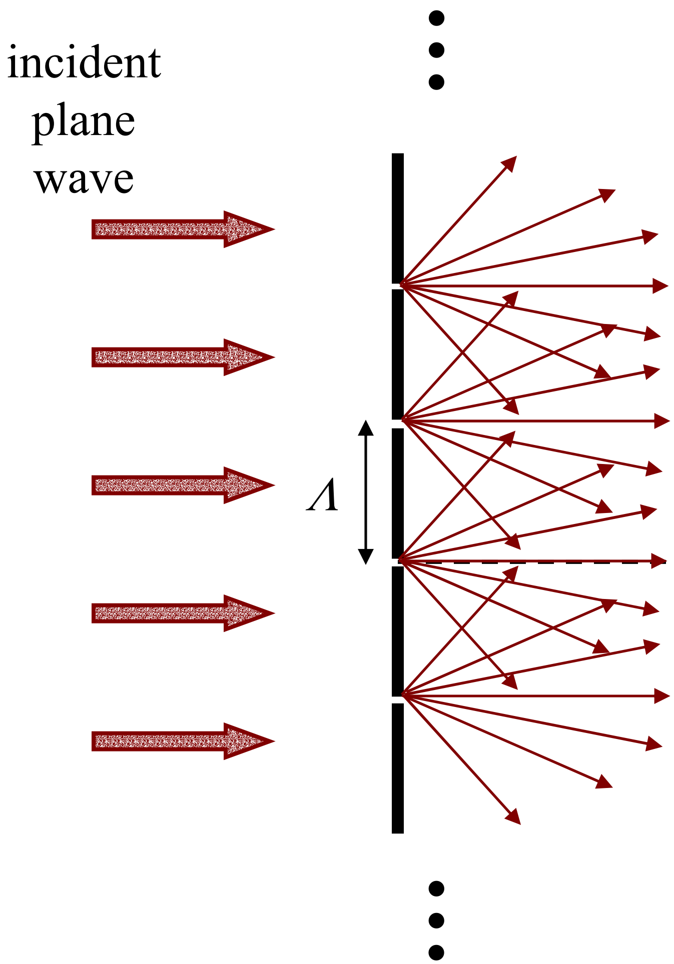
Today

- Diffraction from periodic transparencies: ***gratings***
- Grating dispersion

- Wave optics description of a lens: quadratic phase delay
- Lens as Fourier transform engine

Diffraction from periodic array of holes

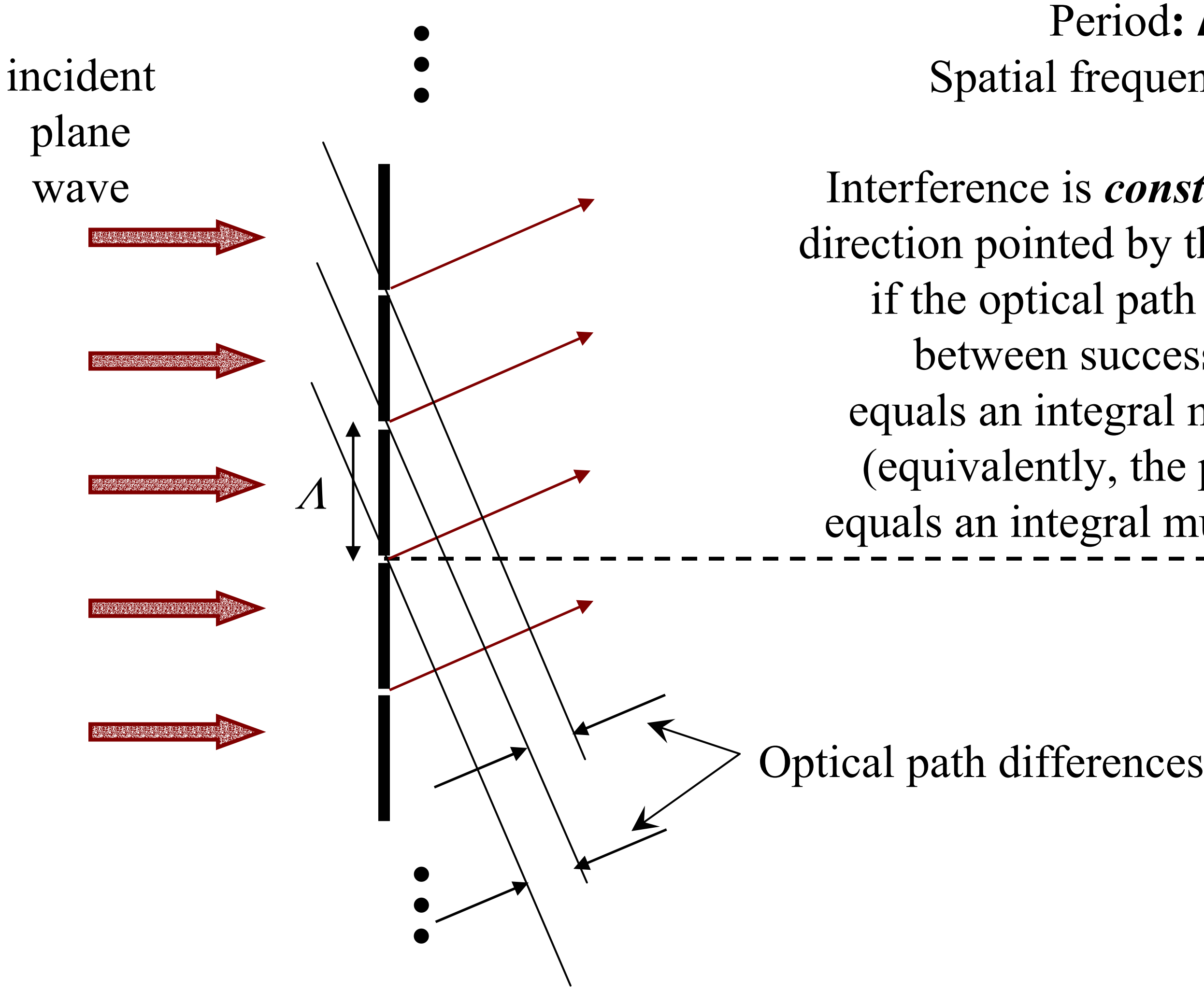


Period: $\boldsymbol{\Lambda}$

Spatial frequency: $\mathbf{1}/\Lambda$

A spherical wave is generated at each hole; we need to figure out how the periodically-spaced spherical waves ***interfere***

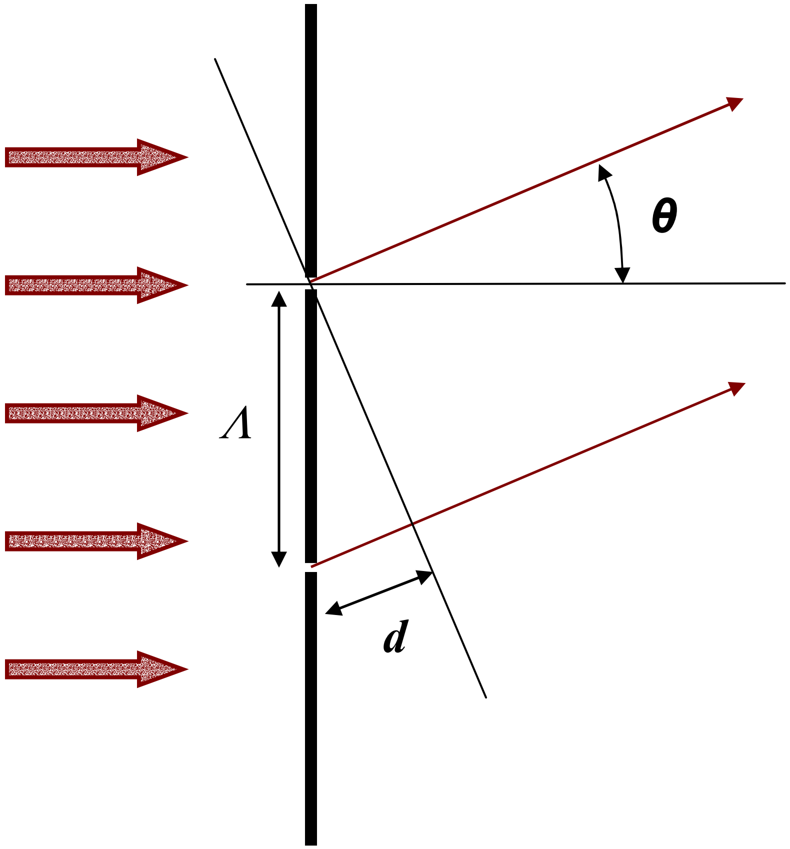
Diffraction from periodic array of holes



Period: **Λ**
Spatial frequency: **1**/Λ

Interference is ***constructive*** in the direction pointed by the parallel rays if the optical path difference between successive rays equals an integral multiple of **λ** (equivalently, the phase delay equals an integral multiple of **2π**)

