MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

FINAL EXAM SOLUTIONS

Problem 1 (i) There are many choices for u^o. We can achieve the desired result, namely, $\dot{s} = d'(A + bk')x = 0$ when $d'x = 0$, if we pick k such that $\lambda d' = d'(A + bk')$. Letting $\lambda = 0$ yields $u = \begin{pmatrix} a_o & a_1 - d_o & a_2 - d_1 & \ldots & a_{n-1} - d_{n-2} \end{pmatrix} x$.

(ii) Using the u^o that was found above, we have that $\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & -d_o & -d_1 & \cdots & -d_{n-2} \end{bmatrix} x.$
First, note that no vector in S can evolve in the direction of the eigenvalue at the origin, since the left eigenvector associated with this eigenvalue is d' and $d'x = 0 \; \forall x \in S$. Alternatively, note that the vectors in the nullspace of $A + bk'$, namely $x' = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \end{pmatrix}$ and $c \neq 0$, will not belong to S as long as $d_o \neq 0$. So, the necessary and sufficient conditions becomes that the roots of $s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_o$ (i.e. the eigenvalues of $A + bk'$ except the one we placed at zero by our choice of u^o) are in the open LHP.

(iii) First, note that $\dot{s} = \begin{pmatrix} -a_o & -a_1 + d_o & -a_2 + d_1 & \ldots & -a_{n_1} + d_{n-2} \end{pmatrix} x + u^o(x) + \tilde{u}(x) = \tilde{u}(x)$, where the last equality follows from our choice of u^o of part (i). Now, we can verify that if $\tilde{u}(x) = -\frac{1}{2}c\frac{d'x}{|d'x|}$ for $d'x \neq 0$ and $\tilde{u}(x) = 0$ for $d'x = 0$ we have the desired result. Namely, when $d'x \neq 0$, $2s\dot{s} = 2d'x\tilde{u}(x) \leq -c|d'x|$.

(iv) Assume, without loss of generality, that $s_o = d'x(0) > 0$; then, $\frac{d'x}{|d'x|} = 1$ until $s(t_f) = 0$, at which time the input changes. Thus, $\dot{s} = -\frac{1}{2}c$; integrating, we have that $s_f = s_o - \frac{1}{2}ct_f$. So, $s_f = 0$ when $t_f = \frac{2s_o}{c}$.

(v) With the choice of u made above, we have that $\dot{x} = (A + bk')x - \frac{c}{2}\frac{d'x}{|d'x|}b$. Consider the Lyapunov function $W = x'Px$ where $P > 0$ such that $(A+bk')'P + P(A+bk') \leq 0$ since $A+bk'$ has an eigenvalue at the origin. Then, we have, $\dot{W} = x'((A + bk')'P + P(A + bk'))x - 2c\frac{d'x}{|d'x|}x'Pb \leq -2c\frac{d'x}{|d'x|}x'Pb$. Note that if we chose P such that its last column is d, we have $\dot{W} \leq -2c|d'x|$.

Problem 2

1. It is not possible. To see that we rewrite $(I + P_0K)^{-1}$ as:

$$(I + P_0K)^{-1} = I - \frac{1}{\mathbf{1}'K\mathbf{p}_0 + 1}P_0K$$

   As such, regardless of K, for every w, we can choose x such that $(I + P_0(jw)K(jw))^{-1}x = x$. That implies that $\sup_w \sigma_{max}((I + P_0(jw)K(jw))^{-1}) \geq 1$.

2. Consider the decomposition:

$$(I+PK)^{-1} = (I+P(I+\Delta)K)^{-1} = (I+P_0K)^{-1}(I+P_0\Delta K(I+P_0K)^{-1})^{-1}$$

where $\Delta = diag(\Delta_1, \Delta_2)$. Since $(I+P_0K)^{-1}$ is stable, we only have to check if $det(I + P_0\Delta K(I+P_0K)^{-1}) = 0$ for some choice of Δ_i satisfying $\|\Delta_i\|_{2-ind} < \gamma_i$.

We start by noticing that $det(I+P_0\Delta K(I+P_0K)^{-1}) = det(1+\mathbf{1}'\Delta f)$, where $f(s) = K(s)(I+P_0K)^{-1}\mathbf{p}_0(s)$ is a vector of dimension 2. The condition for stability becomes:

$$\Delta_1(jw)f_1(jw) + \Delta_2(jw)f_2(jw) \neq -1$$

Since we have complete freedom to choose Δ_i, the necessary and sufficient condition for robust stability is just $\gamma_1|f_1(jw)| + \gamma_2|f_2(jw)| < 1$, for every w.

