Overview from last week

- Optical systems act as linear shift-invariant (LSI) filters (we have not yet seen why)

- Analysis tool for LSI filters: Fourier transform
  – decompose arbitrary 2D functions into superpositions of 2D sinusoids *(Fourier transform)*
  – use the transfer function to determine what happens to each 2D sinusoid as it is transmitted through the system *(filtering)*
  – recompose the filtered 2D sinusoids to determine the output 2D function *(Fourier integral, aka inverse Fourier transform)*

Today

- Wave description of optical systems
- Diffraction
  - *very* short distances: near field, we skip
  - intermediate distances: Fresnel diffraction

    *expressed as a **convolution***
  - long distances (∞): Fraunhofer diffraction

    *expressed as a **Fourier transform***

Space and spatial frequency representations

SPACE DOMAIN $g(x,y)$

$$G(u,v)=\int g(x,y)\ \mathrm{e}^{-i2\pi(ux+vy)}\,\mathrm{d}x\mathrm{d}y$$

2D Fourier transform

$$g(x,y)=\int G(u,v)\ \mathrm{e}^{+i2\pi(ux+vy)}\,\mathrm{d}u\mathrm{d}v$$

2D Fourier integral
aka
inverse 2D Fourier transform

SPATIAL FREQUENCY DOMAIN $G(u,v)$

2D linear shift invariant systems

input — output

convolution with ***impulse response***

$g_i(x,y)$ → $g_o(x',y')$

$$g_o(x',y') = \int g_i(x,y)\, h(x'-x, y'-y)\, \mathrm{d}x\mathrm{d}y$$

Fourier transform ↓ — ↑ inverse Fourier transform

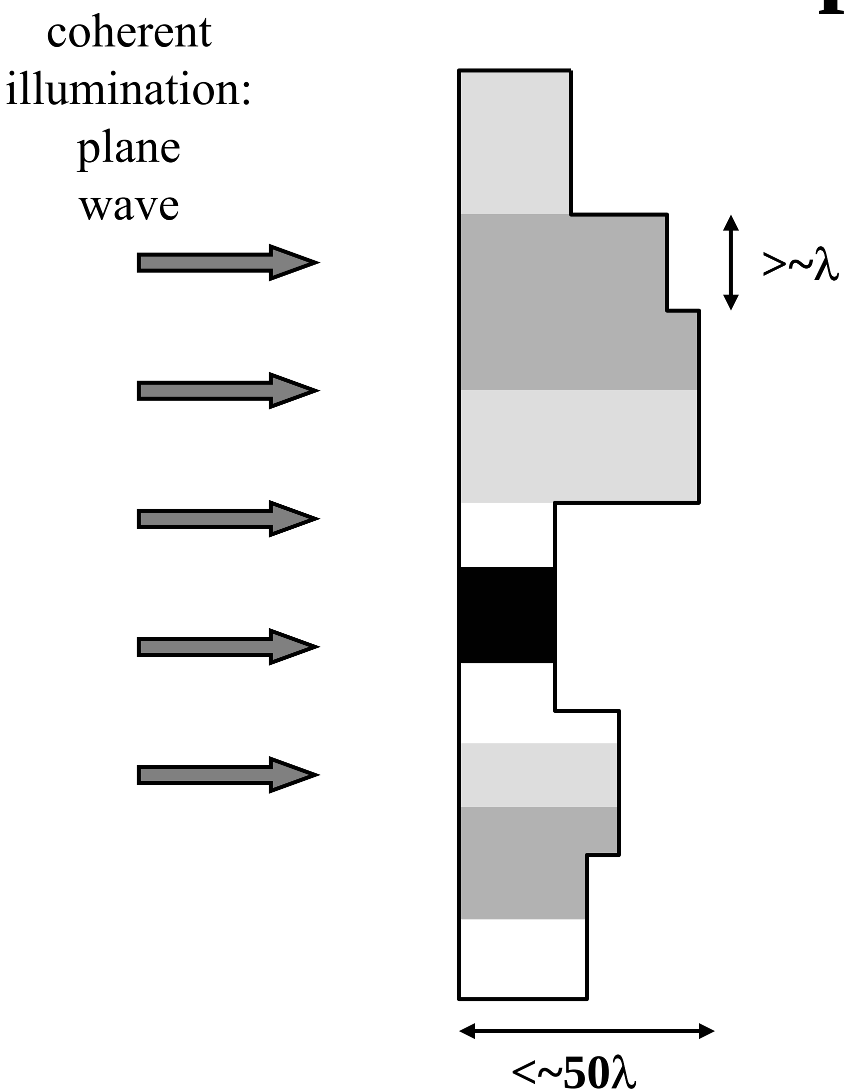
multiplication with ***transfer function***

$G_i(u,v)$ → $G_o(u,v)$

$$G_o(u,v) = G_i(u,v)\, H(u,v)$$

Wave description of optical imaging systems

Thin transparencies



Transmission function:

$$g_{\text{in}}(x,y) = t(x,y)\exp\left\{ i\phi(x,y) \right\}$$

Field before transparency:

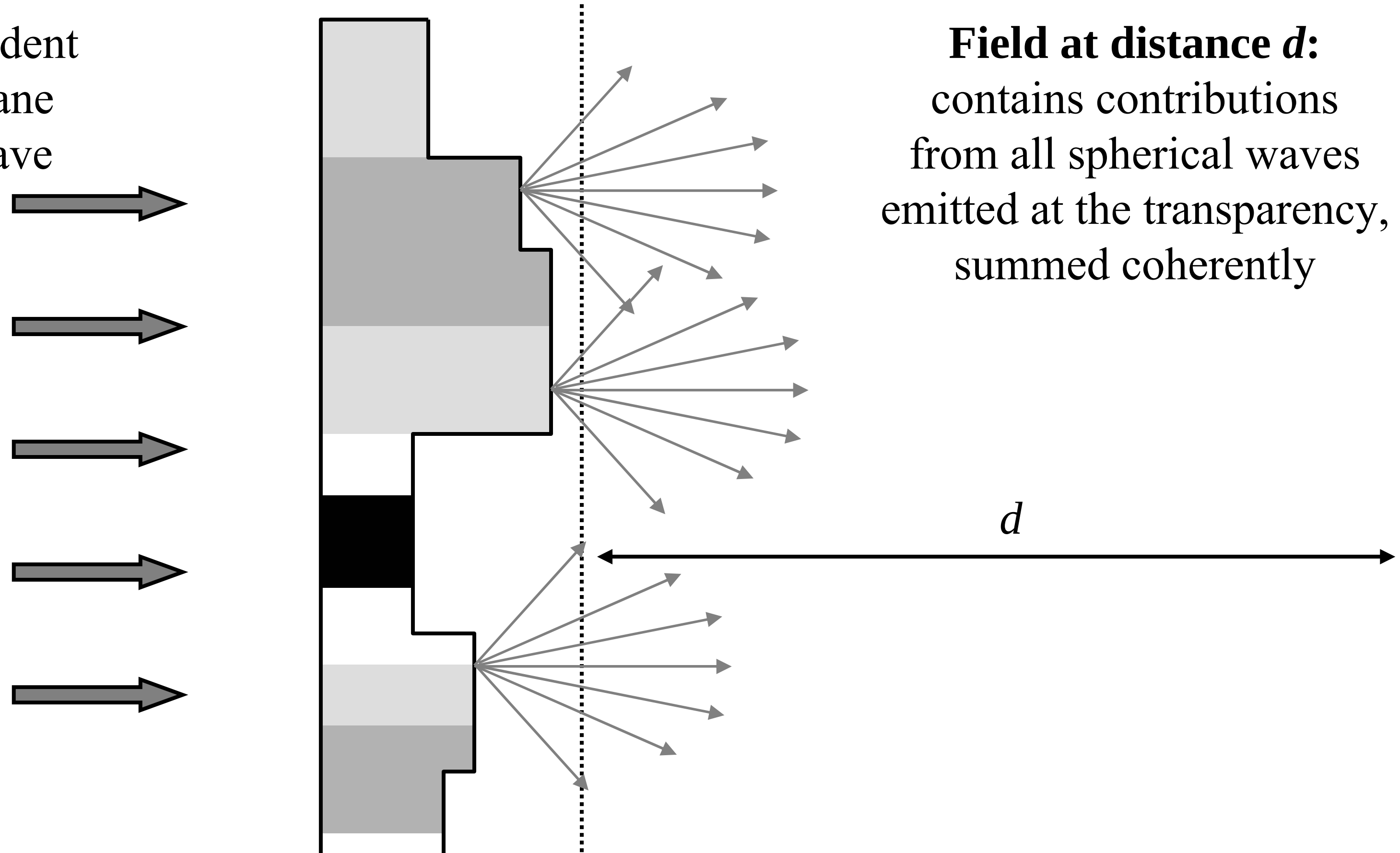
$$a_{-}(x,y,\overset{=0}{\cancel{z}}) = \exp\cancel{\left\{ i2\pi\frac{z}{\lambda} \right\}}$$

