Massachusetts Institute of Technology
6.241 Dynamic Systems

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Out 12/8/03

Take-home Exam
Due 24 hours later

Instructions

1. Please, do not discuss this exam with **anyone**.

2. You have 24 hours to finish the exam. Exams will be available after 12:30pm on Monday. I need all the exams back to me by wed 5pm.

3. During the exam, you can use the course notes, this year's homework and solutions, and any additional handouts given in the class. Please, do not consult with other books or other material. Most likely, you will end-up wasting your time.

4. Please, write down the solution of each problem on a separate page. Be organized and clear. If you write down an argument and you think it is not completely correct, indicate exactly where the problem is. This way I can assess the level of confidence you have in your arguments.

5. We denote the H_∞ (2-induced) norm of a system G as $\|G\|_\infty$.

Happy Holidays

Problem 1 (25 points)

This problem addresses a different method for stabilizing a system using state feedback. The approach presented here extends easily to a class of nonlinear systems. The idea of this method is to pick a $n-1$ dimensional subspace in $\mathbb{R}^n$ and then to construct a control input so that the trajectories of the system arrive to this subspace in finite time. If the subspace is selected so that the dynamics of the system are stable once restricted to the subspace, then stabilization will be guaranteed.

Given a linear time-invariant SISO system (A, b, c) in the reachable canonical form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} \end{pmatrix}$$

Define a subspace of $\mathbb{R}^n$ as:

$$S = \{x | d'x = 0,\, d' = \begin{pmatrix} d_0 & d_1 & \dots & d_{n-2} & 1 \end{pmatrix} \in \mathbb{R}^n\}$$

1. Show that there exists a control input $u^o(x) = k'x$ such that S is an invariant subspace of $\mathbb{R}^n$ i.e., if $x(0) \in S$ then $x(t) \in S$ for all $t \geq 0$. (Hint: Define $s = d'x$, and compute $\dot{s}$).

2. With u^o, give a necessary and sufficient condition for the dynamics on S to be asymptotically stable. This will imply that if $x(0) \in S$, then $\lim_{t\to\infty} x(t) = 0$.

3. We need to construct a control input as a function of the state x, $u(x)$, such for any $x(0) \in \mathbb{R}^n$, there exists a finite time t_f such that $x(t_f) \in S$. If $u(x) = u^o(x) \forall x \in S$, then once the trajectory enter S, it stays there. If d is selected such that the dynamics in S are stable, then, the trajectory will converge to 0 in the limit.

   One way to do this is to consider the system descried by $s = d'x$. Define $V(s) = s^2$ as a Lyapunov function candidate. Show how to select $u(x) = u^o(x) + \tilde{u}(x)$ such that
   $$\frac{dV}{dt} = 2s\dot{s} \leq -c|s|, \qquad c > 0.$$
   Note that $\tilde{u}$ may be discontinuous.

4. Show that with $u(x)$ as above, for any initial condition, there exists a t_f such that $x(t_f) \in S$. (**Hint:** consider the equation $s(t_f) - s(0) = \int_0^{t_f} \dot{s} dt$ and use the previous construction to bound $\dot{s}$.)

5. Show that the system with $u(x)$ is asymptotically stable.

Problem 2 (25 points)

Consider the following system:

$$P = \begin{pmatrix} \frac{1}{s-1}(1+\Delta_1) & \frac{1}{s-1}(1+\Delta_2) \\ \frac{1}{s+1}(1+\Delta_1) & \frac{1}{s+1}(1+\Delta_2) \end{pmatrix}, \qquad \|\Delta_i\|_\infty \le \gamma_i$$

Denote the nominal system with $\Delta_i = 0$, $i = 1, 2$ by P_0. Notice that for every s, $P_0(s)$ is a rank 1 matrix that can be written as $P_0(s) = \mathbf{p}_0(s)\mathbf{1}'$ where

$$\mathbf{p}_0(s) = \begin{pmatrix} \frac{1}{s-1} \\ \frac{1}{s+1} \end{pmatrix}$$

and $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1. Does there exist a controller K such that

   $$\|(I + P_0 K)^{-1}\|_\infty < 1?$$

   Explain your answer.

2. If K stabilizes P_0, give a necessary and sufficient condition for the stability of the closed loop system for all Δ_i with $\|\Delta_i\|_\infty \le \gamma_i$ and $i = 1, 2$.

3. Consider that $\gamma = max\{\gamma_1, \gamma_2\}$. Show that there exists $\gamma_0 > 0$ and $f(.)$ such that

   $$\|(I + PK)^{-1}\|_\infty < \|(I + P_0 K)^{-1}\|_\infty \frac{1}{1 - f(\gamma)}, \gamma < \gamma_0$$

   where $f(\gamma) \to 0$ as $\gamma \to 0$.

