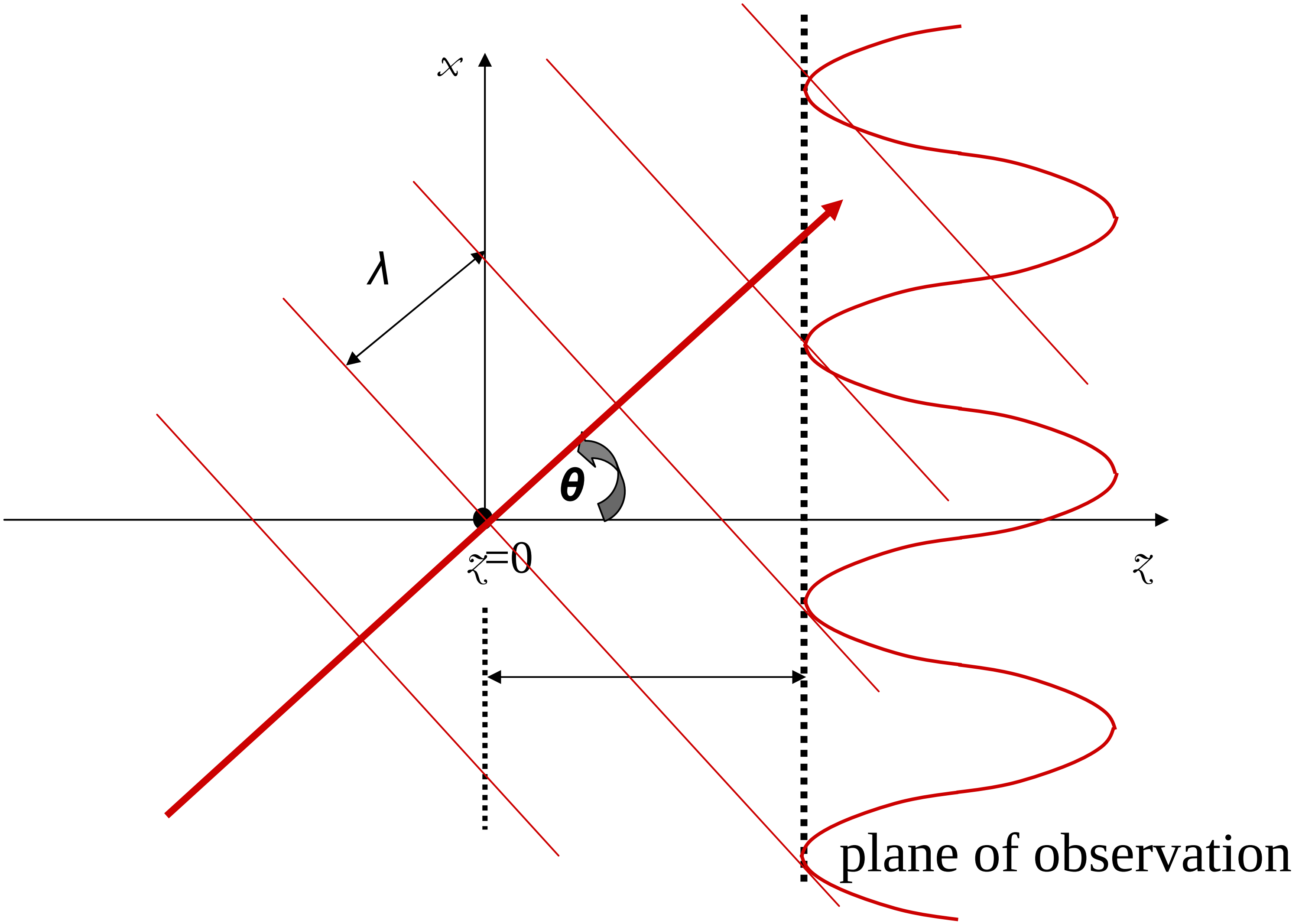
Recall: plane wave propagation



Spatial frequency ⇔ angle of propagation?



Spatial frequency ⇔ angle of propagation?



Spatial frequency ⇔ angle of propagation?

The cross-section of the optical field with the optical axis is a sinusoid of the form

$$E_0 \exp\left(i2\pi \frac{\sin\theta}{\lambda} x + \phi_0 \right)$$

i.e. it looks like

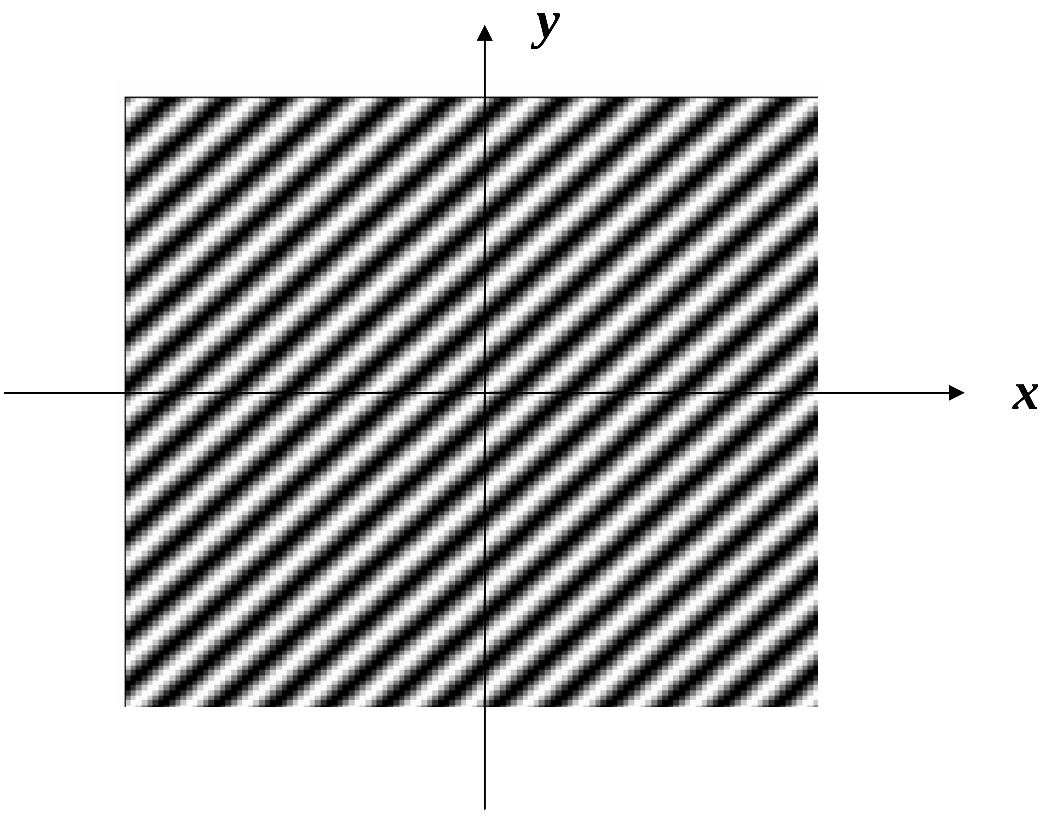
$$E_0 \exp\left(i2\pi\, u\, x + \phi_0\right) \quad \text{where} \quad u \equiv \frac{\sin\theta}{\lambda}$$

is called the **spatial frequency**

2D sinusoids

$$E_0 \cos\left[2\pi\left(ux + vy\right)\right]$$

$$\left(\begin{array}{c}\text{corresp. phasor}\\ E_0 \exp\left[i2\pi\left(ux + vy\right)\right]\end{array}\right)$$