Field after transparency:

$$a_{+}(x,y,\overset{=0}{\cancel{z}}) = g_{\text{in}}(x,y)\exp\cancel{\left\{ i2\pi\frac{z}{\lambda} \right\}}$$

assumptions: transparency at z=0
transparency thickness can be ignored

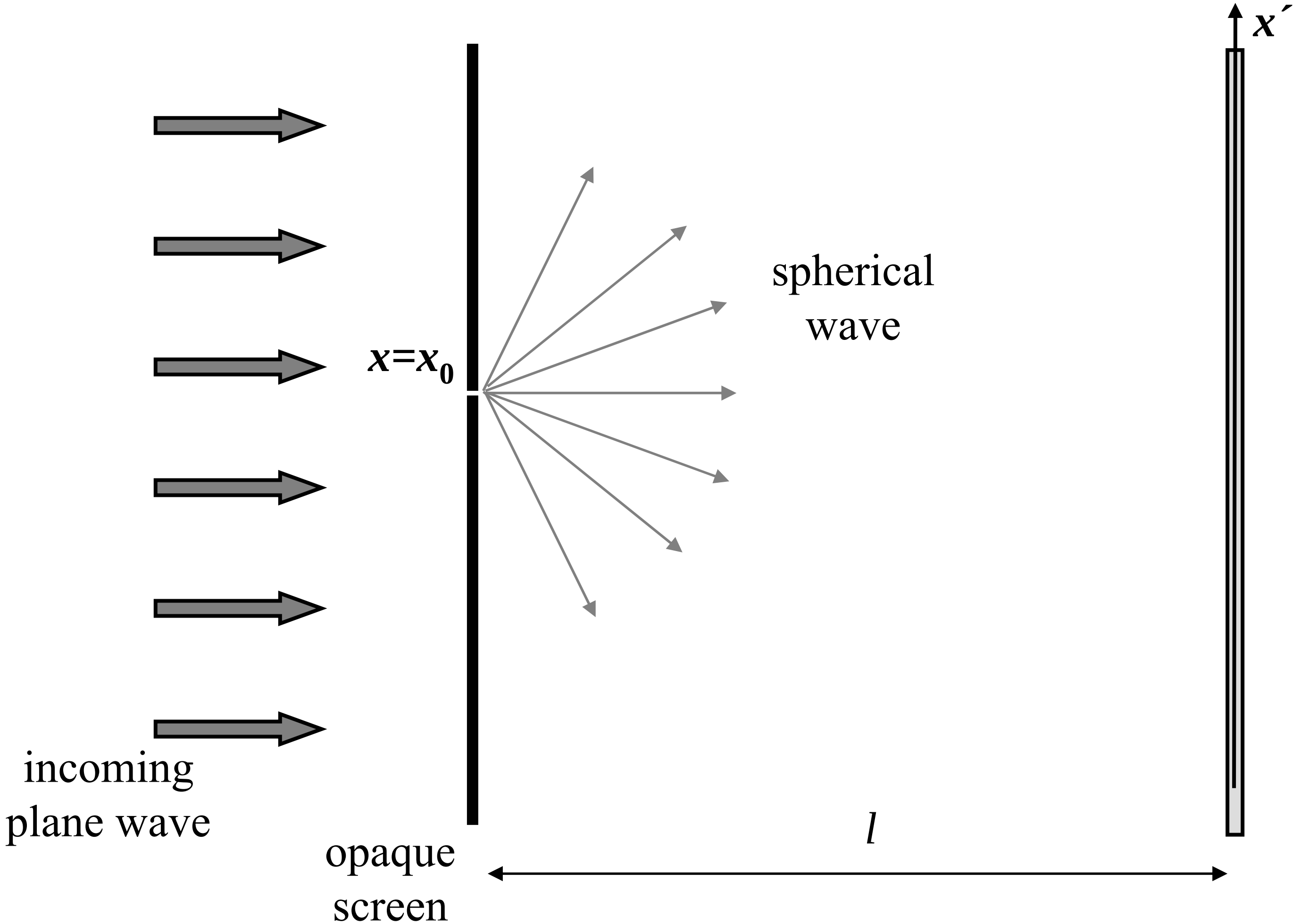
Diffraction: Huygens principle



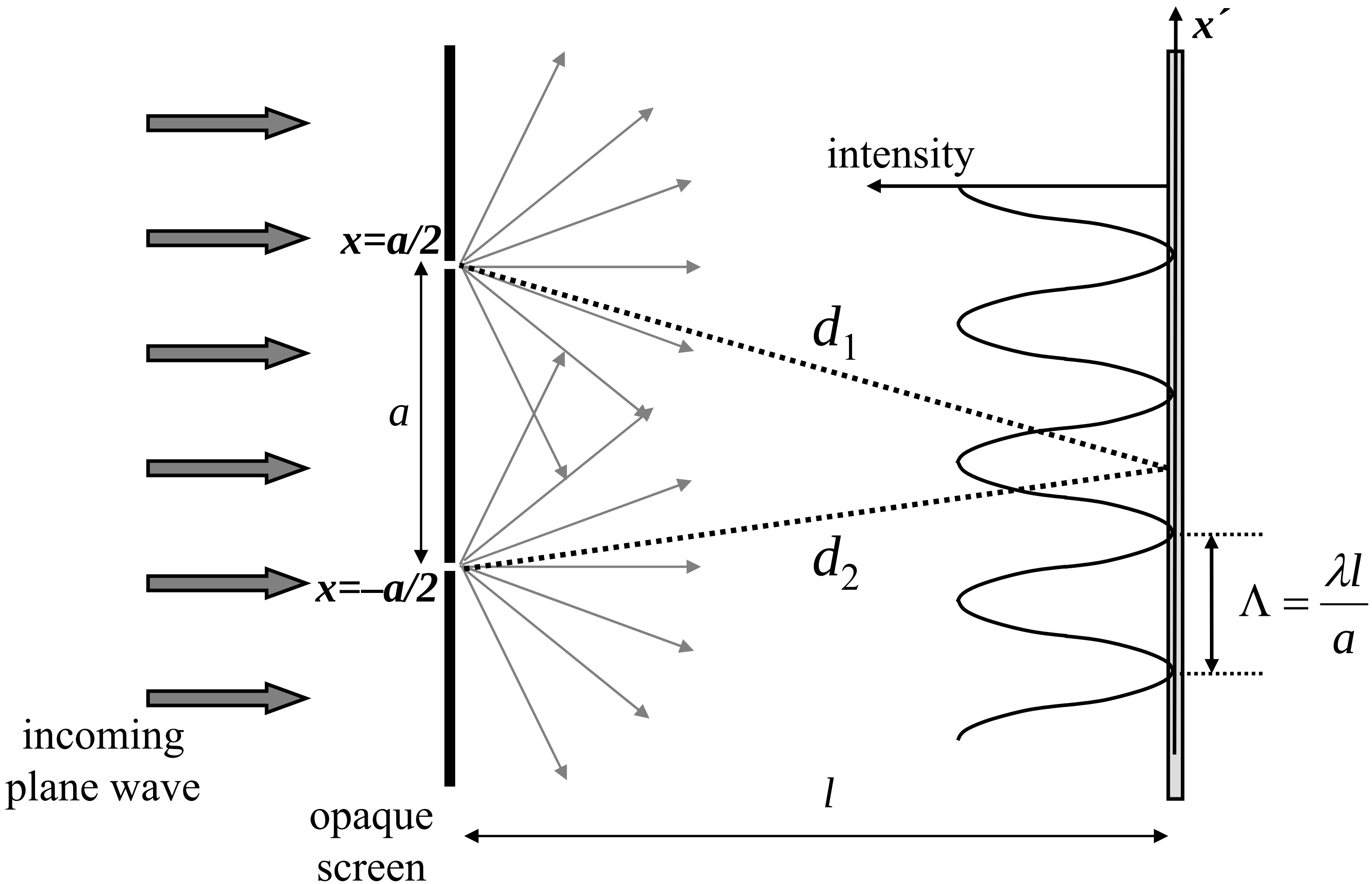
Field after transparency:

$$a_{+}(x, y, z) = g_{\text{in}}(x, y)$$

Huygens principle: one point source



Simple interference: two point sources



Two point sources interfering: math…

(paraxial approximation)

Amplitude:

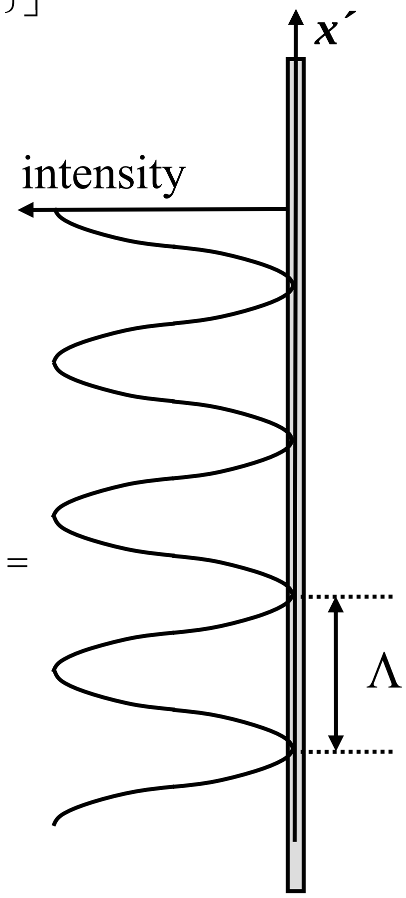
$$e(x',y') = -\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}\right\}\left[\exp\left\{i\pi\frac{(x'-a/2)^2+y'^2}{\lambda l}\right\}+\exp\left\{i\pi\frac{(x'+a/2)^2+y'^2}{\lambda l}\right\}\right]=$$

$$=\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}+i\pi\frac{x'^2+\frac{a^2}{4}+y'^2}{\lambda l}\right\}\left(\exp\left\{-i2\pi\frac{ax'}{2\lambda l}\right\}+\exp\left\{i2\pi\frac{ax'}{2\lambda l}\right\}\right)$$

$$=\frac{2}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}+i\pi\frac{x'^2+\frac{a^2}{4}+y'^2}{\lambda l}\right\}\cos\left(\pi\frac{ax'}{\lambda l}\right).$$

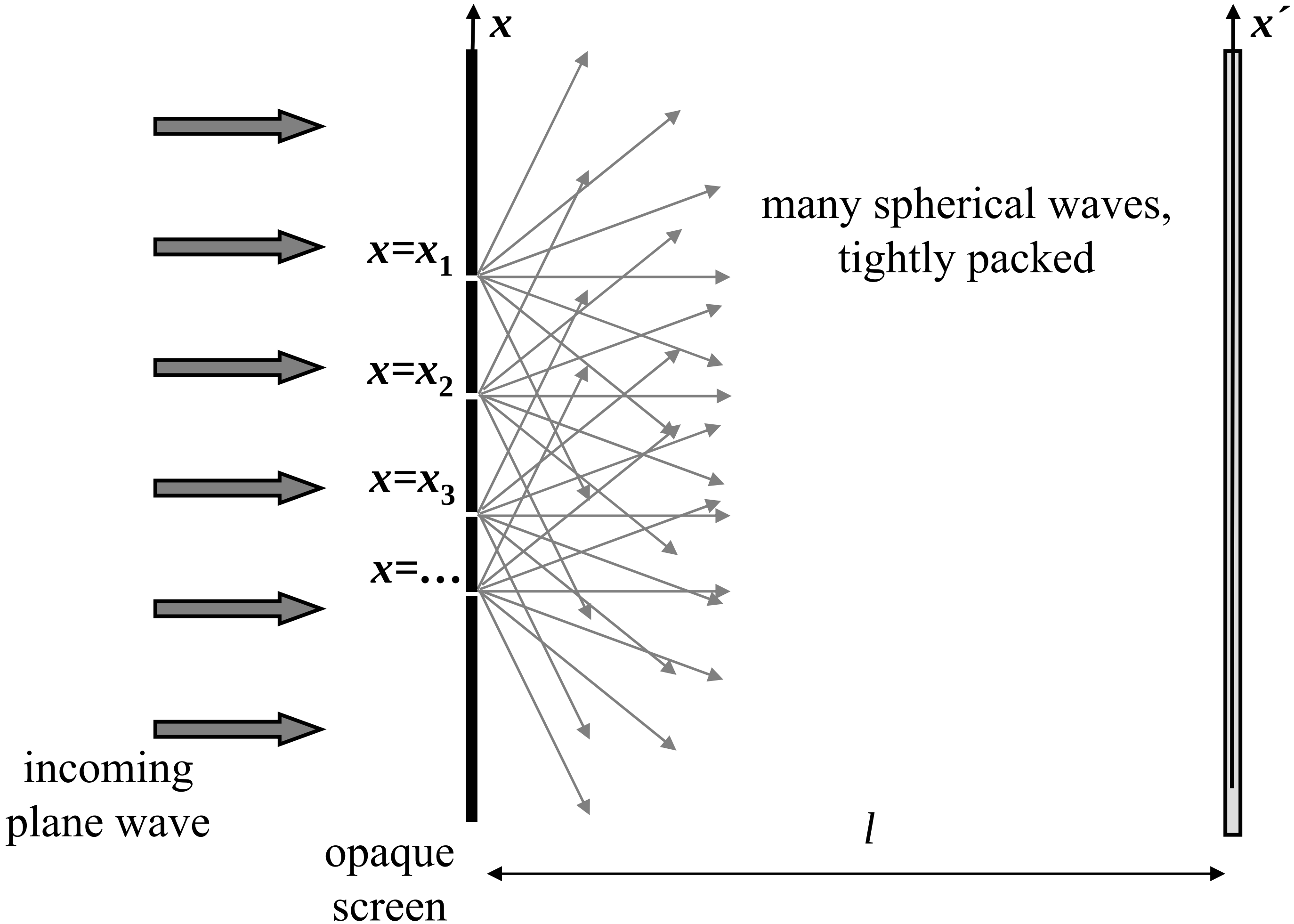
$$\text{Intensity}: I(x',y') = |e(x',y')|^2 = \left|\frac{2}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}+i\pi\frac{x'^2+\frac{a^2}{4}+y'^2}{\lambda l}\right\}\cos\left(\pi\frac{ax'}{\lambda l}\right)\right|^2 =$$

$$=\frac{4}{(\lambda l)^2}\cos^2\left(\pi\frac{ax'}{\lambda l}\right)=\frac{2}{(\lambda l)^2}\left(1+\cos\left\{2\pi\left(\frac{a}{\lambda l}\right)x'\right\}\right).$$