Diffraction from periodic array of holes



Period: **Λ**
Spatial frequency: **1/Λ**

From the geometry
we find

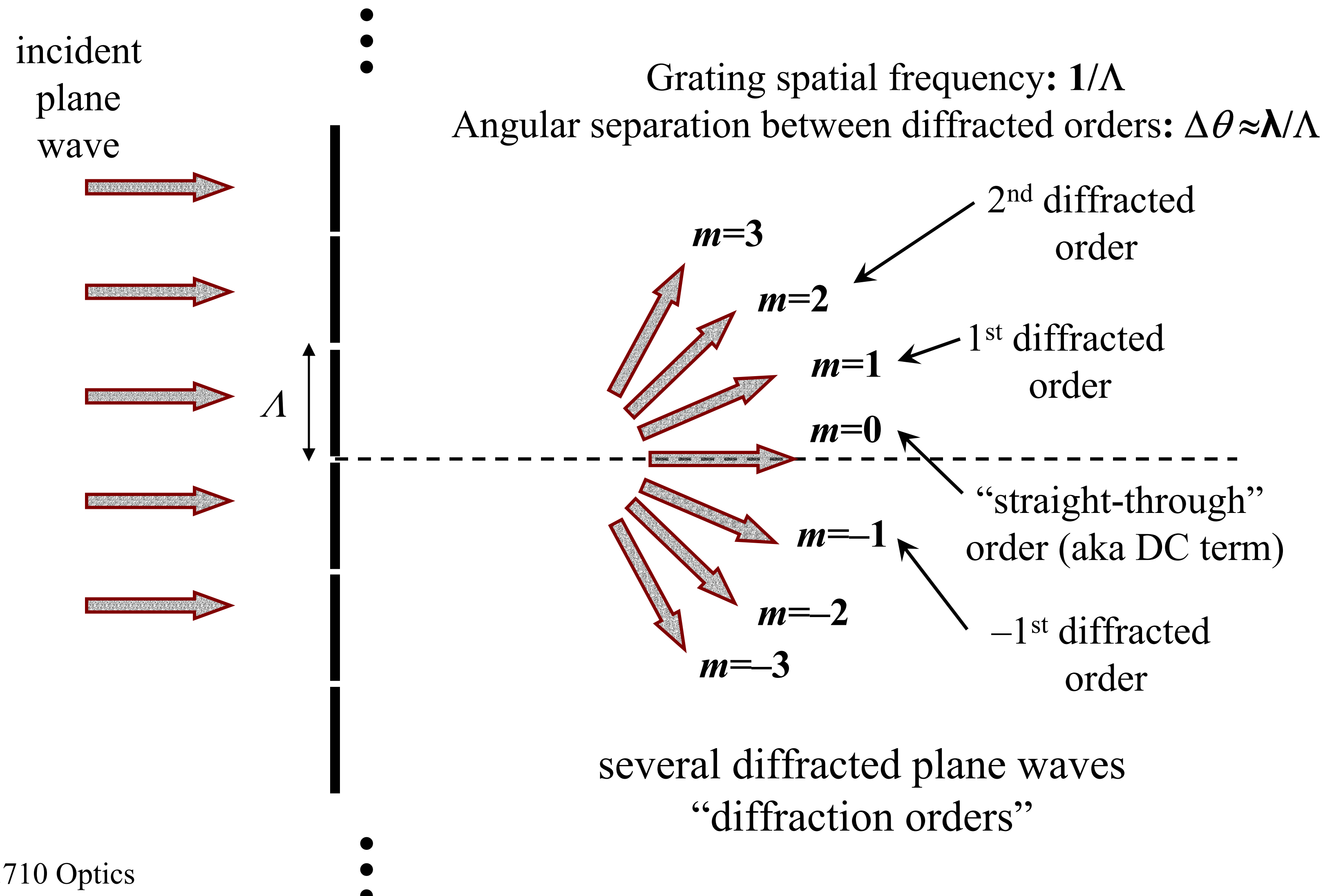
$$d = \Lambda \sin\theta$$

Therefore, interference is
constructive iff

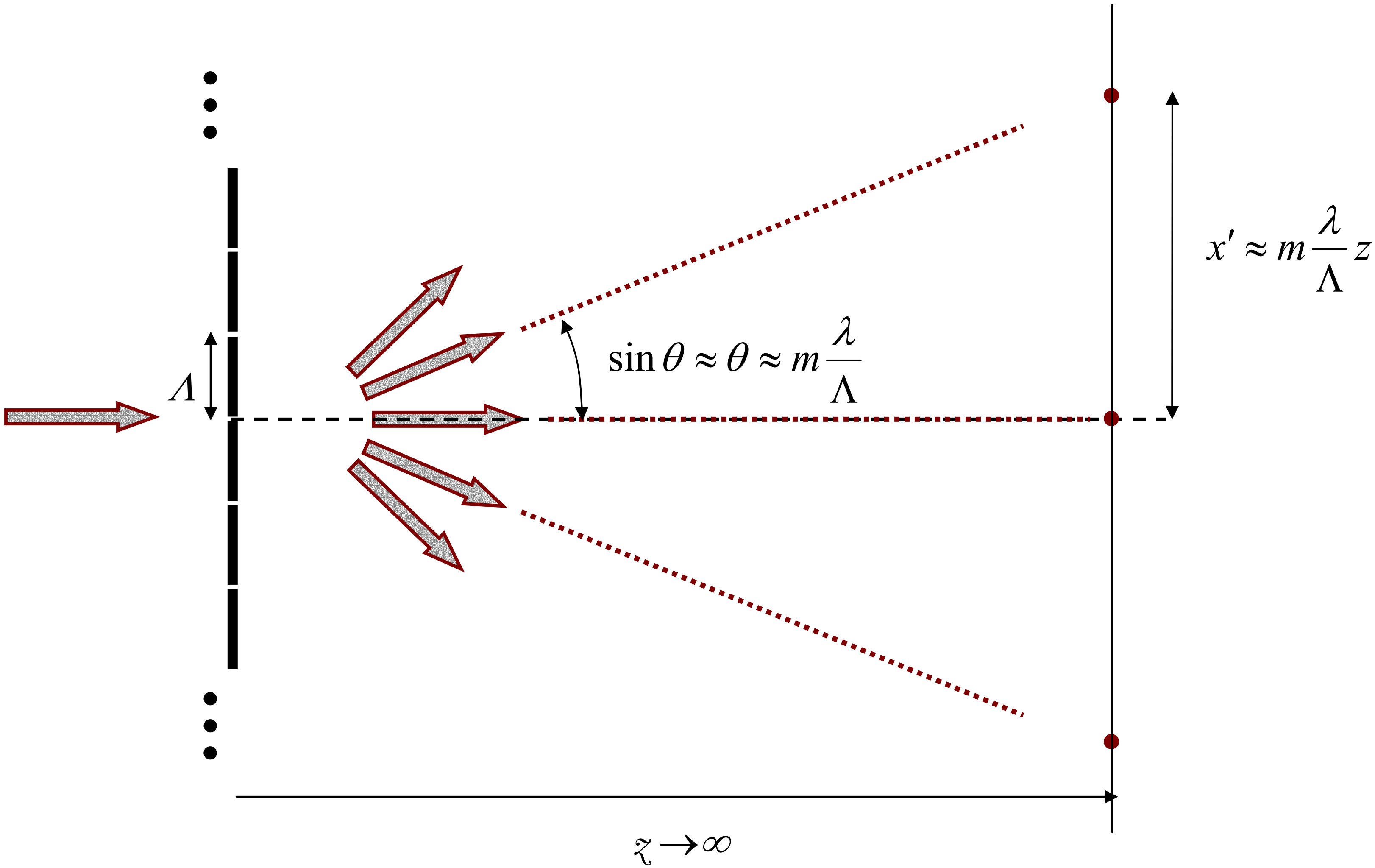
$$\Lambda \sin\theta = m\lambda \Leftrightarrow$$

$$\Leftrightarrow \sin\theta = m\frac{\lambda}{\Lambda}$$

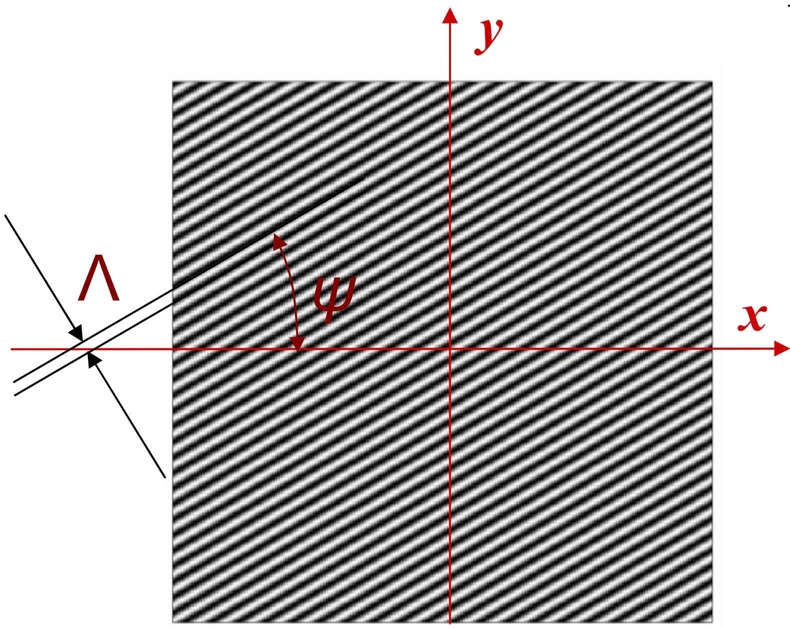
Diffraction from periodic array of holes