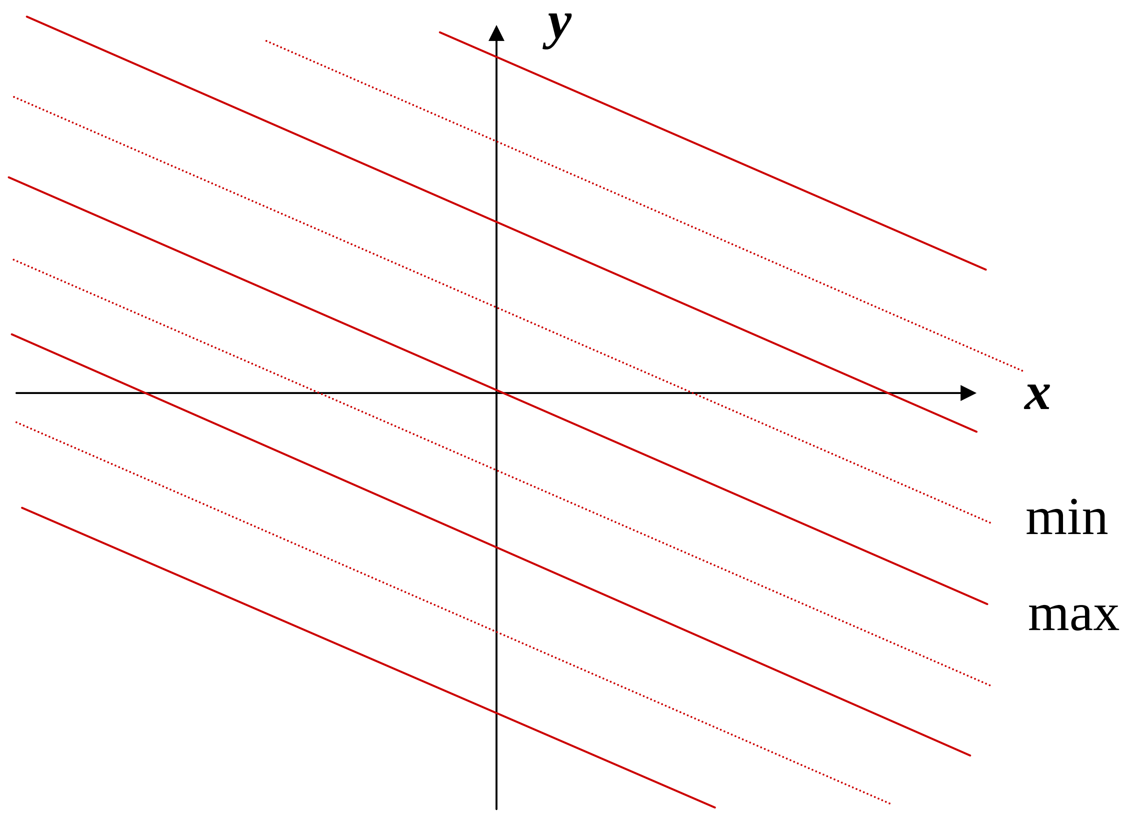


2D sinusoids

$$E_0 \cos\left[2\pi\left(ux + vy\right)\right]$$

corresp. phasor

$$\left(E_0 \exp\left[i2\pi\left(ux + vy\right)\right] \right)$$



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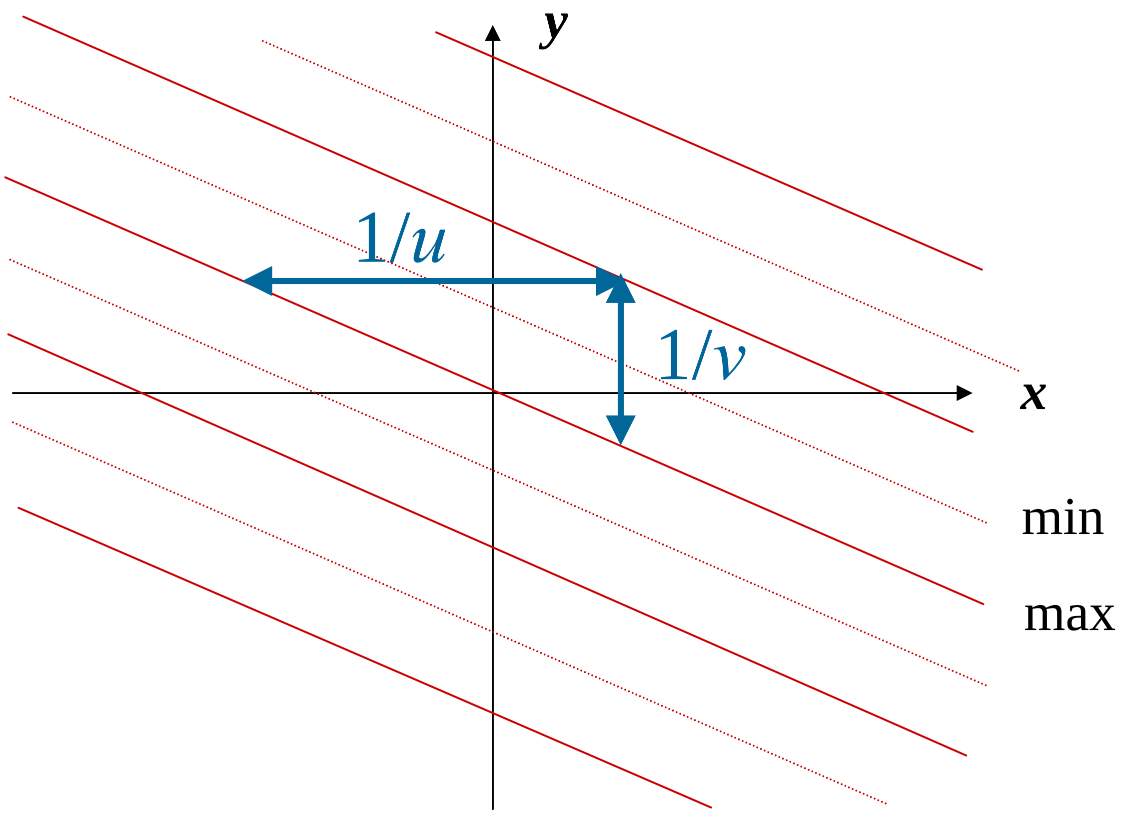