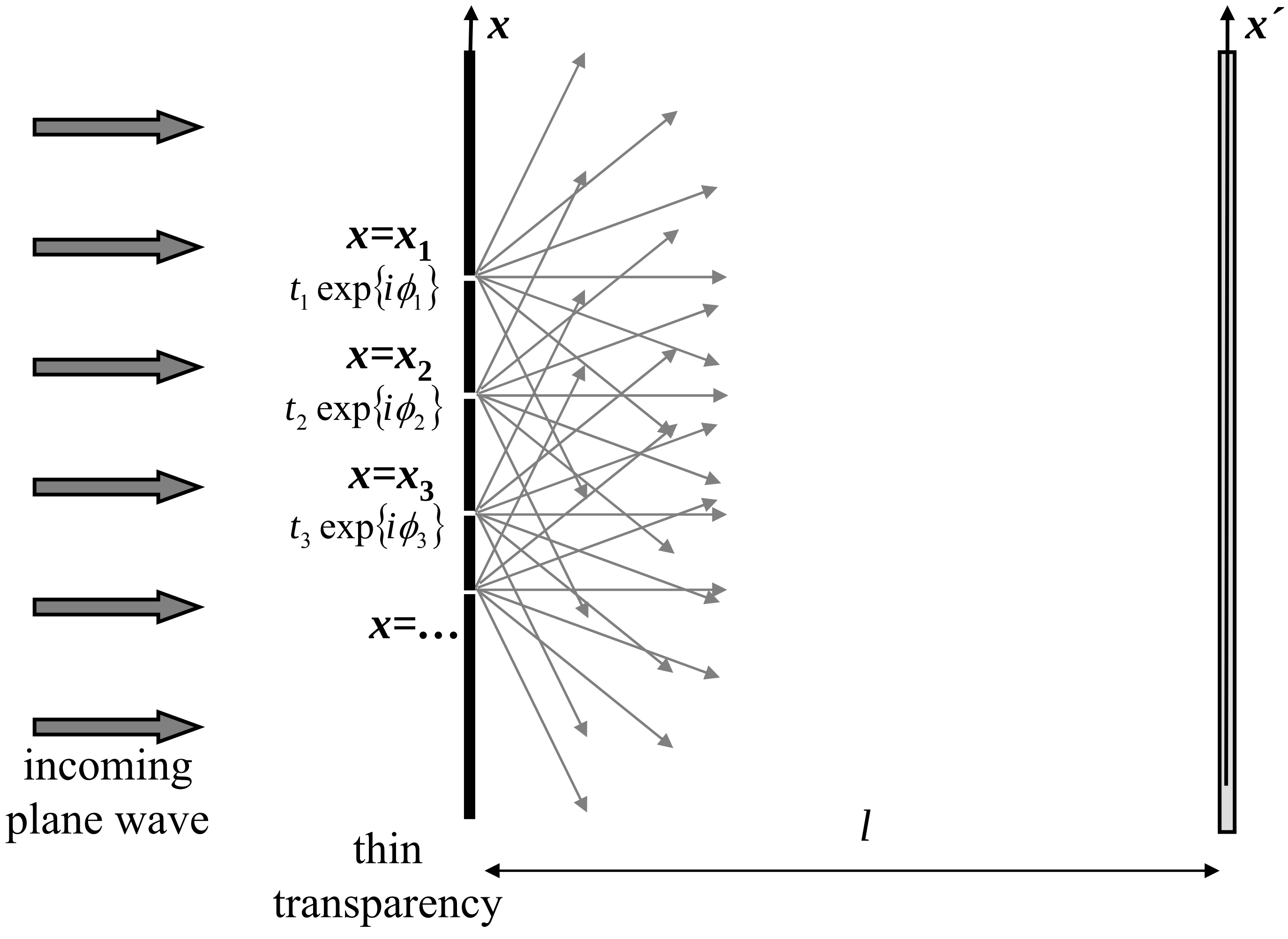


$$\Lambda = \frac{\lambda l}{a}$$

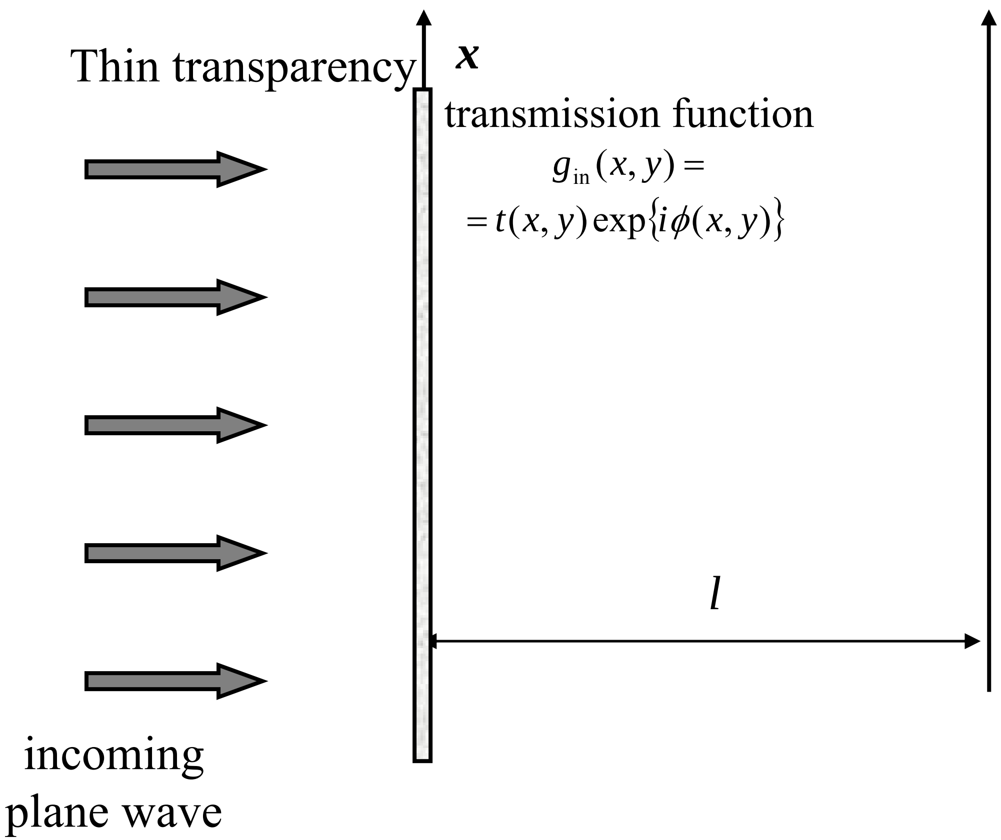
Diffraction: many point sources



Diffraction: many point sources, attenuated & phase-delayed



Diffraction: many point sources attenuated & phase-delayed, math



$$\text{field}(x') = \begin{pmatrix}\text{spherical}\\ \text{wave}\\ \text{from } x_1\end{pmatrix} + \begin{pmatrix}\text{spherical}\\ \text{wave}\\ \text{from } x_2\end{pmatrix} + \begin{pmatrix}\text{spherical}\\ \text{wave}\\ \text{from } x_3\end{pmatrix} + \ldots =$$

$$= \frac{t_1}{i\lambda l}\exp\left\{ i\phi_1 + i2\pi\frac{l}{\lambda} + i\pi\frac{(x'-x_1)^2 + y'^2}{\lambda l}\right\} +$$

$$\frac{t_2}{i\lambda l}\exp\left\{ i\phi_2 + i2\pi\frac{l}{\lambda} + i\pi\frac{(x'-x_2)^2 + y'^2}{\lambda l}\right\} +$$

$$\frac{t_3}{i\lambda l}\exp\left\{ i\phi_3 + i2\pi\frac{l}{\lambda} + i\pi\frac{(x'-x_3)^2 + y'^2}{\lambda l}\right\}\ldots$$

continuous limit

$$\text{field}(x', y') = \frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}\right\}\iint \underbrace{t(x,y)\exp\{i\phi(x,y)\}}_{g_{\text{in}}(x,y)}\exp\left\{i\pi\frac{(x'-x)^2 + (y'-y)^2}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y.$$