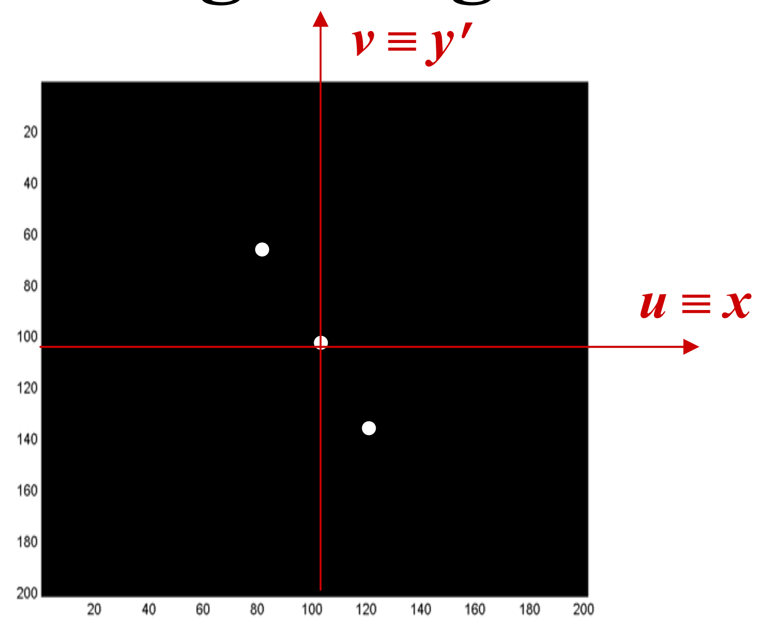


Fraunhofer diffraction from periodic array of holes



Sinusoidal amplitude grating





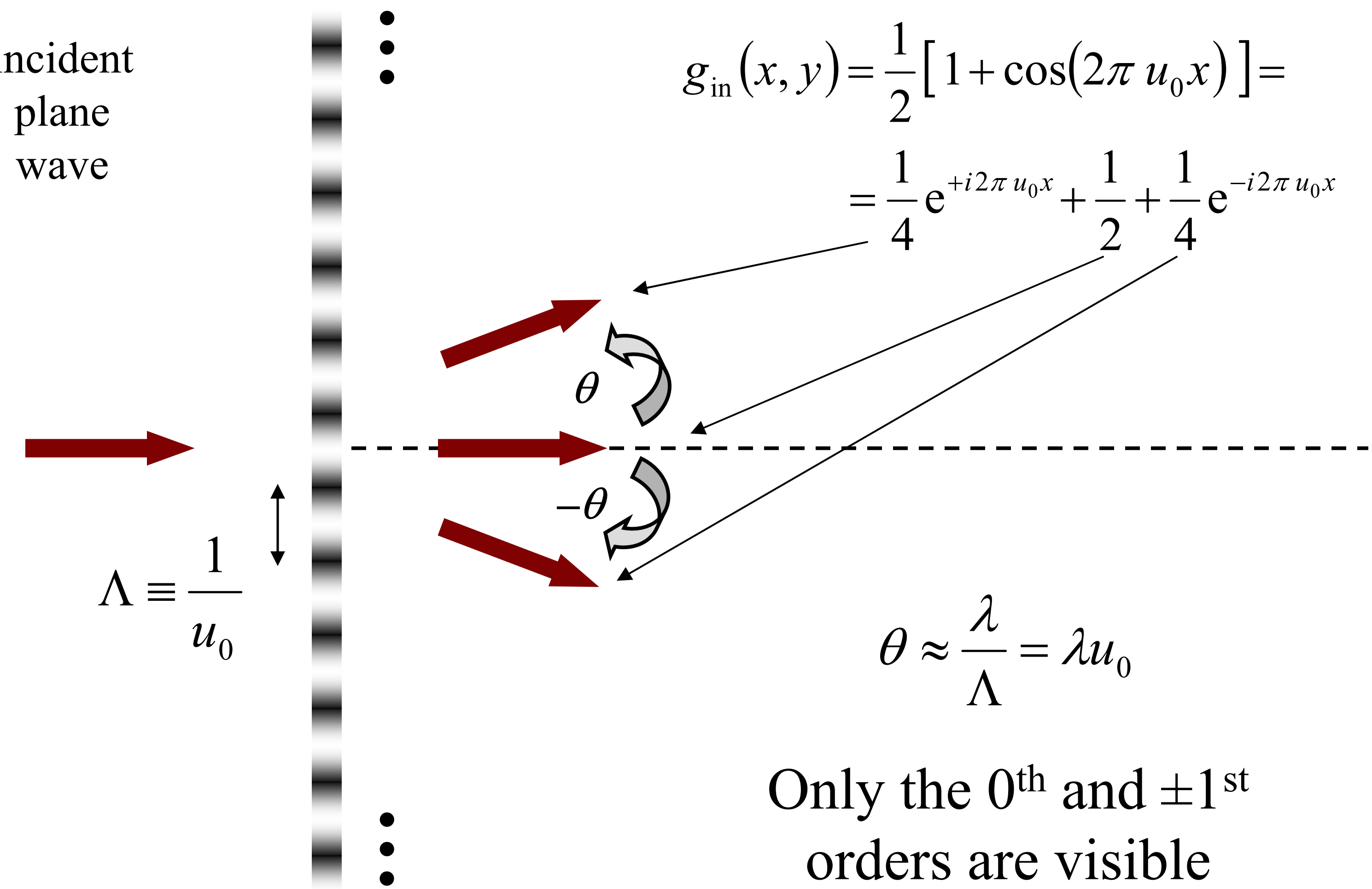
$$g_{\text{in}}(x,y)=\frac{1}{2}\left[1+\cos\left(2\pi\left(u_0x+v_0y\right)\right)\right]$$

$$\Lambda=\frac{1}{\sqrt{u_0^2+v_0^2}},\quad \tan\psi=\frac{u_0}{v_0}$$

$$g_{\text{out}}(x,y)=\frac{\mathrm{e}^{i2\pi z/\lambda}}{i\lambda z}\exp\left\{\frac{(x')^2+(y')^2}{\lambda z}\right\}$$

$$\times\left[\frac{1}{2}\delta(x')\delta(y')+\frac{1}{4}\delta(x'-\lambda z u_0)\delta(y'-\lambda z v_0)+\frac{1}{4}\delta(x'+\lambda z u_0)\delta(y'+\lambda z v_0)\right]$$

Sinusoidal amplitude grating



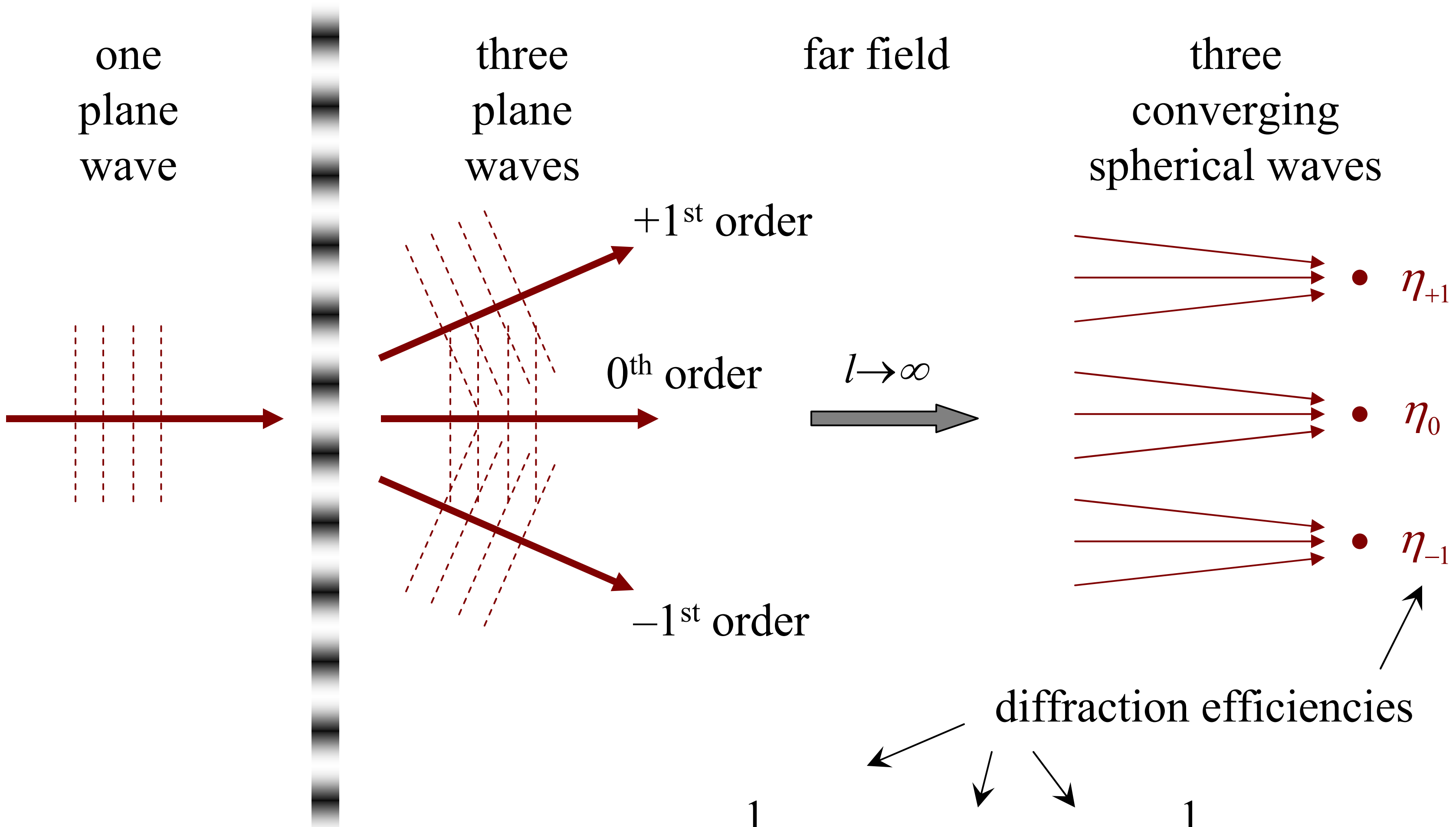
$$g_{\text{in}}(x, y) = \frac{1}{2}\left[1 + \cos(2\pi\, u_0 x)\right] =$$

$$= \frac{1}{4}\mathrm{e}^{+i2\pi\, u_0 x} + \frac{1}{2} + \frac{1}{4}\mathrm{e}^{-i2\pi\, u_0 x}$$