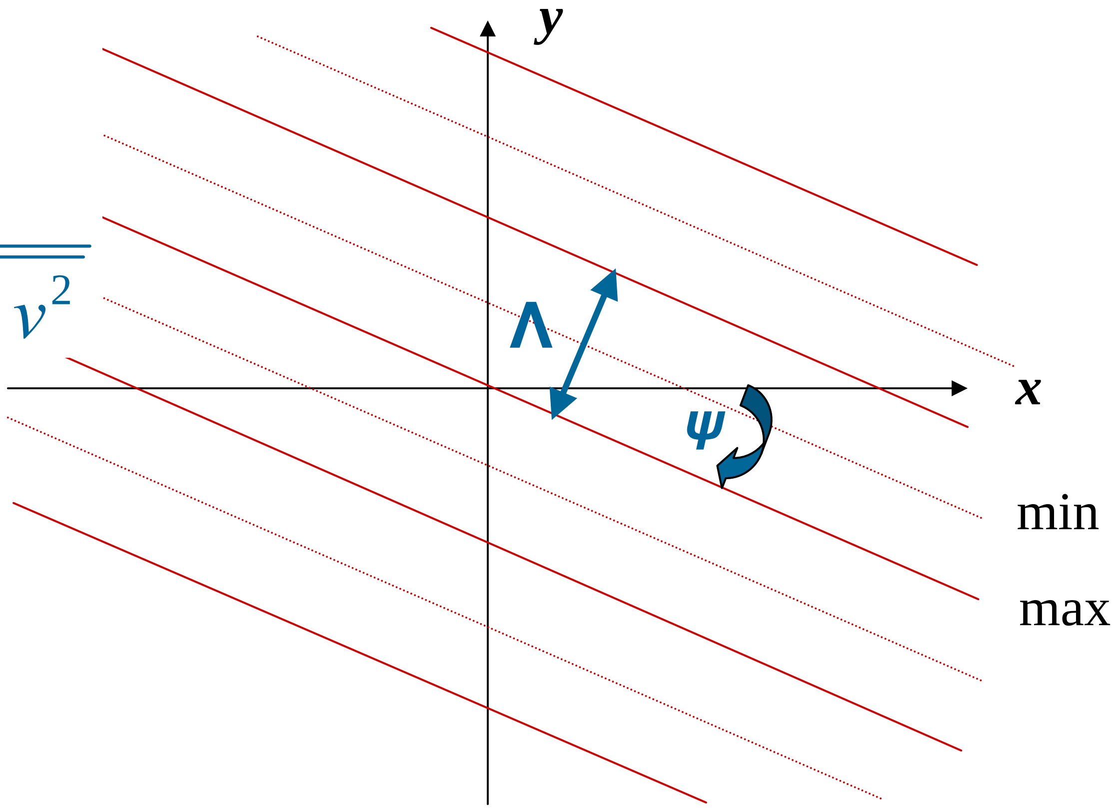
2D sinusoids

$$E_0 \cos[2\pi(ux + vy)]$$

$$\left(\begin{array}{c}\text{corresp. phasor}\\ E_0 \exp[i2\pi(ux+vy)]\end{array}\right)$$