Fresnel diffraction

$$g_{\text{out}}(x', y') = \frac{1}{i\lambda l} \exp\left\{i2\pi \frac{l}{\lambda}\right\} \iint g_{\text{in}}(x, y) \exp\left\{i\pi \frac{(x'-x)^2 + (y'-y)^2}{\lambda l}\right\} \mathrm{d}x\mathrm{d}y.$$

The diffracted field is the *convolution* of the transparency with a spherical wave

$$\underbrace{g_{\text{out}}(x', y')}_{} = \underbrace{g_{\text{in}}(x, y)}_{} * \underbrace{\left[\frac{1}{i\lambda l} \exp\left\{i2\pi \frac{l}{\lambda}\right\} \underbrace{\exp\left\{i\pi \frac{x^2 + y^2}{\lambda l}\right\}}_{}\right]}_{}$$

CONSTANT: ***NOT*** interesting

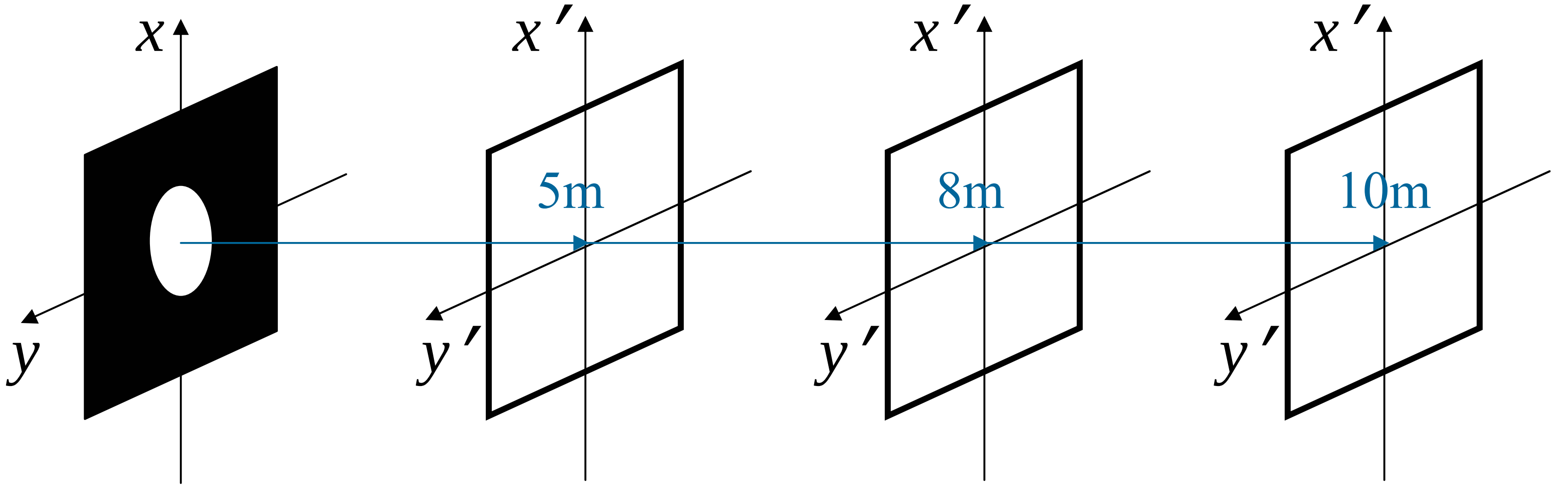
FUNCTION OF LATERAL COORDINATES: ***Interesting!!!***

amplitude distribution at output plane

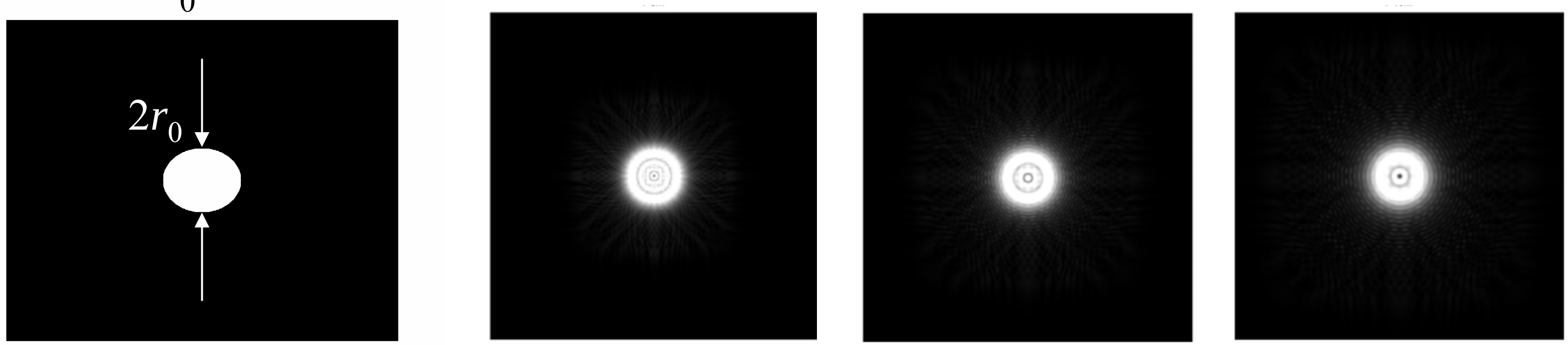
transparency transmission function (complex $te^{i\phi}$)

spherical wave @$z=l$ (aka Green's function)