$$\theta \approx \frac{\lambda}{\Lambda} = \lambda u_0$$

Only the 0th and ±1st
orders are visible

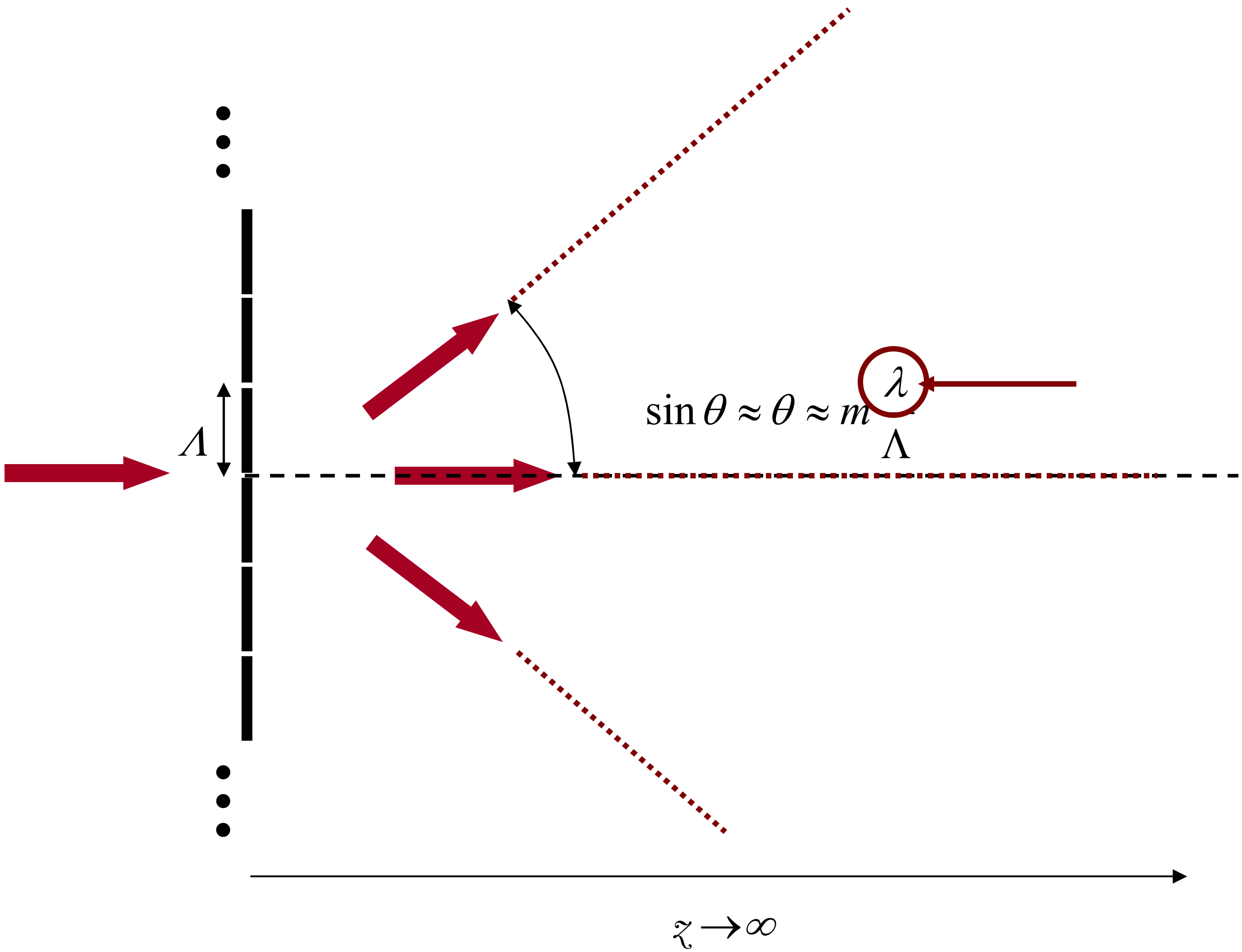
Sinusoidal amplitude grating



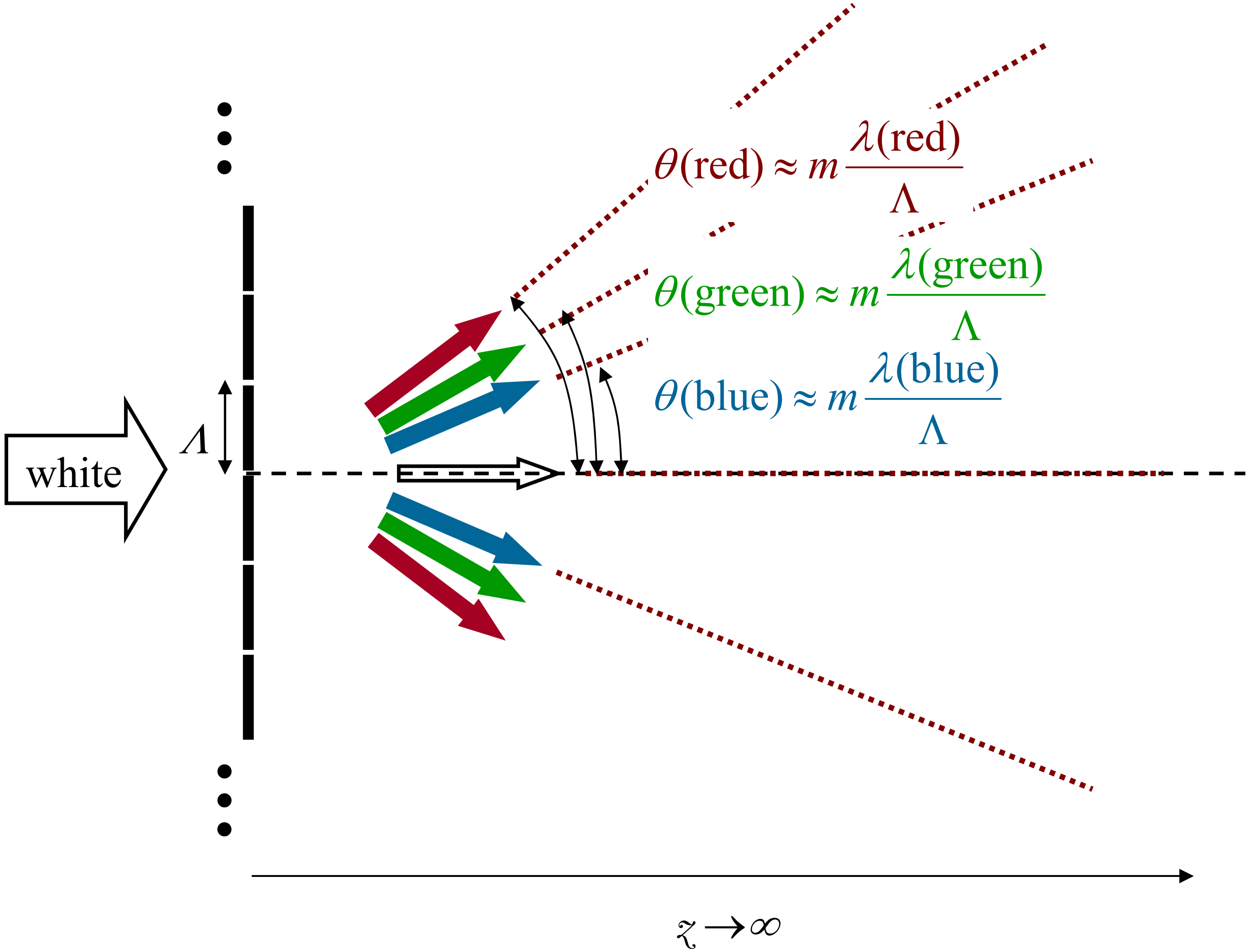
$$\eta_0 = \frac{1}{4} = 25\%, \quad \eta_{+1} = \eta_{-1} = \frac{1}{16} = 6.25\%$$

Dispersion

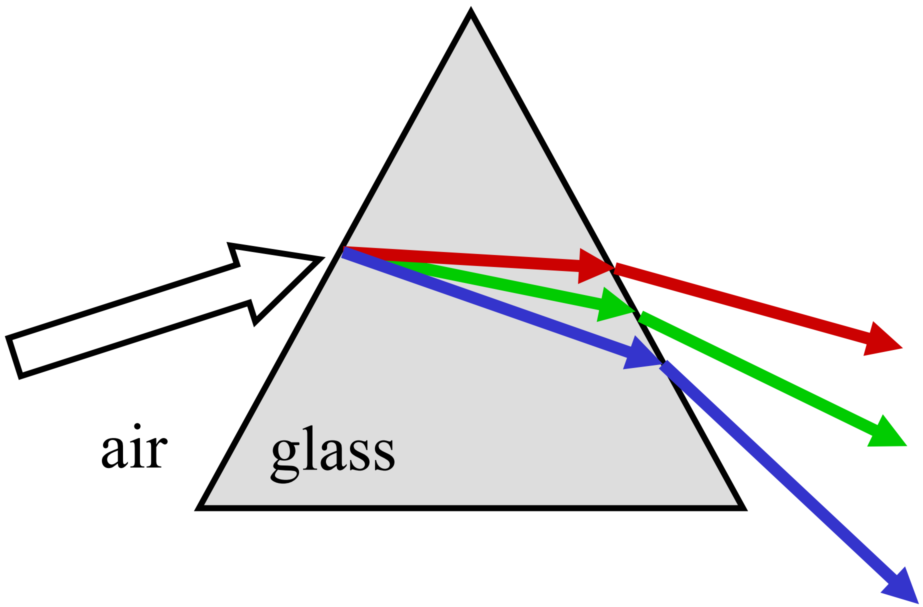
Diffraction from a grating

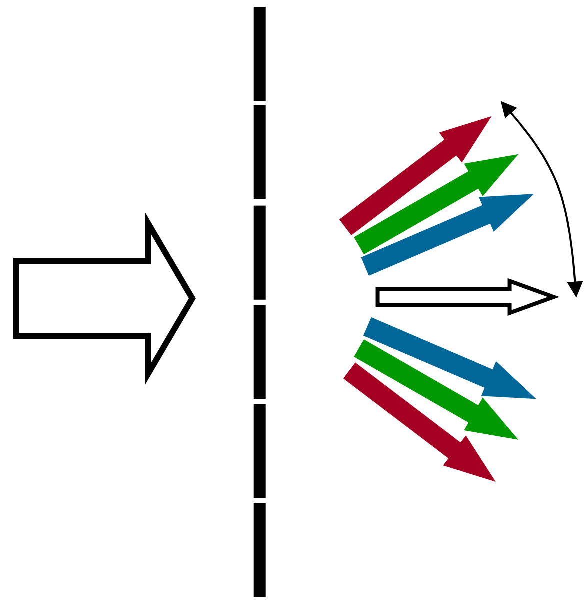


Dispersion from a grating



Prism dispersion *vs* grating dispersion





Blue light is refracted at *larger* angle than red:

normal dispersion

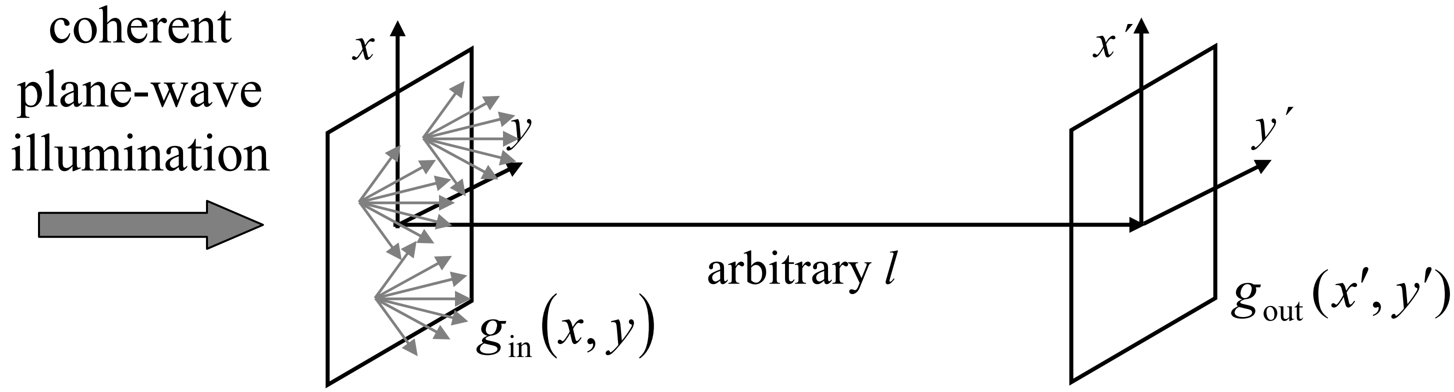
Blue light is diffracted at *smaller* angle than red:

anomalous dispersion

The ideal thin lens as a Fourier transform engine

Fresnel diffraction

Reminder



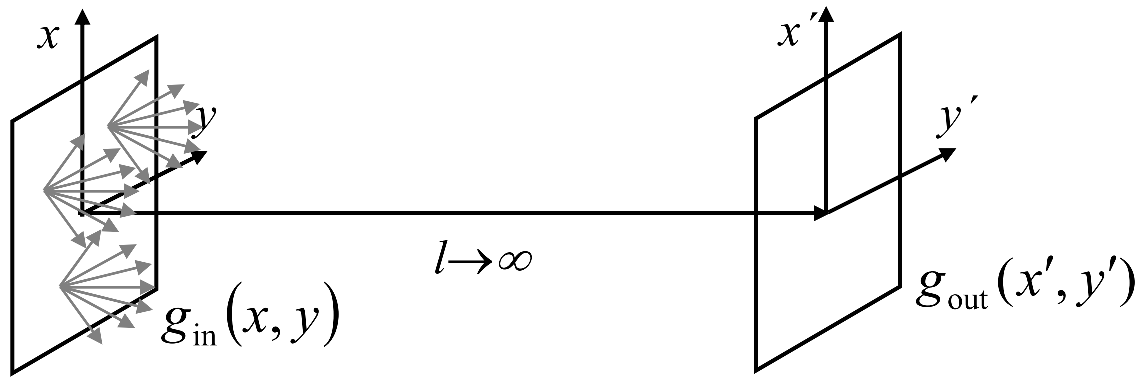
$$g_{\text{out}}(x',y';l)=\frac{1}{i\lambda l}\exp\left\{i2\pi\frac{l}{\lambda}\right\}\iint g_{\text{in}}(x,y)\exp\left\{i\pi\frac{(x'-x)^2+(y'-y)^2}{\lambda l}\right\}\mathrm{d}x\mathrm{d}y.$$

The diffracted field is the *convolution* of the transparency with a spherical wave