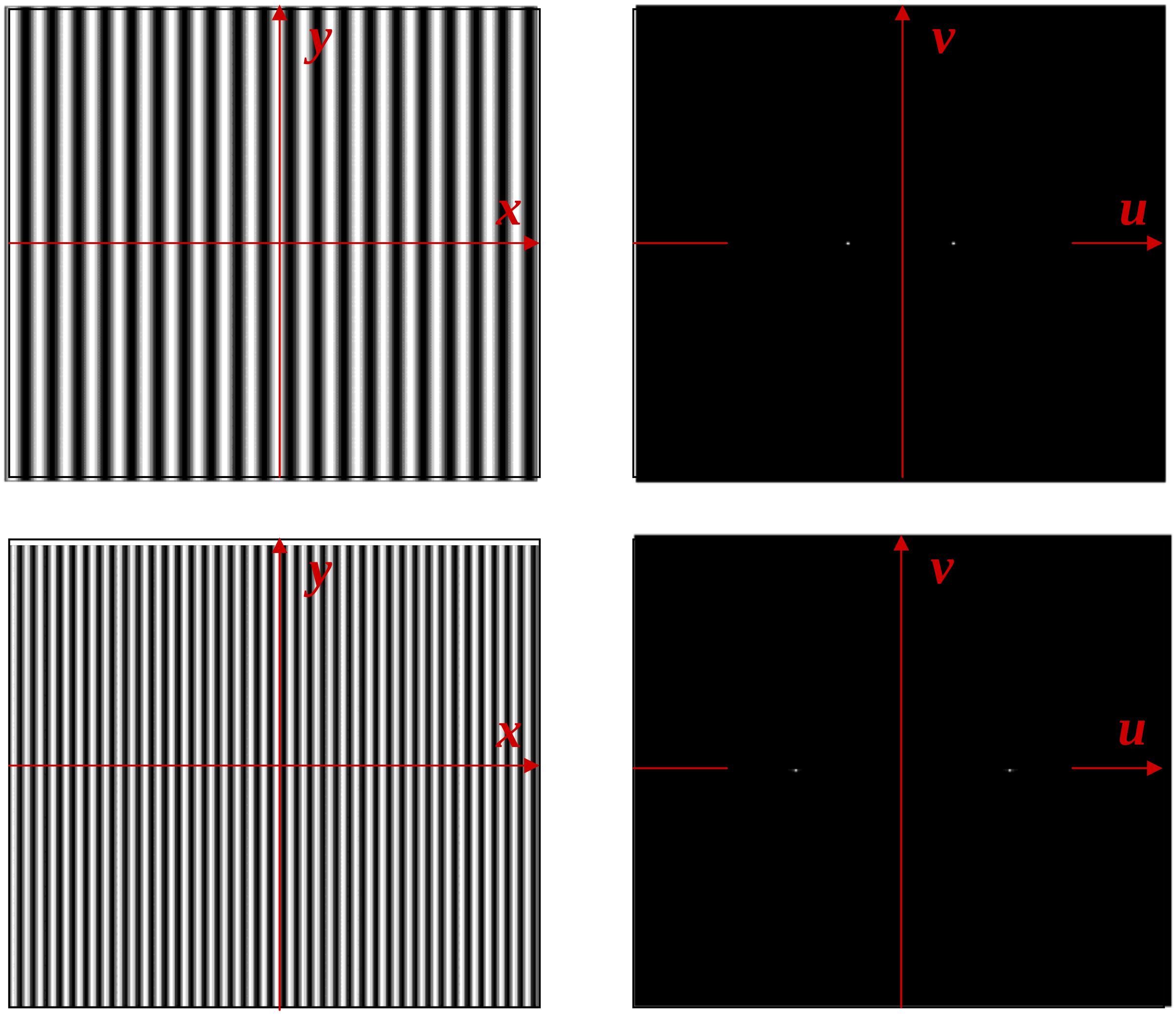
$$\tan\psi = \frac{u}{v}$$

$$\Lambda = \frac{1}{\sqrt{u^2+v^2}}$$



Periodic Grating /1: vertical

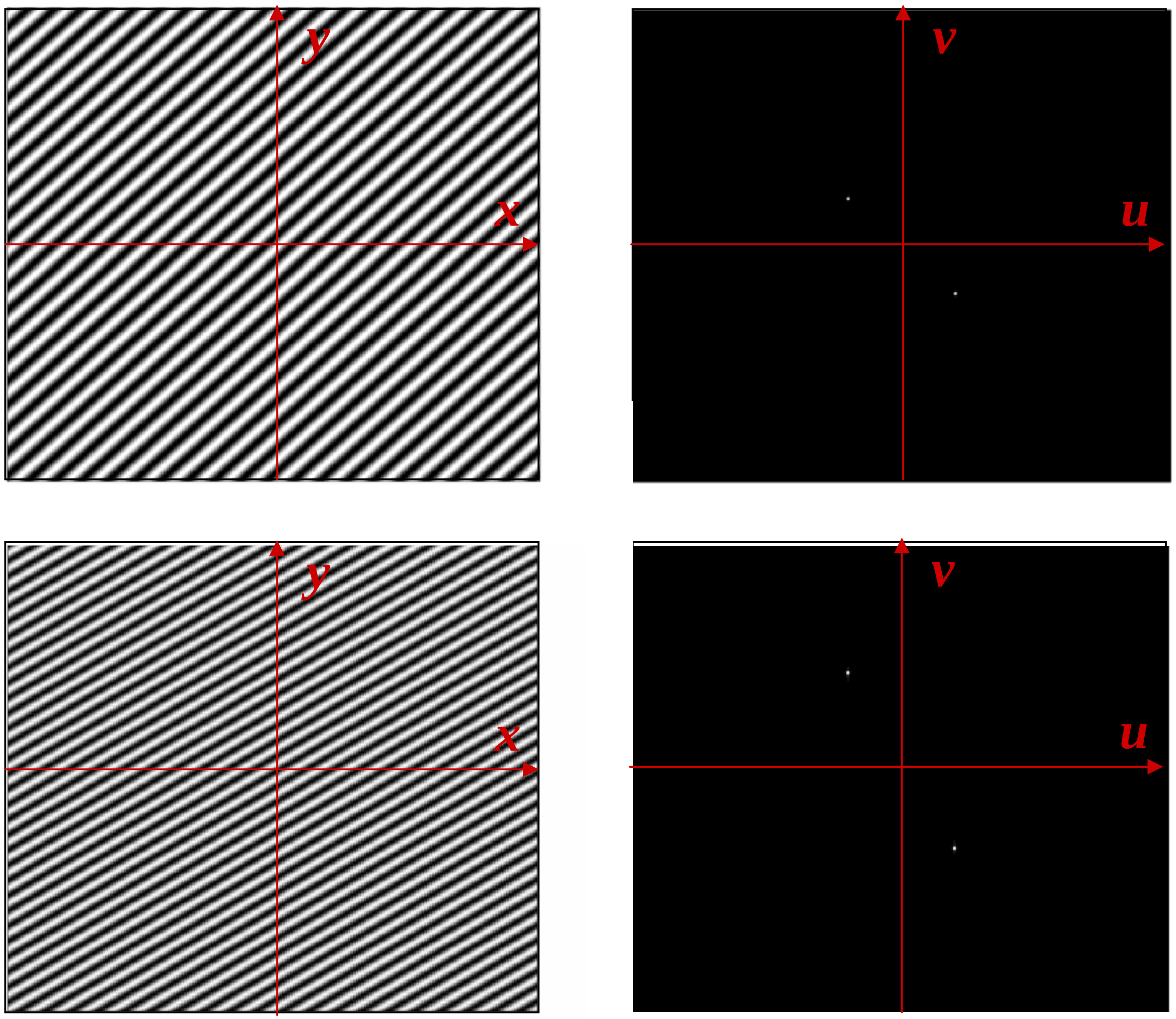
Space domain



Frequency (Fourier) domain

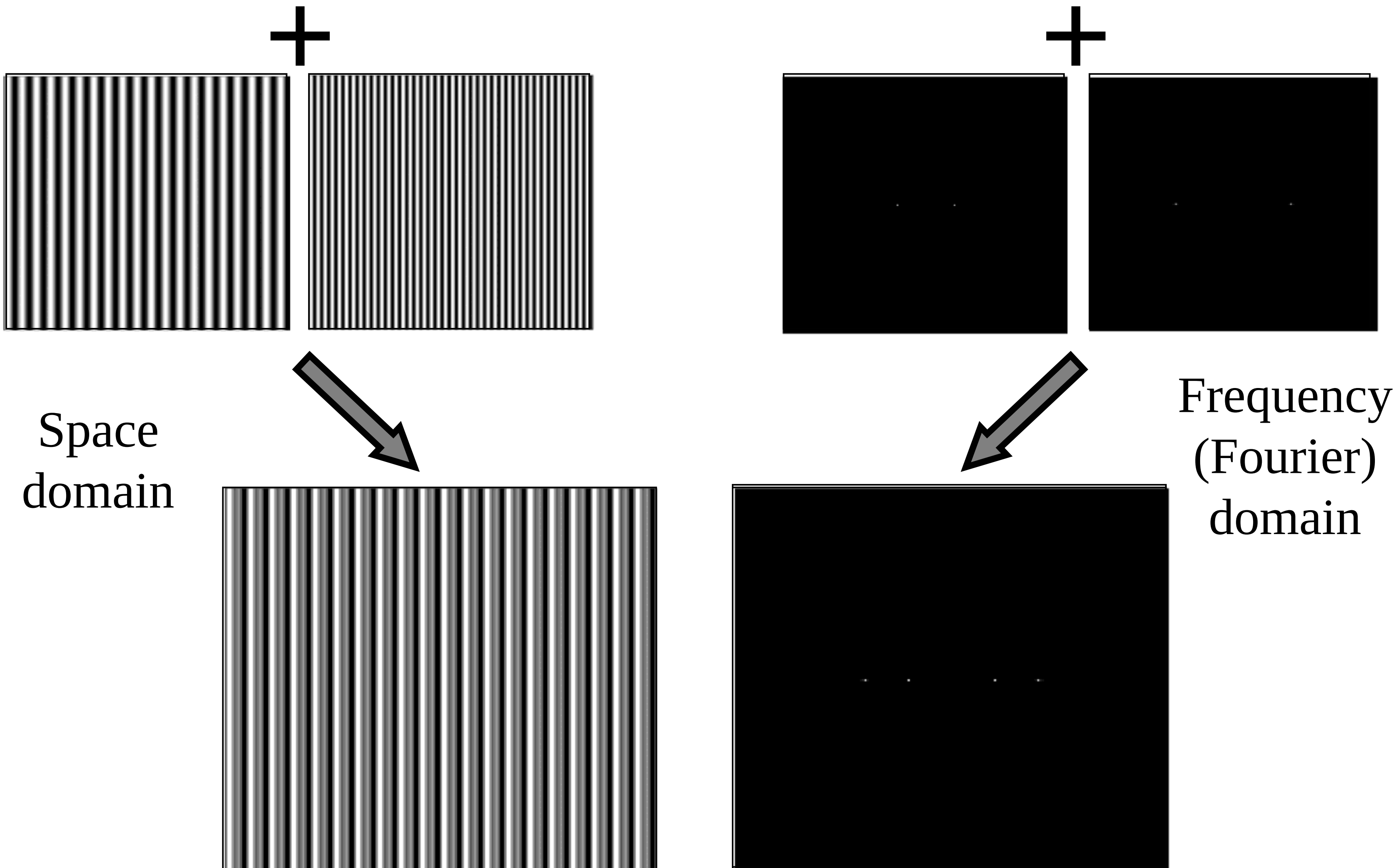
Periodic Grating /2: tilted

Space domain



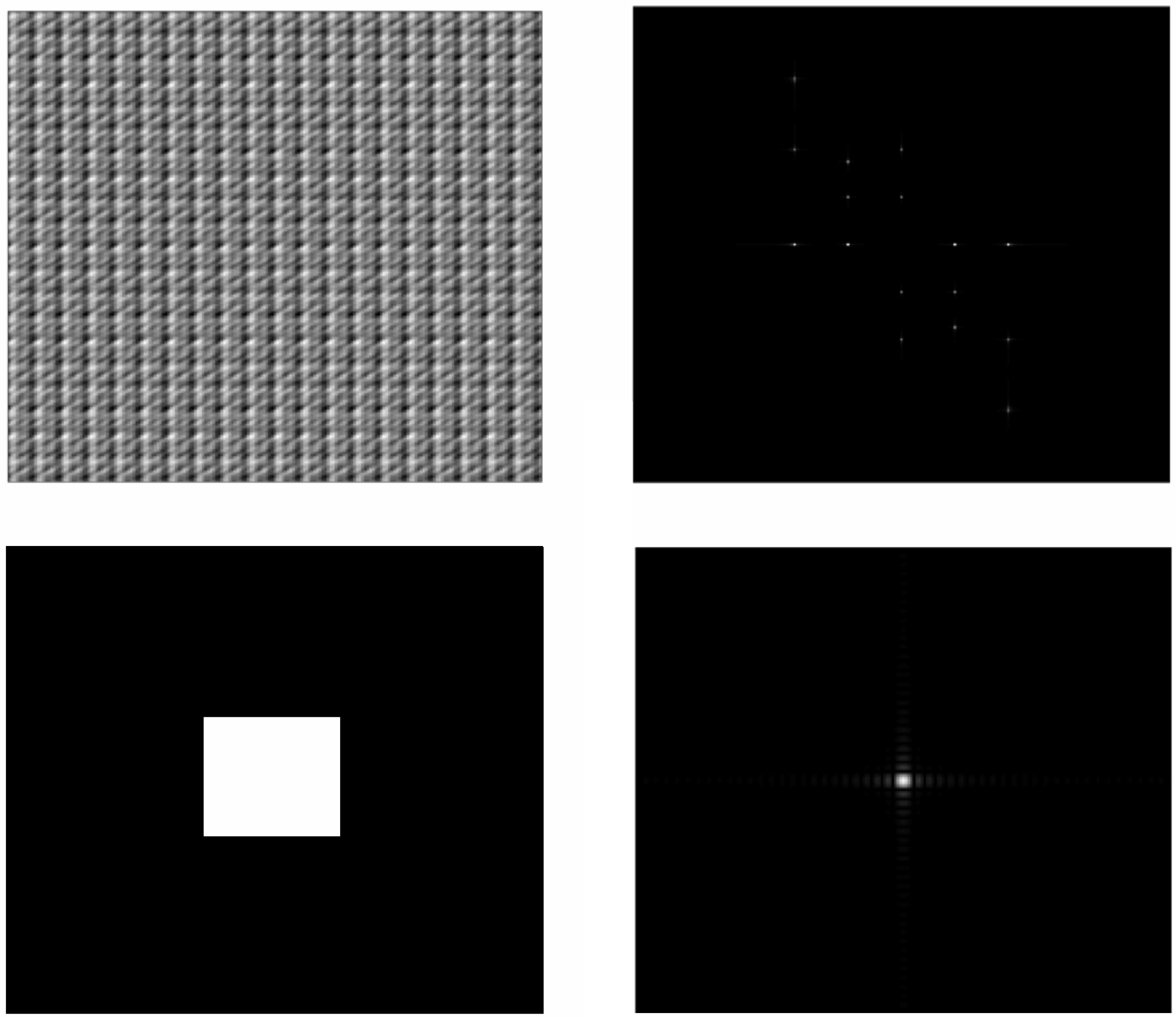
Frequency (Fourier) domain

Superposition: multiple gratings



More superpositions