Example: circular aperture

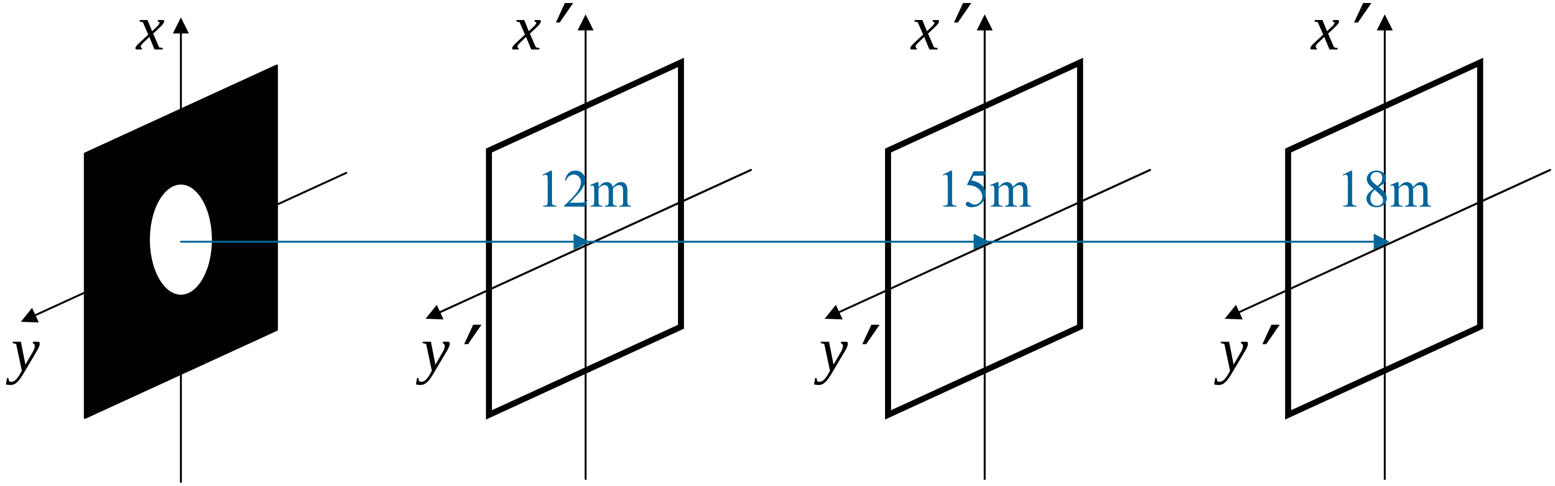


r_0=10mm

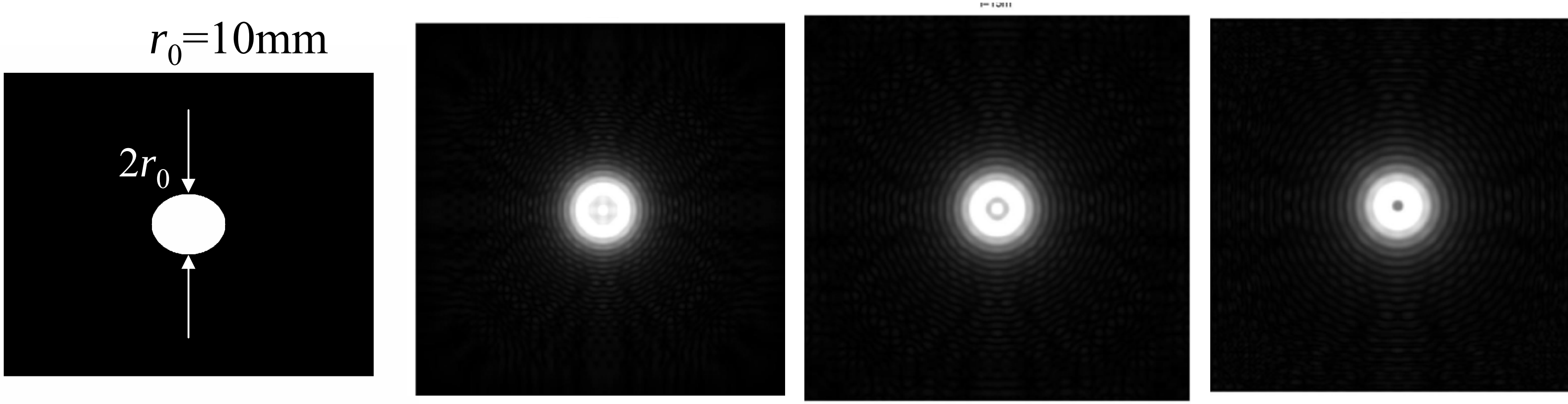


input field $g_{in}(x,y)$ $g_{out}(x,y;5m)$ $g_{out}(x,y;8m)$ $g_{out}(x,y;10m)$

Example: circular aperture

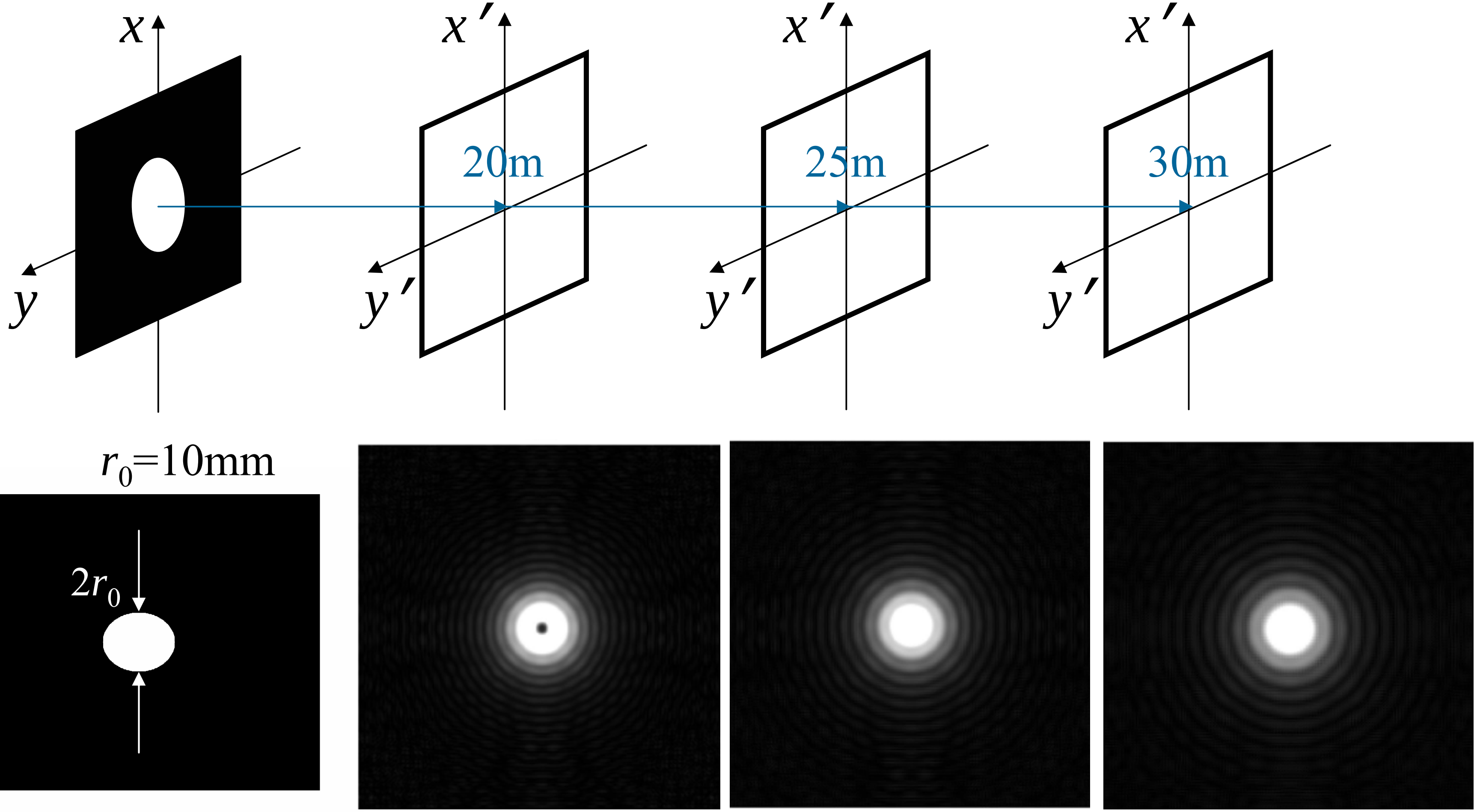


r_0=10mm



input field $g_{in}(x,y)$ | $g_{out}(x,y;12m)$ | $g_{out}(x,y;15m)$ | $g_{out}(x,y;18m)$