Q: how can we "undo" the convolution optically**?**

Fraunhofer diffraction

Reminder



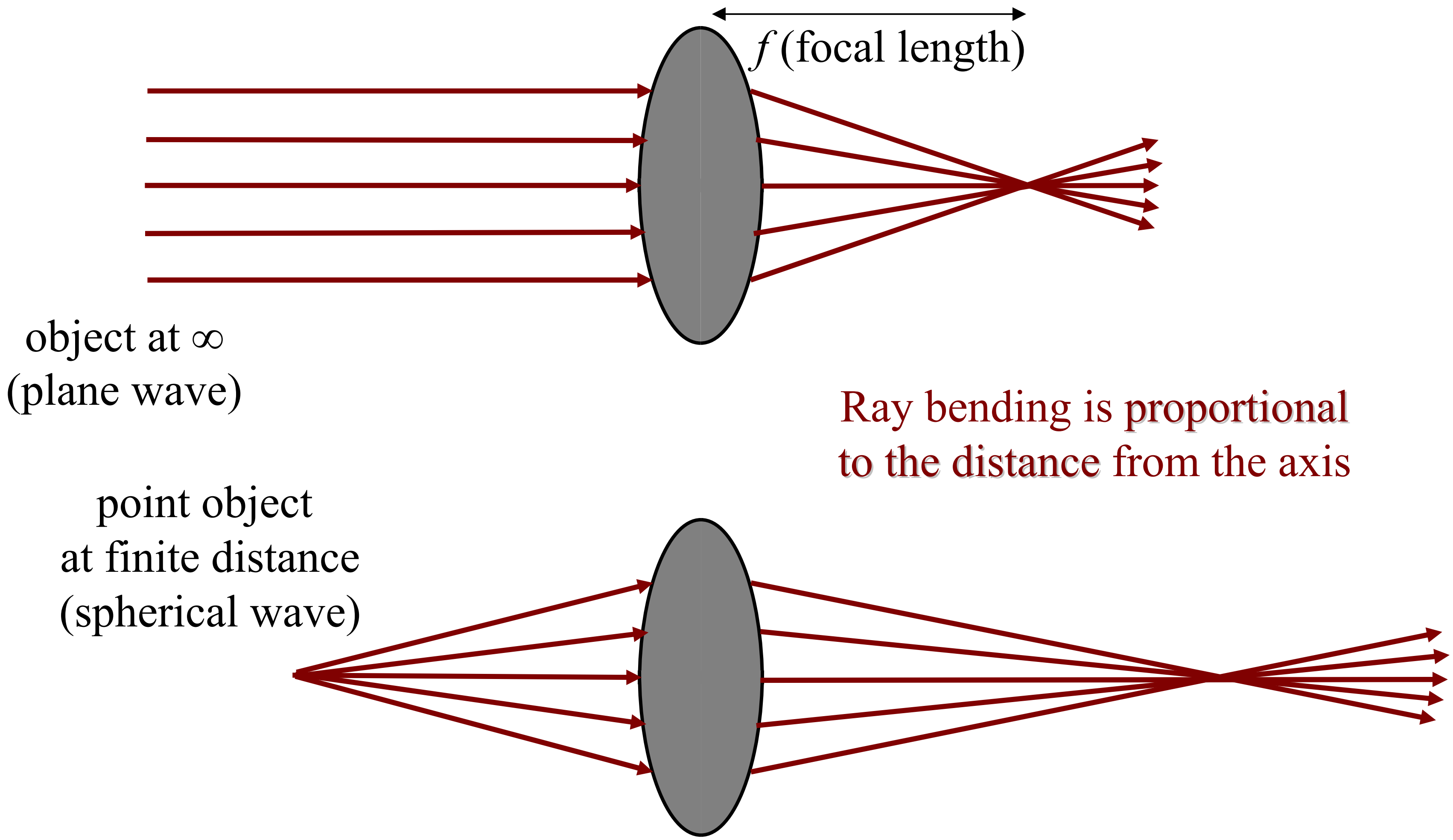
$$g_{\text{out}}(x', y'; l) \propto \int g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda l}\right) + y\left(\frac{y'}{\lambda l}\right)\right]\right\} \mathrm{d}x\mathrm{d}y$$

The **"far-field"** (i.e. the diffraction pattern at a large longitudinal distance l equals the **Fourier transform** of the original transparency calculated at spatial frequencies

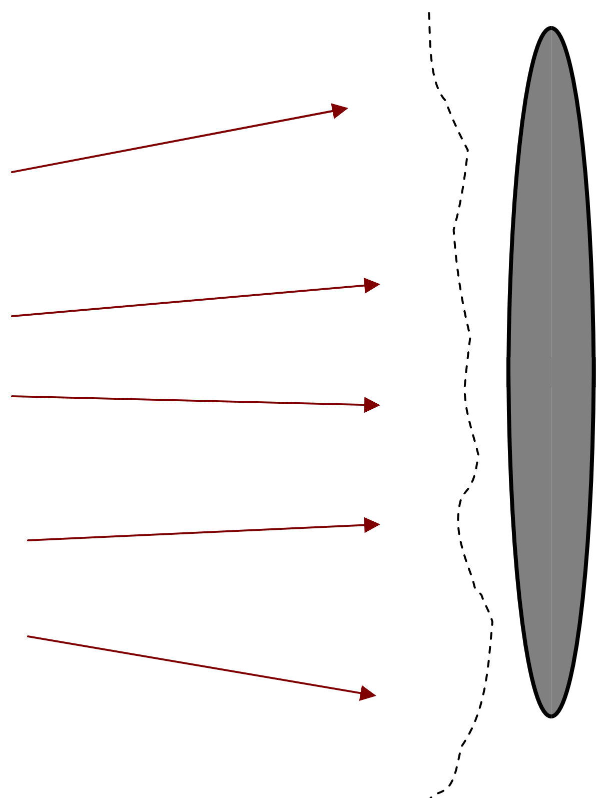
$$f_x = \frac{x'}{\lambda l} \qquad f_y = \frac{y'}{\lambda l}$$

Q: is there another optical element who can perform a Fourier transformation without having to go too far (to ∞) **?**

The thin lens (geometrical optics)



The thin lens (wave optics)



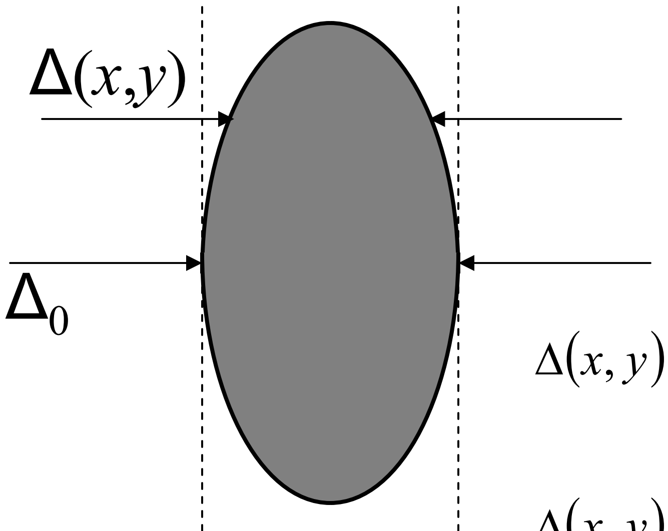
incoming wavefront

$a(x,y)$

outgoing wavefront

$$a(x,y)\ \underbrace{t(x,y)e^{i\varphi(x,y)}}_{\text{(thin transparency approximation)}}$$

The thin lens transmission function



$$\Delta(x,y) = \Delta_0 - R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1}}\right) + R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2}}\right)$$

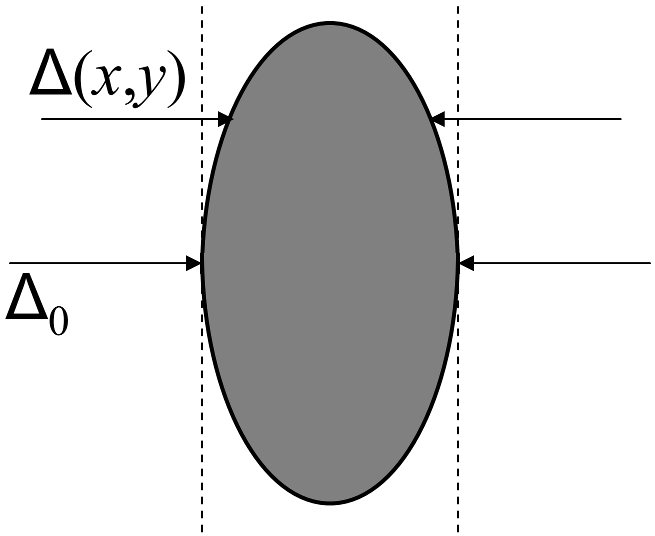
$$\Delta(x,y) \approx \Delta_0 - R_1\left(1-\left[1-\frac{x^2+y^2}{2R_1}\right]\right) + R_2\left(1-\left[1-\frac{x^2+y^2}{2R_2}\right]\right)$$

$$\Delta(x,y) \approx \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$a_{\text{lens}}(x,y) = \exp\left\{i\frac{2\pi}{\lambda}\Delta_0 + \frac{2\pi}{\lambda}(n-1)\Delta(x,y)\right\}$$

$$a_{\text{lens}}(x,y) \approx \exp\left\{i\frac{2\pi n}{\lambda}\Delta_0\right\}\exp\left\{-i\frac{2\pi}{\lambda}(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{x^2+y^2}{2}\right\}$$