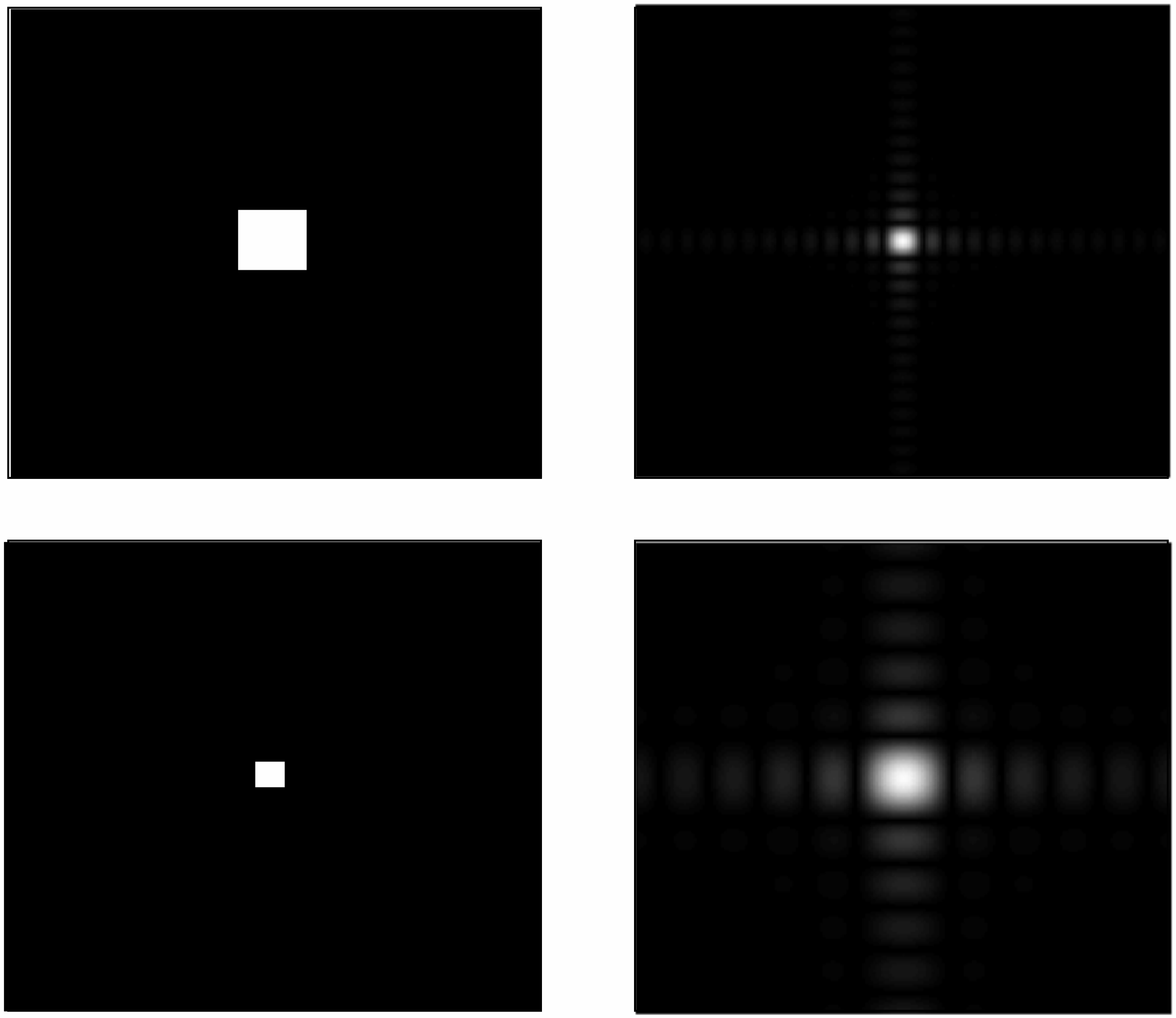
Space domain



Frequency (Fourier) domain

Size of object *vs* frequency content

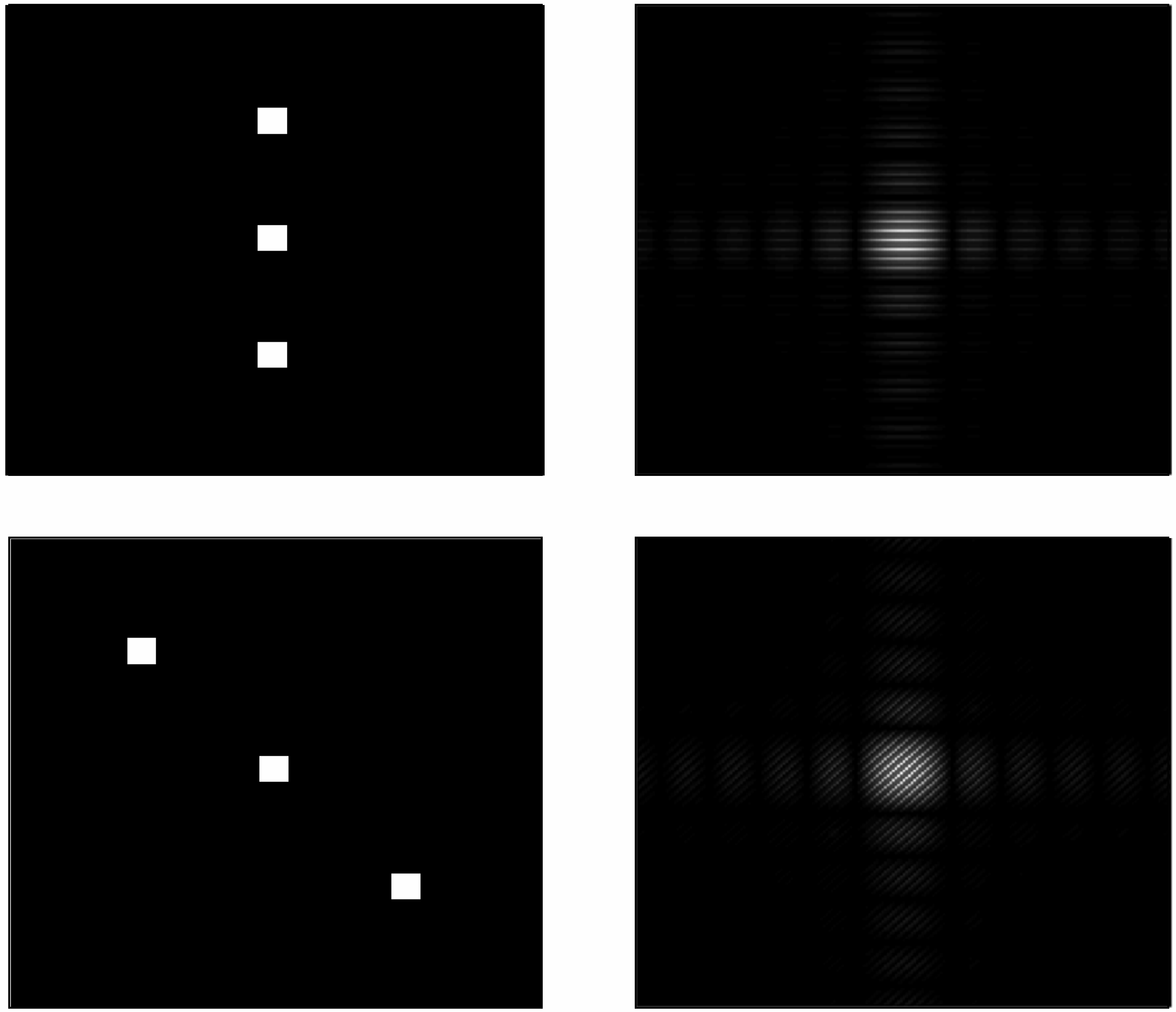
Space domain



Frequency (Fourier) domain

Superimposing shifted objects

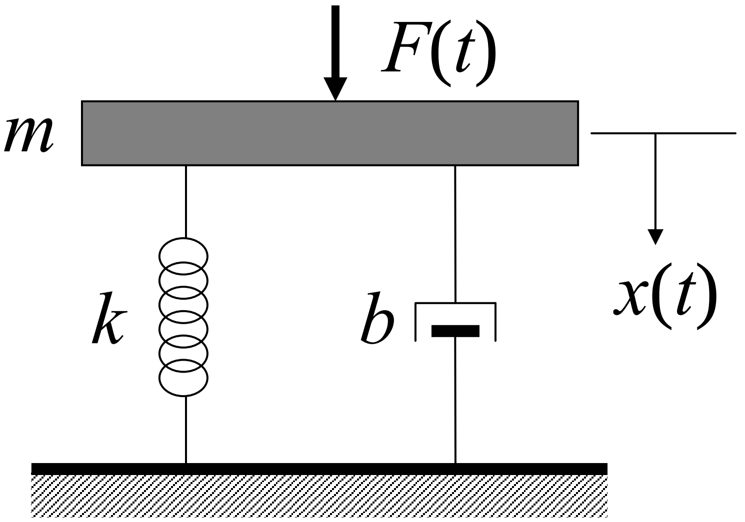
Space domain



Frequency (Fourier) domain

Linear shift invariant systems in the time domain

Linear shift invariant systems

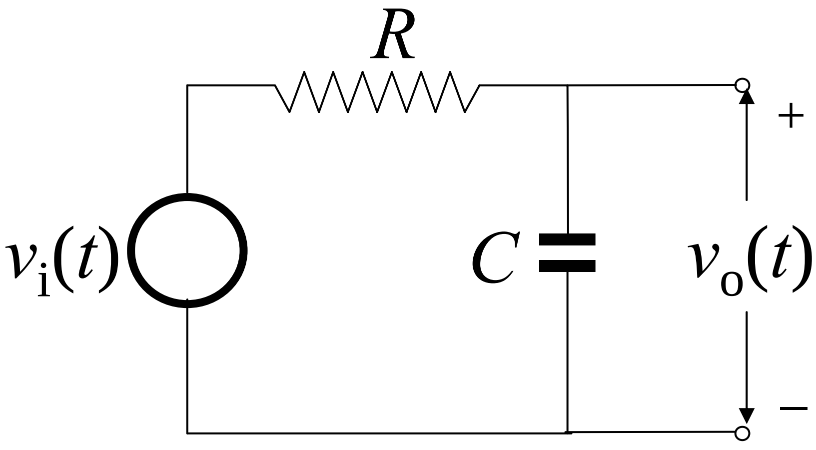