Example: circular aperture



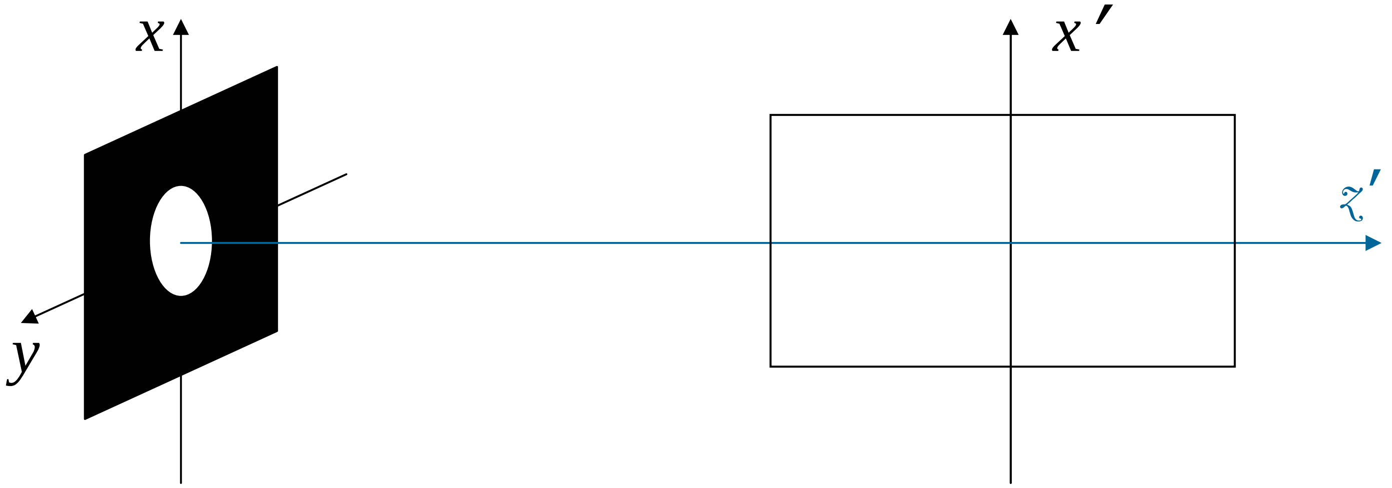
input field $g_{in}(x,y)$

$g_{out}(x,y;20\text{m})$

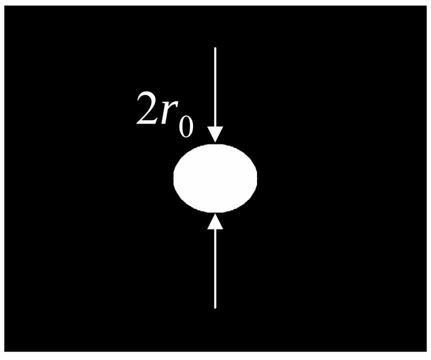
$g_{out}(x,y;25\text{m})$

$g_{out}(x,y;30\text{m})$

Example: circular aperture



r_0=??



input field $g_{in}(x,y)$

Image removed due to copyright concerns

(from Hecht, *Optics*, 4^{th} edition, page 494)

Fraunhofer diffraction

propagation distance l is "very large"

$$g_{\text{out}}(x',y')=\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}\right\}\iint g_{\text{in}}(x,y)\exp\left\{i\pi\frac{(x'-x)^2+(y'-y)^2}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y,$$

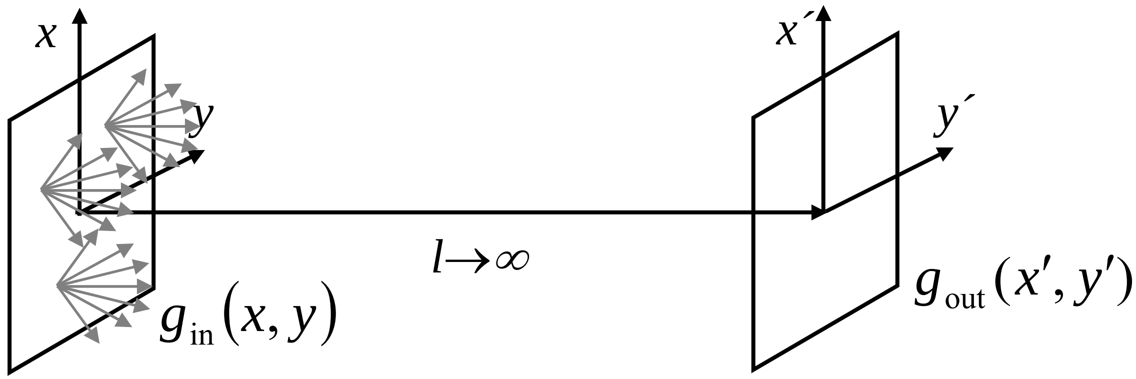
$$g_{\text{out}}(x',y')=\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}\right\}\iint g_{\text{in}}(x,y)\exp\left\{i\pi\frac{(x')^2+x^2-2xx'+(y')^2+y^2-2yy'}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y,$$

$$g_{\text{out}}(x',y')=\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}+i\pi\frac{(x')^2+(y')^2}{\lambda l}\right\}\iint g_{\text{in}}(x,y)\exp\left\{i\pi\frac{x^2+y^2}{\lambda l}\right\}\exp\left\{-i2\pi\frac{xx'+yy'}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y,$$

$$g_{\text{out}}(x',y')\approx\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}+i\pi\frac{(x')^2+(y')^2}{\lambda l}\right\}\iint g_{\text{in}}(x,y)\exp\left\{-i2\pi\frac{xx'+yy'}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y$$

approximation valid if $x^2+y^2<<\lambda l \Leftrightarrow l>>\frac{(x^2+y^2)_{\max}}{\lambda}$

Fraunhofer diffraction

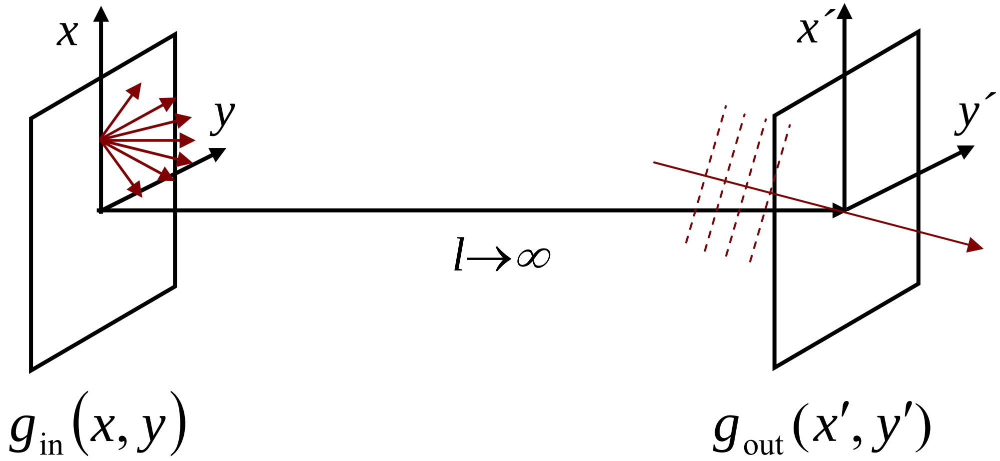


$$g_{\text{out}}(x', y'; l) \propto \int g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda l}\right) + y\left(\frac{y'}{\lambda l}\right) \right] \right\} \mathrm{d}x\mathrm{d}y$$