   Explain this answer.

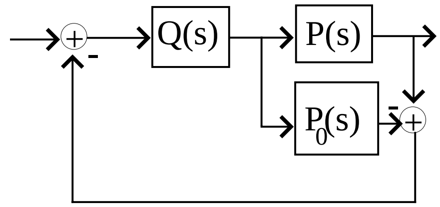

Figure 1: Feedback interconnection for Problem 4.

Problem 3 (25 points)

The diagram in Figure 1 depicts a controller configuration implemented on a process P assuming a model P_0 is available, and Q is a system to be designed. This type of controller is quite common in process control application. Nevertheless, many designers found it hard to come up with a Q to stabilize the system in cases when P was unstable. The purpose of this problem is to explain this observation for SISO plants.

1. Suppose $P_0 = \frac{1}{s-1}$, $P = \frac{2}{s-1}$,and $Q = 2$. Is the closed loop system stable? Justify your answer

2. Show that if the closed loop system is stable, then necessarily P_0 and P do not have common unstable poles.

3. Assume that $P = P_0$ stable. Show that the system is stable for any stable Q. Show that any stabilizing controller of P can be implemented by an appropriate choice of Q. Hence this parametrization of stabilizing controllers is complete.

4. Suppose $P = P_0 + \Delta$, where P_0 and Δ are both stable, and $\|\Delta\|_\infty < 1$. Under what conditions on Q is the closed loop system stable?

Problem 4 (25 points)

An alternative to the "small gain approach" in analyzing I/O stability and stability robustness is the passivity approach. Let h be a stable, causal, LTI, MIMO, system with input/output relation given by $y = h * u$. Let $H(s)$ be the Laplace Transform of h. Then, H is passive if

$$\int_0^\infty u'y \geq \gamma \|u\|_2^2 \qquad \text{for all } u \in L_2[0,\infty),\ \gamma > 0$$

1. Show that H is passive if and only if

   $$\sigma_{min}[H(jw) + H'(jw)] > \gamma \forall\, w$$

   In the SISO case, this translates to $ReH(jw) > \gamma$.

   Hint If $y(t)$ and $u(t)$ are vectors of real functions, then

   $$\int_0^\infty u'ydt = 2\int_0^\infty Re[U(jw)'Y(jw)]dw = \int_0^\infty U(jw)'Y(jw) + Y(jw)'U(jw)dw$$

   where $Y(jw) = \int_0^\infty y(t)e^{-jwt}dt$ and $U(jw) = \int_0^\infty u(t)e^{-jwt}dt$.

2. (weak feedback result) Show that if H is passive, then the closed loop system with H and any positive definite gain matrix (see Figure 2) is 2-stable. (First convince yourself that this is true from a Nyquist argument in the SISO).

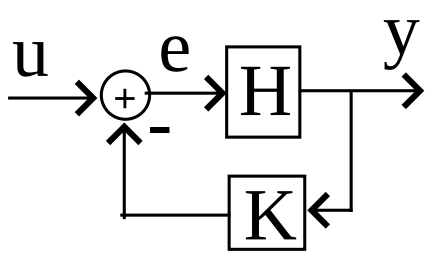

Figure 2: Feedback with positive definite gain matrix $K > 0$.

3. Let H be SISO and suppose it is in the feedback with an uncertain constant gain, $k \in [k_1,\, k_2]$. Use passivity to derive a sufficient condition for stability.

   Hint: Parametrize $k \in [k_1,\, k_2]$ as a function of $\bar{k} \in [0,\, \infty]$ and prove the diagram equivalence depicted in Figure 3. Use the fact that $\bar{k}$ is a non-negative constant.

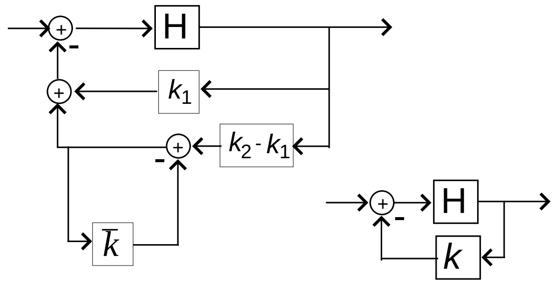

Figure 3: Depiction of two equivalent block diagrams, where $k = k_1 + \frac{k_2 - k_1}{1 + \bar{k}}$.

4. Show H is passive if and only if G, defined as $G(jw) = (I - H(jw))(I + H(jw))^{-1}$, satisfies $\|G\|_\infty < 1$. This shows the equivalence between small gain and passivity results.