$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$H(\omega) = \frac{1}{-\omega^2 m - i\omega b + k}$$

$$|H(\omega)|^2 = \frac{1}{m^2}\frac{1}{\left(\omega^2 - \omega_{\mathrm{r}}^2\right)^2 + 4\zeta^2\omega_{\mathrm{r}}^2\omega^2}$$

$$\omega_{\mathrm{r}} = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}}$$



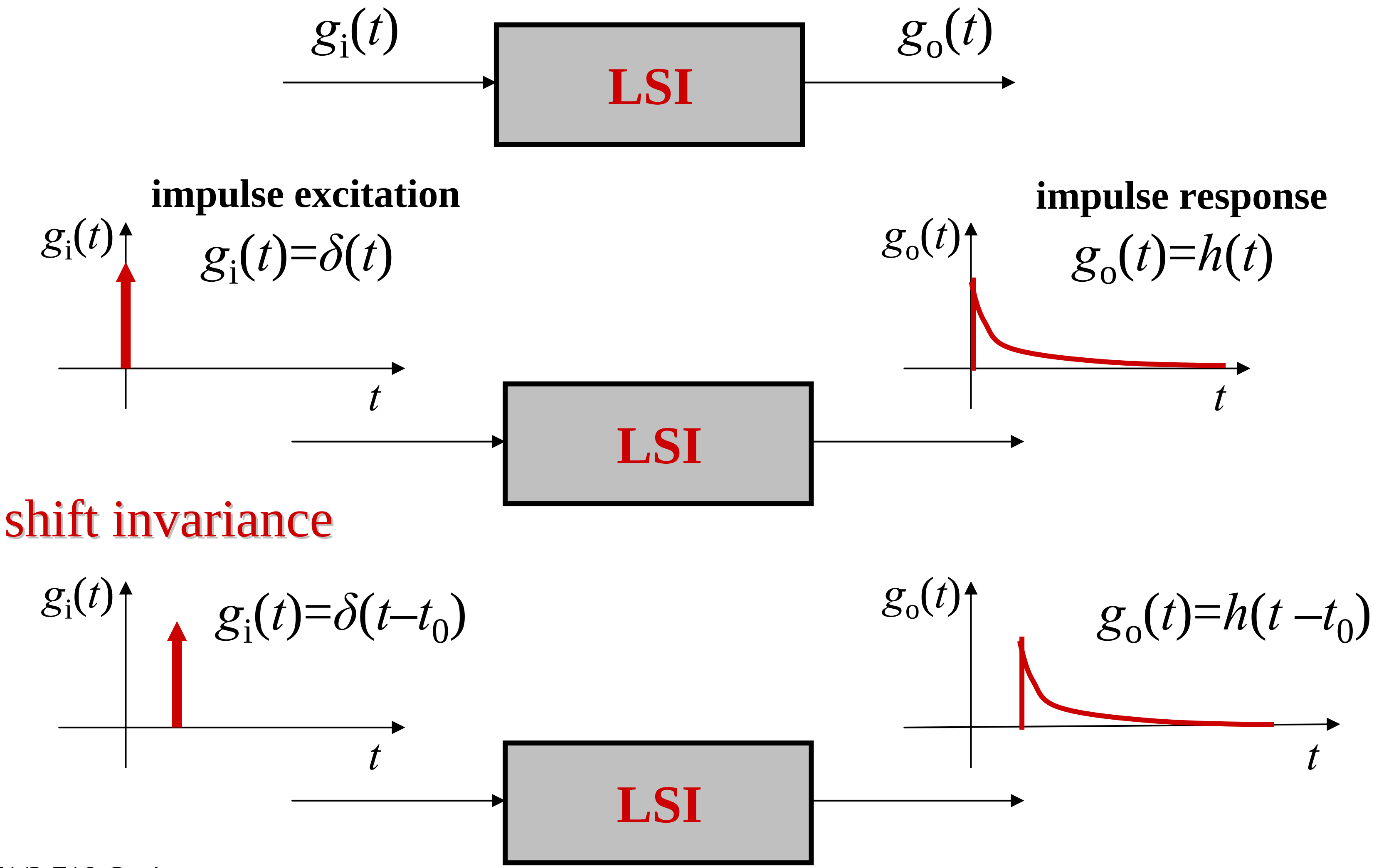
$$RC\dot{v}_{\mathrm{o}} + v_{\mathrm{o}} = v_{\mathrm{i}}(t)$$