The **"far-field"** (i.e. the diffraction pattern at a large longitudinal distance *l* equals the **Fourier transform** of the original transparency calculated at spatial frequencies

$$f_x = \frac{x'}{\lambda l} \qquad f_y = \frac{y'}{\lambda l}$$

Fraunhofer diffraction

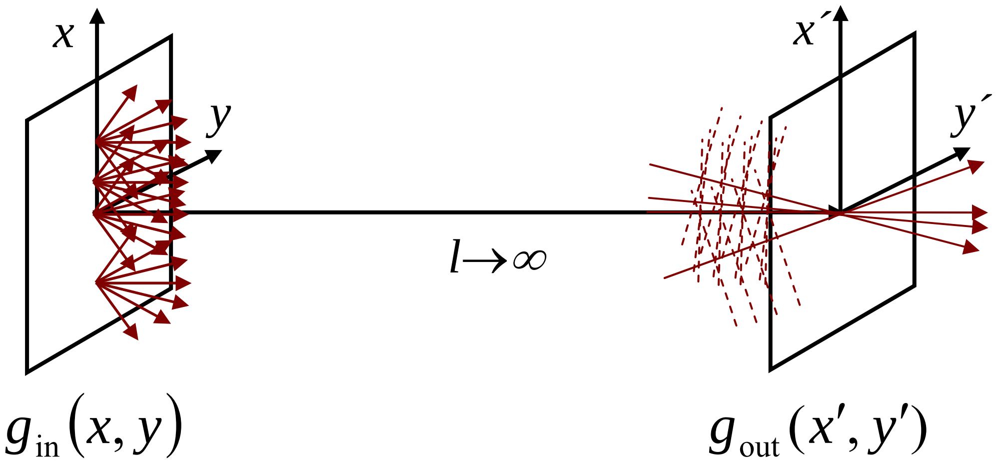


spherical wave originating at x

$l \to \infty$

plane wave propagating at angle $-x/l$
$\Leftrightarrow$ spatial frequency $-x/(\lambda l)$

Fraunhofer diffraction



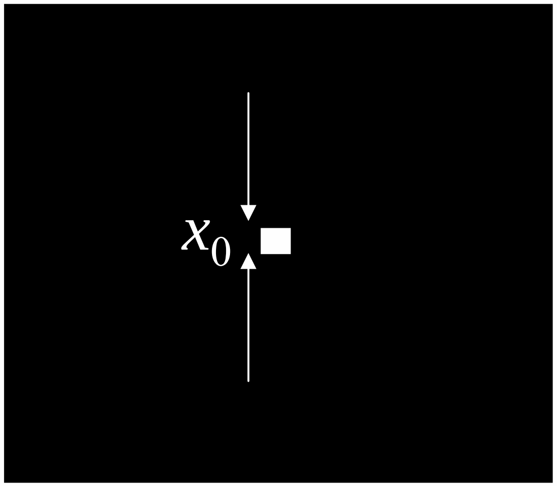
superposition of spherical waves originating at various points along x

$l \to \infty$ ⟶

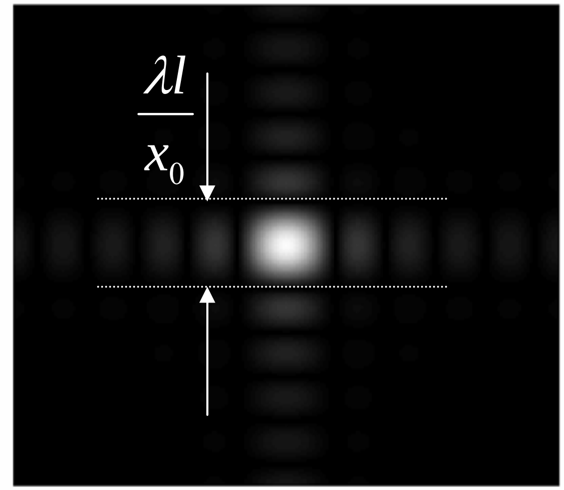
superposition of plane waves propagating at corresponding angles $-x/l$
⇔ spatial frequencies $-x/(\lambda l)$

$$g_{\text{out}}(x',y';l) \propto \int g_{\text{in}}(x,y) \exp\left\{-i2\pi\left[x\left(\frac{x'}{\lambda l}\right)+y\left(\frac{y'}{\lambda l}\right)\right]\right\} \mathrm{d}x\mathrm{d}y$$

Example: rectangular aperture



free space propagation by $l \to \infty$



input field

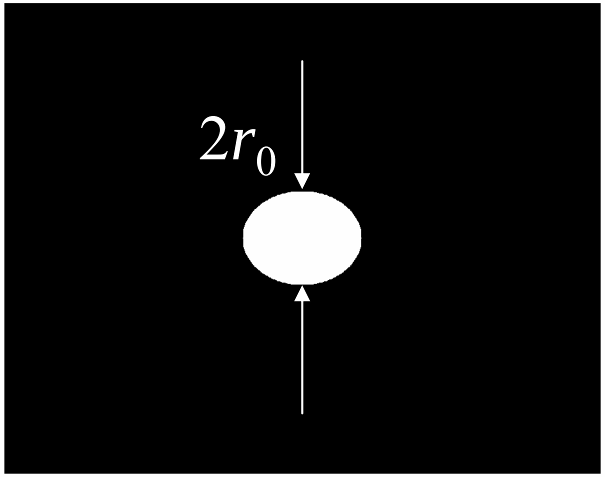
$$g_{\text{in}}(x, y) = \text{rect}\left(\frac{x}{x_0}\right)\text{rect}\left(\frac{y}{y_0}\right)$$

far field

$$g_{\text{out}}(x, y) = \frac{e^{i2\pi l/\lambda}}{i\lambda l}\exp\left\{\frac{(x')^2 + (y')^2}{\lambda l}\right\}$$

$$\times x_0 y_0 \,\text{sinc}\left(\frac{x_0 x'}{\lambda l}\right)\text{sinc}\left(\frac{y_0 y'}{\lambda l}\right)$$

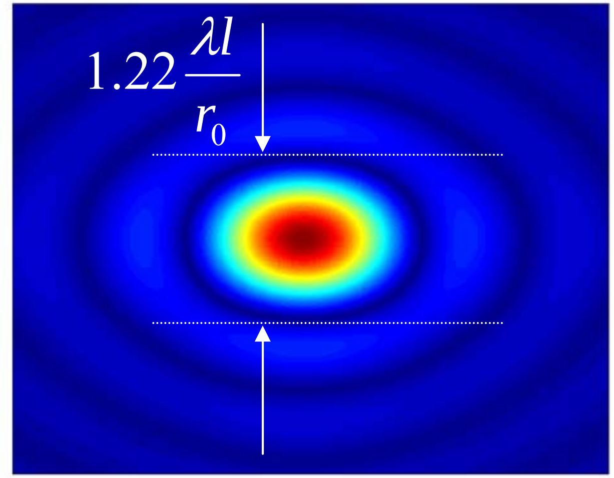
Example: circular aperture



input field

free space propagation by

$l \to \infty$



far field

$$g_{\text{in}}(x,y) = \operatorname{circ}\left(\frac{\sqrt{x^2+y^2}}{r_0}\right)$$

$$g_{\text{out}}(x,y) = \frac{e^{i2\pi l/\lambda}}{i\lambda l}\exp\left\{\frac{(x')^2+(y')^2}{\lambda l}\right\} \times \pi r_0^2 \left[2\frac{J_1\left(\frac{2\pi r_0\sqrt{(x')^2+(y')^2}}{\lambda l}\right)}{\frac{2\pi r_0\sqrt{(x')^2+(y')^2}}{\lambda l}} \right]$$

also known as **Airy pattern**, or

$$\operatorname{jinc}\left(\frac{2\pi r_0\sqrt{(x')^2+(y')^2}}{\lambda l}\right)$$