The thin lens transmission function

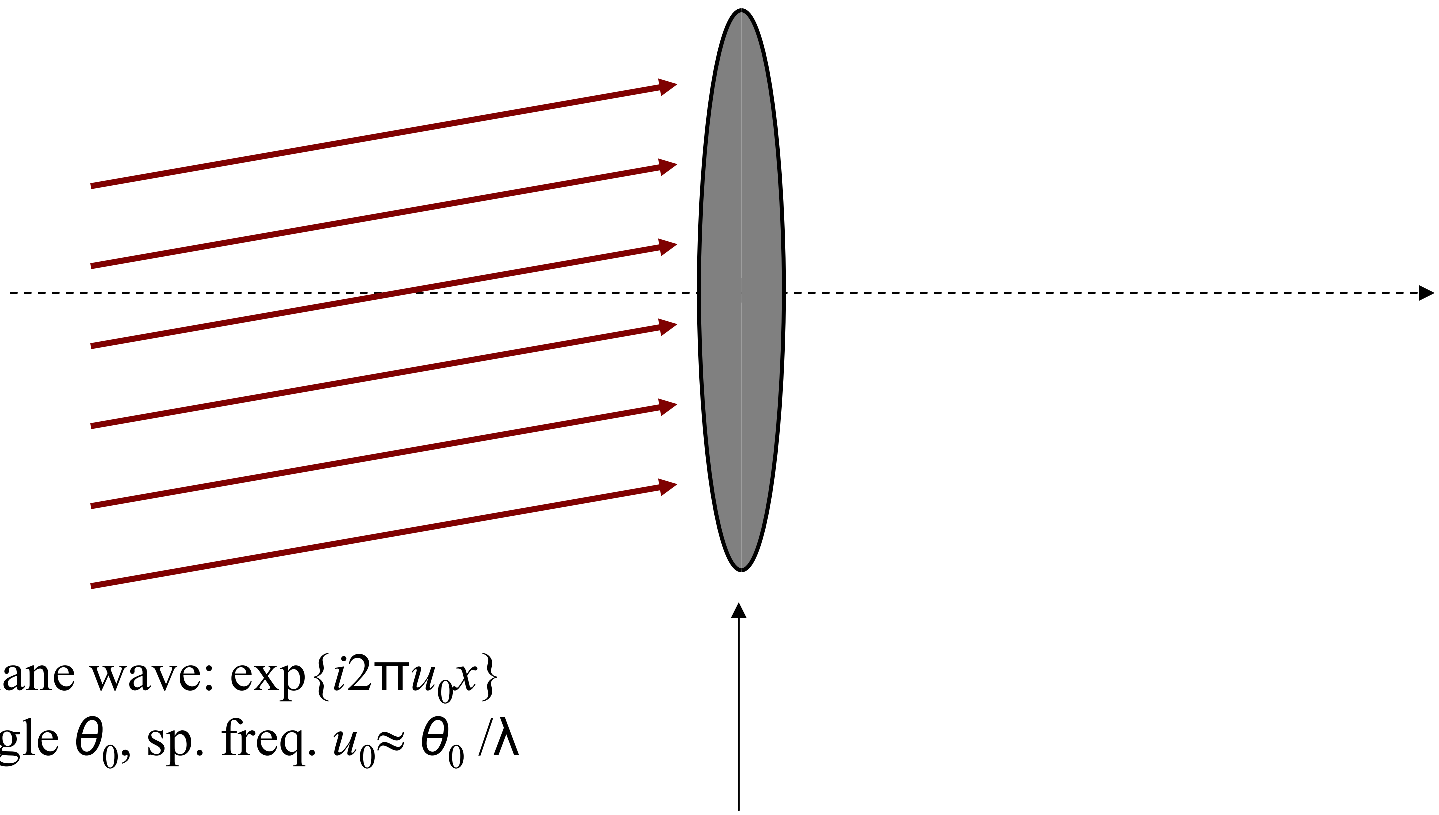


this constant-phase term can be omitted

$$a_{\text{lens}}(x,y) \approx \boxed{\exp\left\{ i\frac{2\pi n}{\lambda}\Delta_0 \right\}} \exp\left\{ -i\pi\frac{x^2+y^2}{\lambda f} \right\}$$

where $\dfrac{1}{f} \equiv (n-1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$ is the focal length

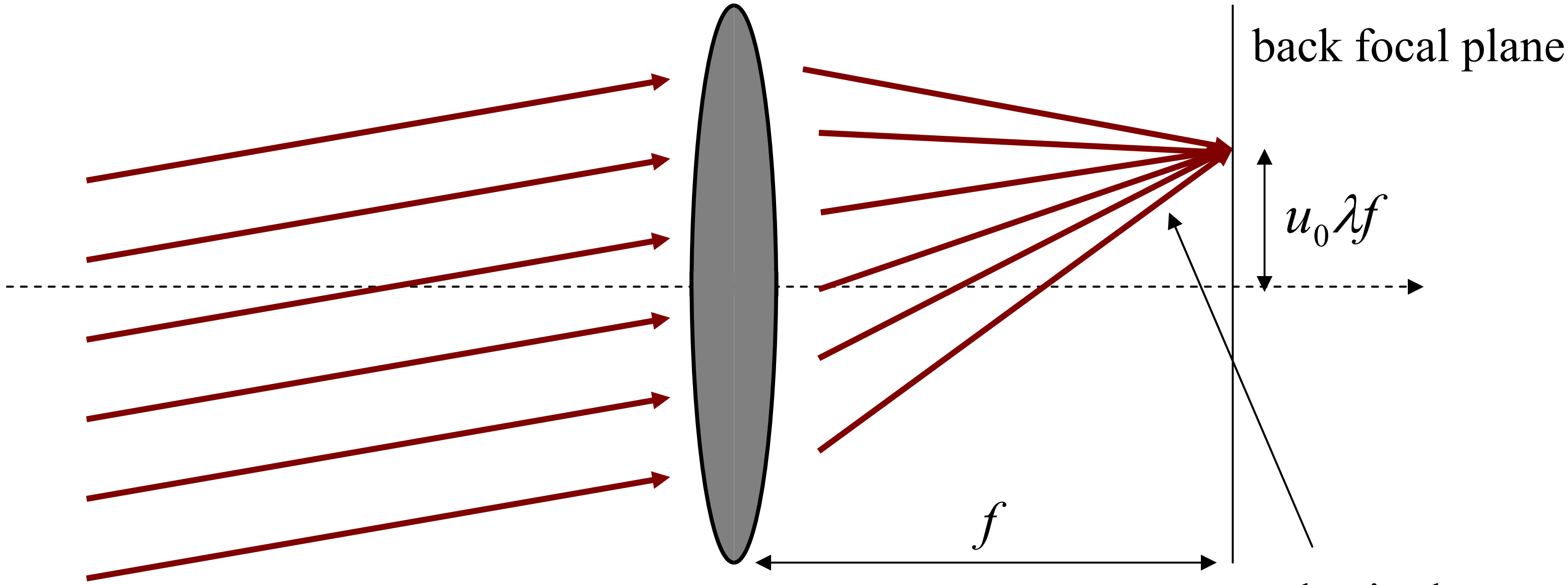
Example: plane wave through lens



plane wave: $\exp\{i2\pi u_0 x\}$
angle θ_0, sp. freq. $u_0 \approx \theta_0 / \lambda$

lens, $\quad a_{\text{lens}}(x, y) = \exp\left\{-i\pi \frac{x^2 + y^2}{\lambda f}\right\}$

Example: plane wave through lens

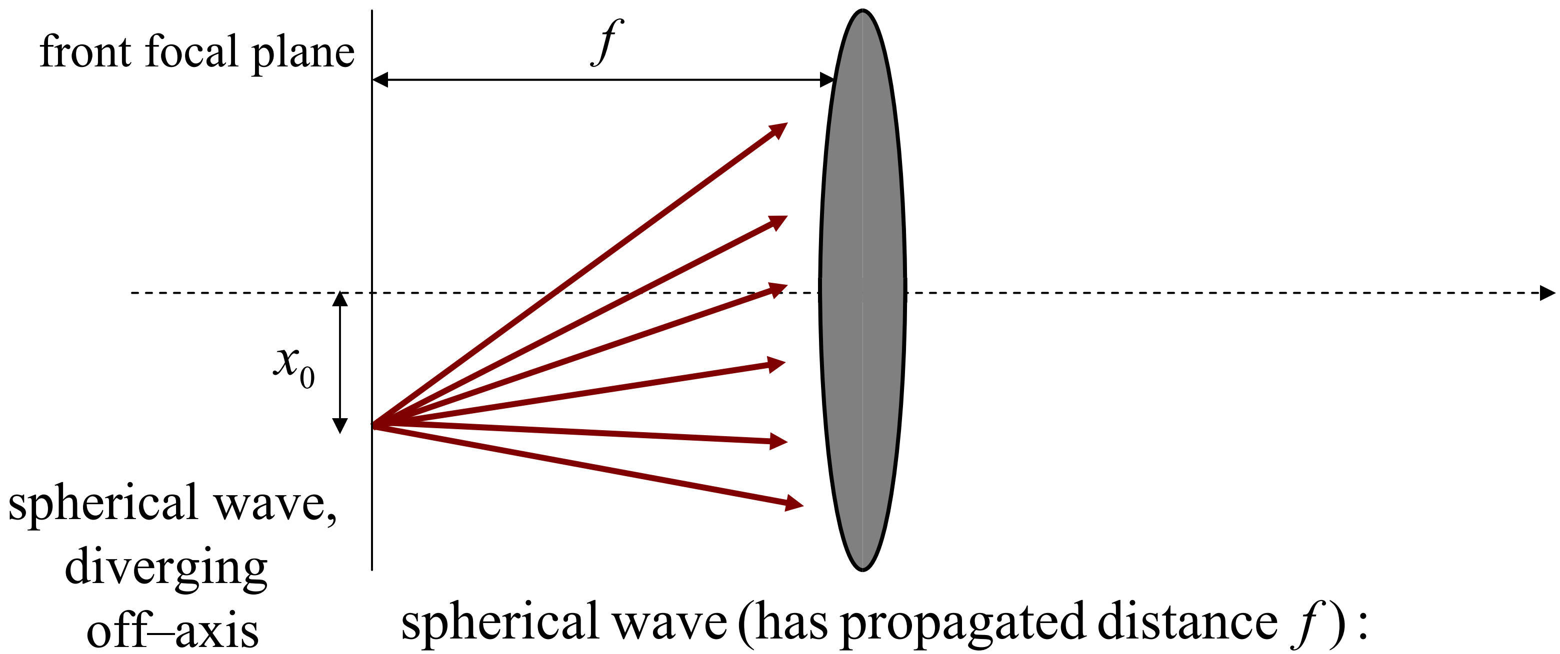


wavefront after lens :

$$a_+(x,y)=\exp\{i2\pi u_0 x\}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

$$a_+(x,y)=\underbrace{\boxed{\exp\{i\pi u_0^2\lambda f\}}}_{\text{ignore}}\exp\left\{-i\pi\frac{(x-u_0\lambda f)^2+y^2}{\lambda f}\right\}$$