$$H(\omega) = \frac{1}{-i\omega\tau + 1}$$

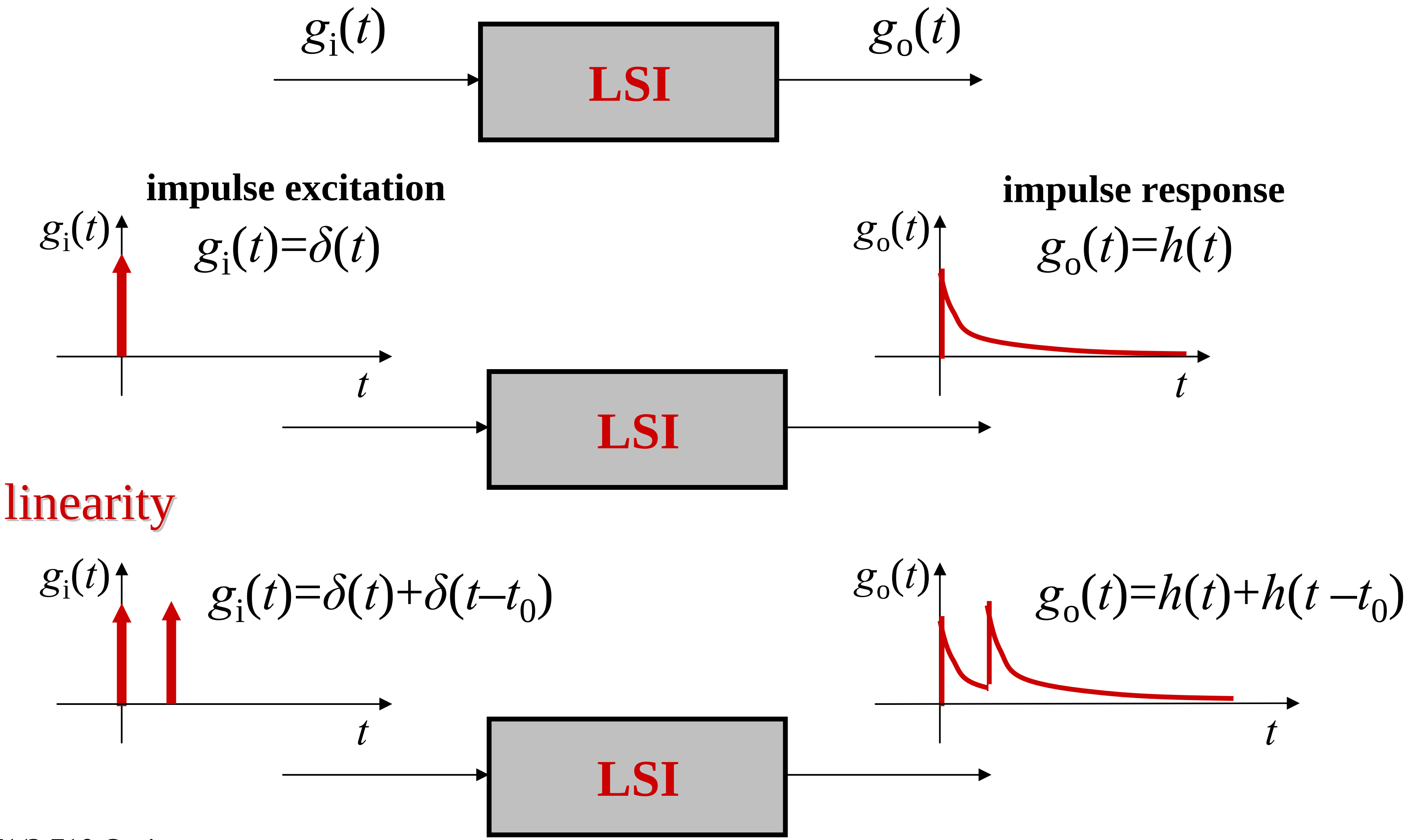
$$|H(\omega)|^2 = \frac{1}{(\omega\tau)^2 + 1}$$

$$\tau = RC$$

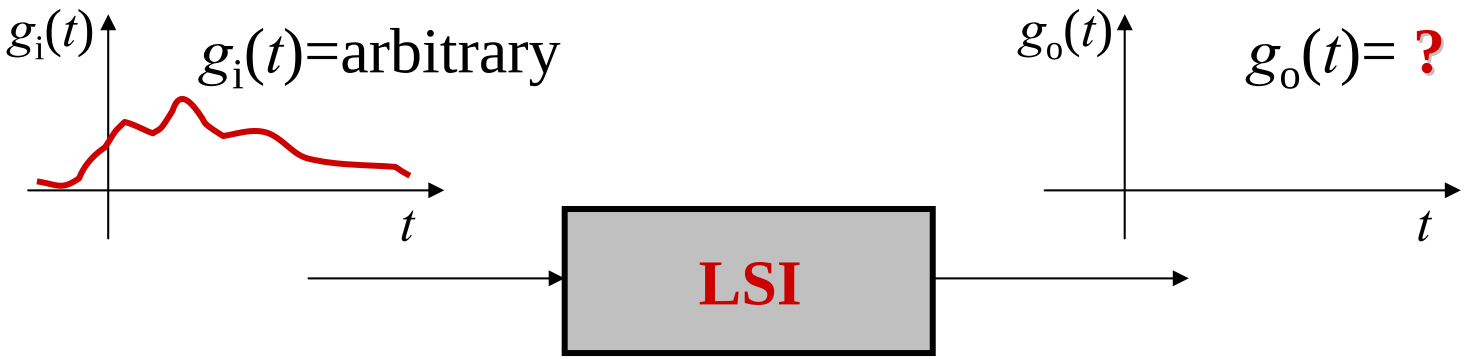
Linear shift invariant systems



Linear shift invariant systems



Linear shift invariant systems

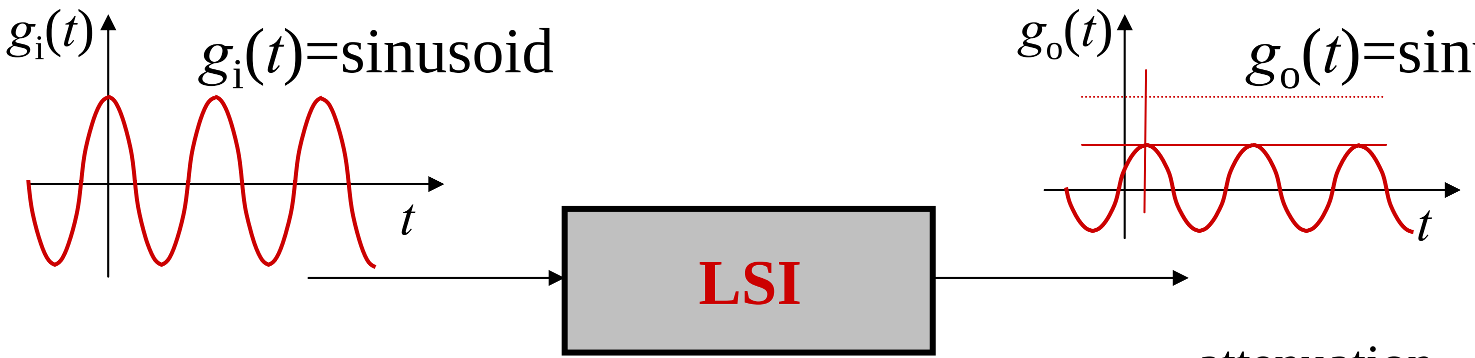


$$g_i(t) = \int g_i(t_0)\,\delta(t - t_0)\,\mathrm{d}t_0 \quad \Rightarrow \quad g_o(t) = \int g_i(t_0)\,h(t - t_0)\,\mathrm{d}t_0$$