Example: spherical wave through lens



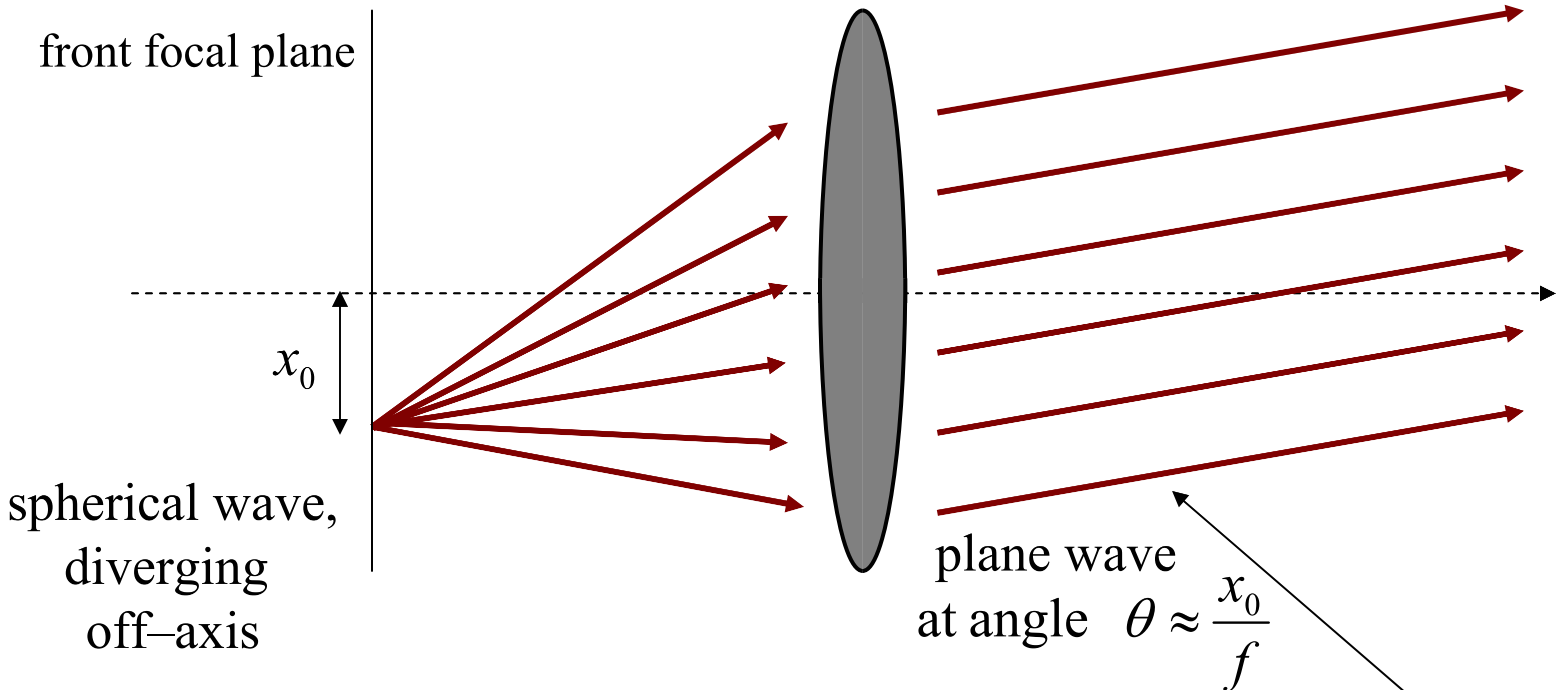
spherical wave (has propagated distance f) :

$$a_{-}(x,y) = \exp\left\{ i2\pi \frac{f}{\lambda} \right\} \exp\left\{ i\pi \frac{(x+x_0)^2 + y^2}{\lambda f} \right\}$$

lens transmission function :

$$a_{\text{lens}}(x,y) = \exp\{ i2\pi n \Delta_0 \} \exp\left\{ -i\pi \frac{x^2+y^2}{\lambda f} \right\}$$

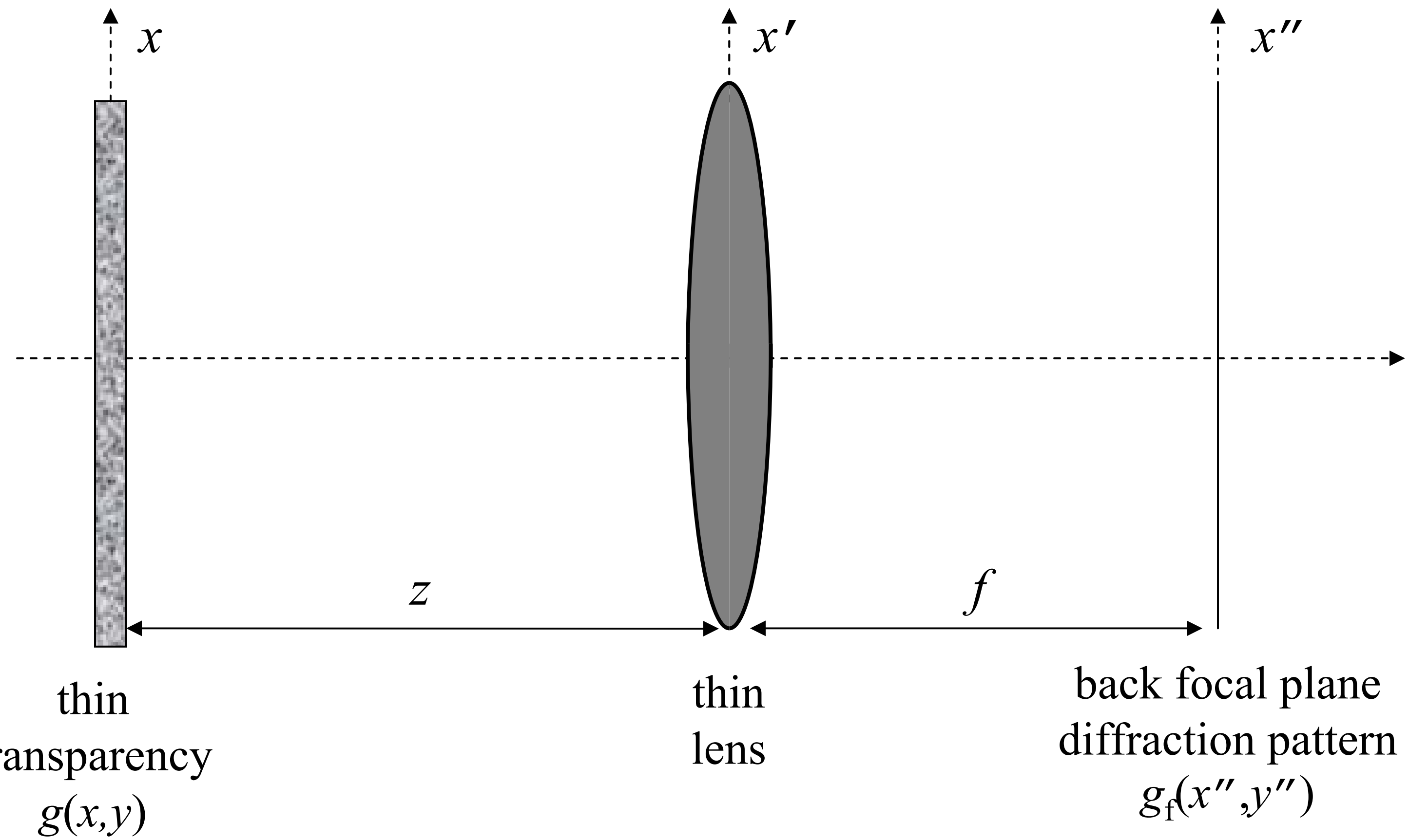
Example: spherical wave through lens



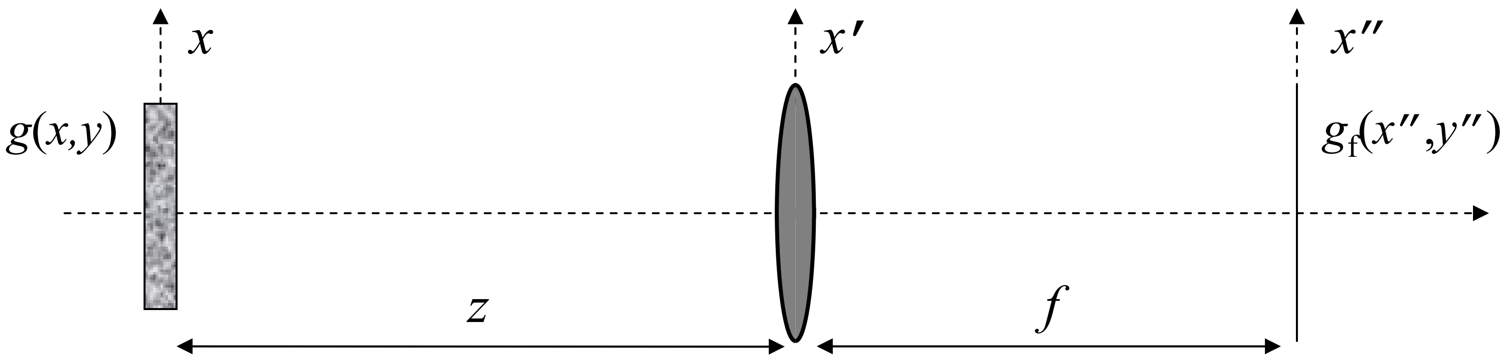
wavefront after lens

$$a_+(x,y) = a_-(x,y) \times a_{\text{lens}}(x,y) = \exp\left\{ \underbrace{i2\pi\left(n\Delta_0 + \frac{f}{\lambda}\right) + i\pi\frac{x_0^2}{\lambda f}}_{\text{ignore}} + i2\pi\frac{x_0 x}{\lambda f} \right\}$$

Diffraction at the back focal plane



Diffraction at the back focal plane



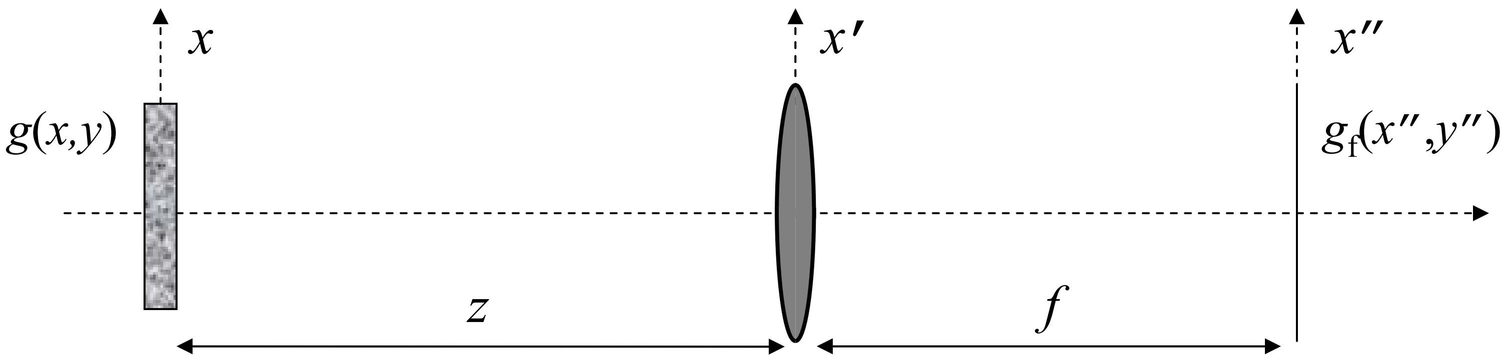
1D calculation

Field before lens $$g_{\text{lens}-}(x') = \int g(x) \exp\left\{ i\pi \frac{(x'-x)^2}{\lambda z} \right\} \mathrm{d}x$$

Field after lens $$g_{\text{lens}+}(x') = g_{\text{lens}-}(x') \exp\left\{ -i\pi \frac{x'^2}{\lambda f} \right\}$$

Field at back f.p. $$g_{\text{f}}(x'') = \int g_{\text{lens}+}(x') \exp\left\{ i\pi \frac{(x''-x')^2}{\lambda f} \right\} \mathrm{d}x'$$