sifting property of the δ-function

convolution integral

Linear shift invariant systems



attenuation

phase delay

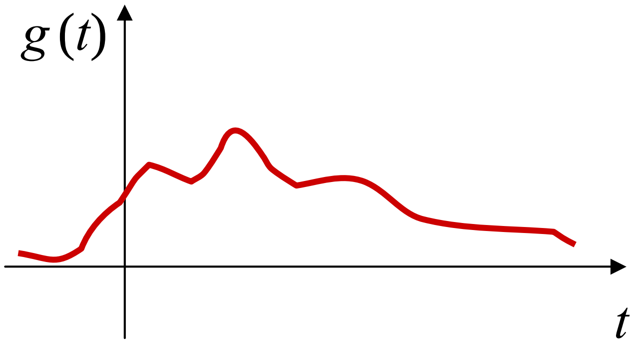
$$g_i(t) = a_0 \cos(\omega_0 t) \Rightarrow g_o(t) = a_0 |H(\omega_0)| \cos(\omega_0 t - \phi(\omega_0))$$

where $H(\omega) = |H(\omega)| \exp[i\phi(\omega)]$

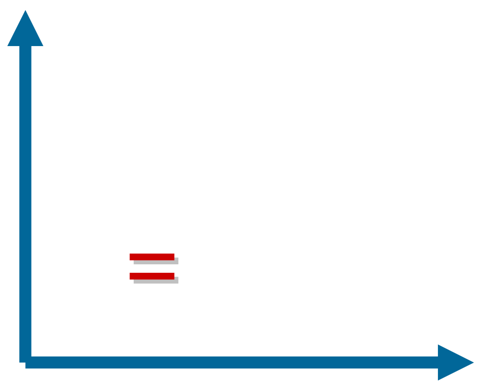
same frequency

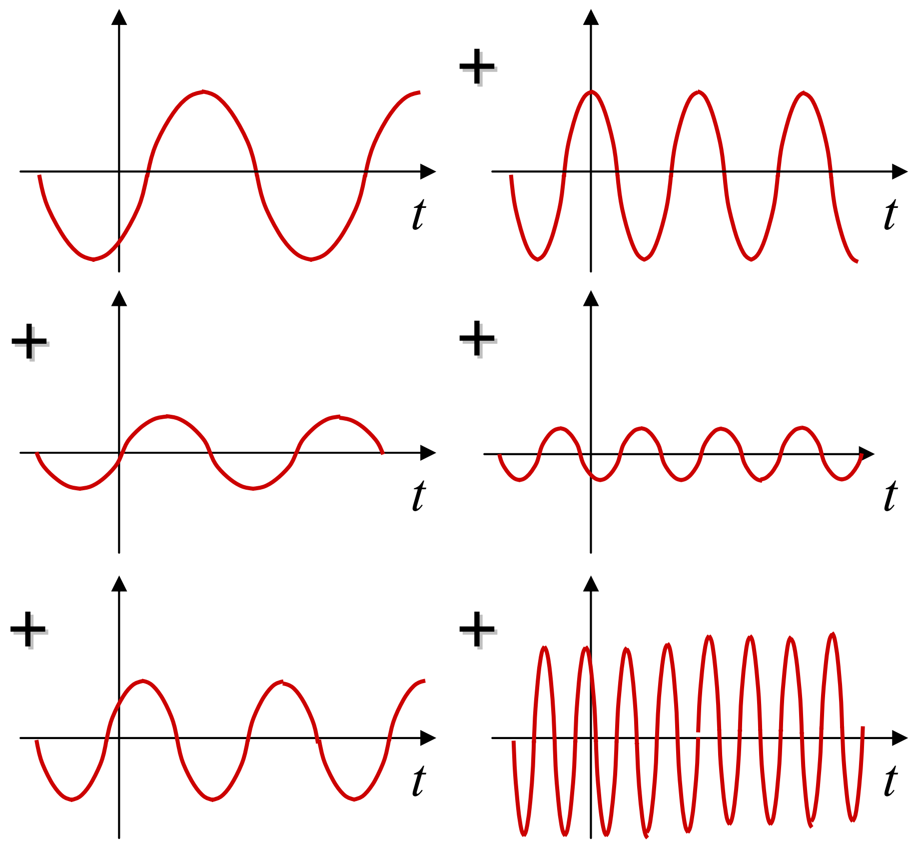
transfer function

From time to frequency: Fourier analysis



Can I express an arbitrary $g(t)$
as a superposition of sinusoids?





The Fourier integral

(aka **inverse Fourier transform**)

$$g(t) = \int G(\nu)\, e^{+i2\pi\nu t}\, d\nu$$

superposition

sinusoids

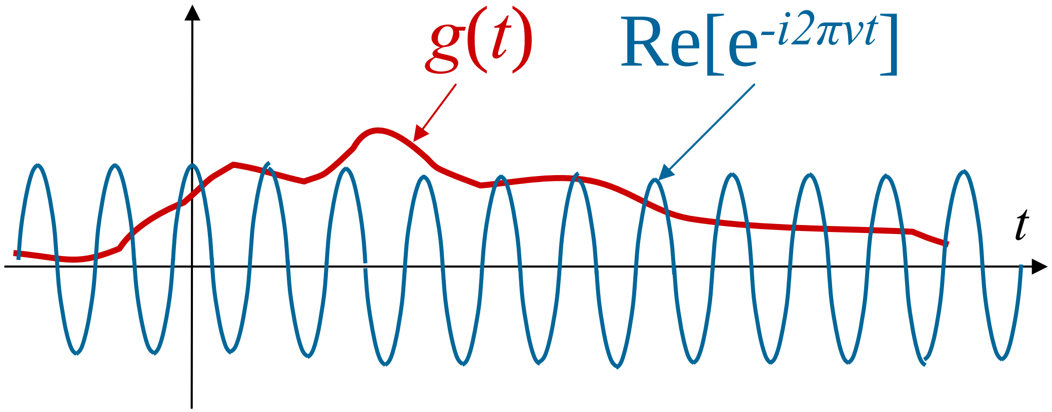
complex weight,
expresses relative amplitude
(magnitude & phase)
of superposed sinusoids

The Fourier transform

The complex weight coefficients $G(\nu)$,
aka **Fourier transform** of $g(t)$
are calculated from the integral

$$G(\nu) = \int g(t)\ \mathrm{e}^{-i2\pi\nu t}\ \mathrm{d}t$$

$$\mathrm{Re}[G(\nu)] = \int$$



$$\mathrm{d}t$$

Fourier transform *pairs*

$$g(t) = \delta(t) \quad \leftrightarrow \quad G(\nu) = 1$$

$$g(t) = \exp(i2\pi\nu_0 t) \quad \leftrightarrow \quad G(\nu) = \delta(\nu - \nu_0)$$

$$g(t) = \cos(2\pi\nu_0 t) \quad \leftrightarrow \quad G(\nu) = \frac{1}{2}\left[\delta(\nu + \nu_0) + \delta(\nu - \nu_0)\right]$$

$$g(t) = \sin(2\pi\nu_0 t) \quad \leftrightarrow \quad G(\nu) = \frac{1}{2i}\left[-\delta(\nu + \nu_0) + \delta(\nu - \nu_0)\right]$$

$$g(t) = \mathrm{rect}\left(\frac{t}{T}\right) \quad \leftrightarrow \quad G(\nu) = T\,\mathrm{sinc}(T\nu)$$

$$g(t) = \exp\left(-\frac{t^2}{2T^2}\right) \quad \leftrightarrow \quad G(\nu) = T\exp\left(-2T^2\nu^2\right)$$

2D Fourier transform *pairs*

Image removed due to copyright concerns

(from Goodman, *Introduction to Fourier Optics*, page 14)