Diffraction at the back focal plane



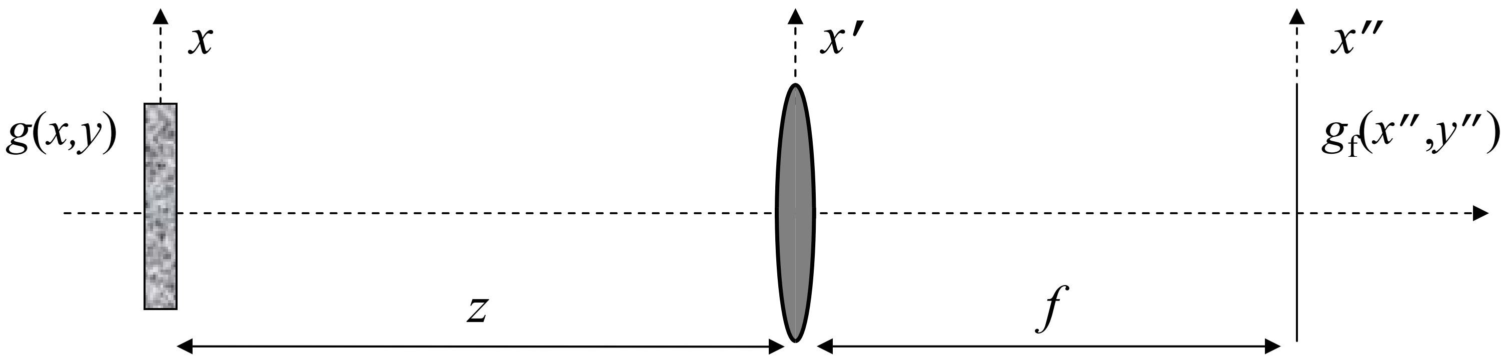
1D calculation

$$g_{\mathrm{f}}(x'') = \exp\left\{ i\pi \frac{x''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\} \int g(x) \exp\left\{-i2\pi \frac{x x''}{\lambda f}\right\} \mathrm{d}x$$

2D version

$$g_{\mathrm{f}}(x'', y'') = \exp\left\{ i\pi \frac{x''^2 + y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\} \iint g(x,y) \exp\left\{-i2\pi \frac{x x'' + y y''}{\lambda f}\right\} \mathrm{d}x\, \mathrm{d}y$$

Diffraction at the back focal plane



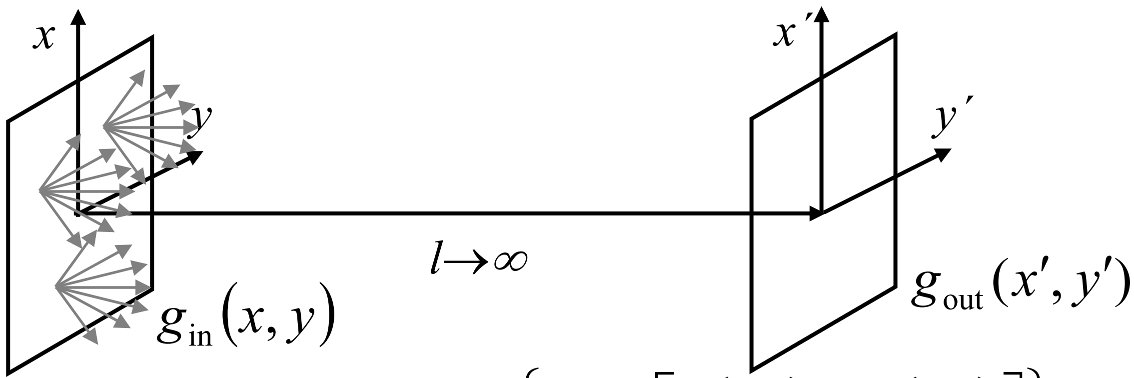
$$g_{\mathrm{f}}(x'', y'') = \exp\left\{ i\pi \frac{x''^2 + y''^2}{\lambda f}\left(1 - \frac{z}{f}\right)\right\} \iint g(x, y) \exp\left\{-i2\pi \frac{xx'' + yy''}{\lambda f}\right\} \mathrm{d}x\, \mathrm{d}y$$

$$\therefore \quad g_{\mathrm{f}}(x'', y'') = \underbrace{\boxed{\exp\left\{ i\pi \frac{x''^2 + y''^2}{\lambda f}\left(1 - \frac{z}{f}\right)\right\}}}_{\text{spherical wave-front}} \underbrace{G\left(\frac{x''}{\lambda f}, \frac{y''}{\lambda f}\right)}_{\text{Fourier transform of } g(x,y)}$$

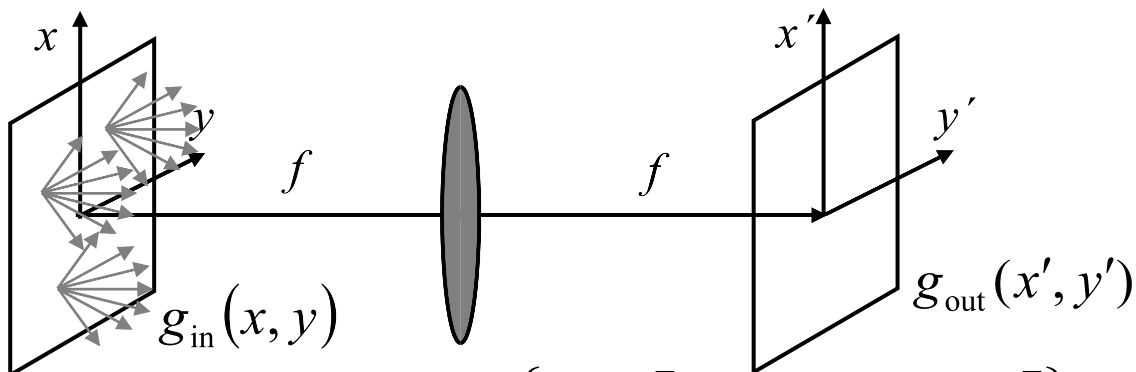
spherical wave-front

Fourier transform of $g(x,y)$

Fraunhofer diffraction *vis-á-vis* a lens

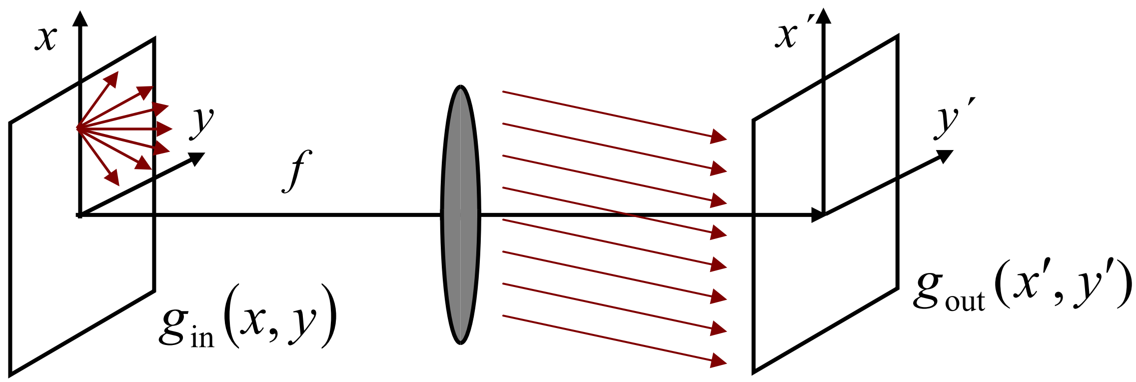


$$g_{\text{out}}(x', y'; l) \propto \iint g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda l}\right) + y\left(\frac{y'}{\lambda l}\right) \right] \right\} \mathrm{d}x\mathrm{d}y$$



$$g_{\text{out}}(x', y'; f) \propto \iint g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda f}\right) + y\left(\frac{y'}{\lambda f}\right) \right] \right\} \mathrm{d}x\mathrm{d}y$$

Spherical – plane wave duality



point source at (x,y)
amplitude $g_{\text{in}}(x,y)$

plane wave oriented

towards $\left(-\dfrac{x}{\lambda f}, -\dfrac{y}{\lambda f}\right)$

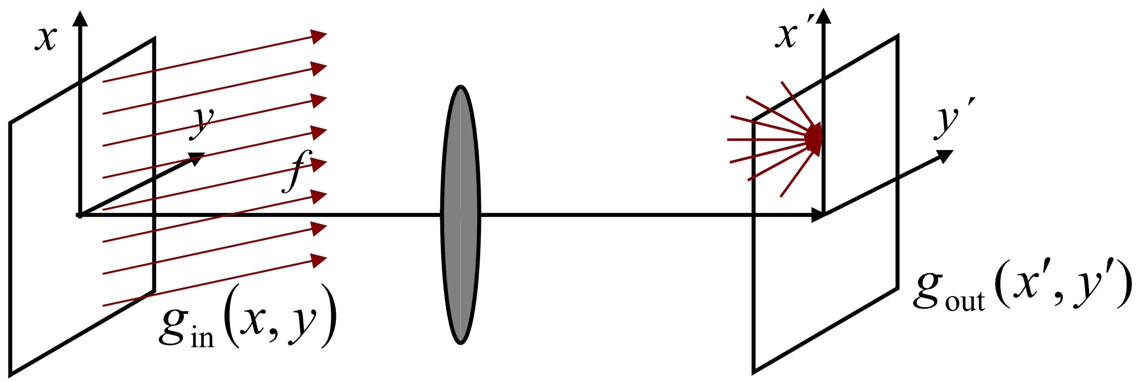
... a superposition ...

... of plane waves corresponding to point sources in the object

each output coordinate (x', y') receives ...

$$g_{\text{out}}(x', y') \propto \iint g_{\text{in}}(x, y) \exp\left\{ i2\pi\left[\left(-\frac{x}{\lambda f}\right)x' + \left(-\frac{y}{\lambda f}\right)y'\right]\right\} \mathrm{d}x\mathrm{d}y$$

Spherical – plane wave duality



a plane wave departing
from the transparency
at angle (θ_x, θ_y) has amplitude
equal to the Fourier coefficient
at frequency $(\theta_x/\lambda, \theta_y/\lambda)$ of $g_{\text{in}}(x,y)$

produces a spherical wave converging

towards $\left(\frac{\theta_x}{\lambda}\times(\lambda f), \frac{\theta_y}{\lambda}\times(\lambda f)\right) = (\theta_x f, \theta_y f)$

each output coordinate
(x', y') receives amplitude equal
to that of the corresponding
Fourier component

$$g_{\text{out}}(x', y') \propto \underbrace{\iint g_{\text{in}}(x,y)\exp\left\{-i2\pi\left[x\left(\frac{x'}{\lambda f}\right)+y\left(\frac{y'}{\lambda f}\right)\right]\right\}\mathrm{d}x\mathrm{d}y}$$

Conclusions

- When a thin transparency is illuminated coherently by a monochromatic plane wave and the light passes through a lens, the field at the focal plane is the Fourier transform of the transparency times a spherical wavefront
- The lens produces at its focal plane the Fraunhofer diffraction pattern of the transparency
- When the transparency is placed exactly one focal distance behind the lens (*i.e.,* $z=f$), the Fourier transform relationship is <u>exact</u>.