3. Using the decomposition of the previous problem, we get:

$$(I+PK)^{-1} = (I+P_0K)^{-1}(I+P_0\Delta K(I+P_0K)^{-1})^{-1}$$

We can further expand to find:

$$(I+PK)^{-1} = (I+P_0K)^{-1}((I+P_0K)^{-1} + (I+P_0K)^{-1}P_0\Delta K(I+P_0K)^{-1})^{-1}(I+P_0K)^{-1}$$

The submultiplicative and triangular inequalities allow us to get:

$$\sigma_{max}((I+PK)^{-1}) \leq \frac{\sigma_{max}((I+P_0K)^{-1})}{1 - \frac{\sigma_{max}(\Delta)\sigma_{max}((I+P_0K)^{-1}P_0)\sigma_{max}(K(I+P_0K)^{-1})}{\sigma_{max}((I+P_0K)^{-1})}}$$

From the definition of γ, it follows:

$$\sigma_{max}((I+PK)^{-1}) \leq \frac{\sigma_{max}((I+P_0K)^{-1})}{1 - \frac{\gamma\sigma_{max}((I+P_0K)^{-1}P_0)\sigma_{max}(K(I+P_0K)^{-1})}{\sigma_{max}((I+P_0K)^{-1})}}$$

Taking sups, we find:

$$\|(I+PK)^{-1}\| \leq \frac{\|(I+P_0K)^{-1}\|}{1 - \gamma \sup_w \frac{\sigma_{max}((I+P_0K)^{-1}P_0)\sigma_{max}(K(I+P_0K)^{-1})}{\sigma_{max}((I+P_0K)^{-1})}}$$

Problem 3 (i) To check stability of the closed loop, we must check stability of the transfer functions from all inputs injected at the inputs of the blocks to all (physical) outputs. In this problem, the transfer function from an input injected at the input of the plant to the output of the plant $(P(I+Q(P-P_o))^{-1}(I-QP_o))$ is unstable: $\frac{s-3}{s+1}\frac{2}{s-1}$.

(ii) Since $P(s)$ and $P_o(s)$ are in a parallel connection, a common unstable pole is an unstable unobservable mode of the closed loop system (see the Claim under the heading "Parallel Connection" in Chapter 30 of the Lecture notes.) Thus, the unstable pole will appear in at least one of the tranfer functions from all inputs to all outputs of the system.

(iii) All transfer function in the loop are stable since Q and P are stable. The transfer function from the output of the plant, to the input of the plant (seen by the plant) is $-(I - QP_o)^{-1}Q$. Hence, $K = -(I - QP_o)^{-1}Q$, and as we vary over stable Q, we obtain K for which the closed loop is stable, provided the interconnection remains well-posed.

(iv) In the block diagram given for the problem, substitute the plant "P(s)", by a block diagram representing $P_o + \Delta$ (drawn as a parallel connection of P_o and Δ). Now, the transfer function seen by Δ turns out to be $-Q$. So, applying the small gain theorem, we must have that $\|Q\|_\infty \leq 1$.

Problem 4

1. Using the hint we get:

$$\int_0^\infty u'ydt = \int_0^\infty U'(jw)(H(jw)' + H(jw))U(jw)dw$$

(Sufficient condition) Assume $\sigma_{min}(H(jw)' + H(jw)) \geq \gamma > 0$ Clearly, from the definition of σ_{min} we have:

$$\begin{aligned}\int_0^\infty u'ydt &= \int_0^\infty U'(jw)(H(jw)' + H(jw))U(jw)dw \geq \\ &\geq \int_0^\infty \sigma_{min}(H(jw)' + H(jw))U'(jw)U(jw)dw \geq \gamma \int_0^\infty U'(jw)U(jw)dw = \gamma\|u\|^2 \quad (1)\end{aligned}$$

(Necessary condition) Assume that there exists w_0 such that $\sigma_{min}(H(jw_0)' + H(jw_0)) < \gamma$. Choose

$$u_T(t) = \begin{cases} v \times cos(w_0 t) & \text{if } t \in [0, T] \\ 0 & otherwise \end{cases}$$

where v is chosen to satisfy $v'(H(jw)' + H(jw))v = \sigma_{min}(H(jw_0)' + H(jw_0))\|v\|^2$

It can be shown that:

$$\lim_{T\to\infty} \frac{\int_0^\infty U_T'(jw)(H(jw)' + H(jw))U_T(jw)dw}{\|u_T\|^2} = \sigma_{min}(H(jw_0)' + H(jw_0))$$

where we used the fact that, as T grows, $\frac{\|U_T(jw)\|^2}{\|u_T\|^2}$ approaches a "dirac" at the frequency w_0.

2. Using the block diagram, we start with

$$\int_0^\infty y'u = \int_0^\infty y'(e + Ky) = \int_0^\infty y'e + \int_0^\infty y'Ky$$

Now notice that $\int_0^\infty y'Ky \geq \sigma_{min}(K)\|y\|^2$ and that passivity of H implies $\int_0^\infty y'e \geq 0$, so that we get:

$$\int_0^\infty y'u \geq \sigma_{min}(K)\|y\|^2$$

But, from Cauchy-Schwartz we have $\int_0^\infty y'u \leq \|u\|\|y\|$. By substitution we get: $\|y\| \leq \frac{1}{\sigma_{min}(K)}\|u\|$

3. The equivalence shown in the block diagram shows that we can equivalenlty analyse the stability of $\bar{H} = \frac{1+Hk_1}{1+Hk_2}$ in feedback with $\bar{k}$. According to the previous exercise a sufficient condition is just the passivity of $\bar{H}$.

4. Using the result proven in problem 1.2, we know that the operator defined by $(I + H(jw))^{-1}$ is 2-stable. Define $\bar{U}(jw) = (I + H(jw))^{-1}U(jw)$, so that we can get $\|y\|^2 = \|\bar{u}\|^2 + \|z\|^2 - 2\int_0^\infty z'\bar{u}$, where $Z(jw) = H(jw)\bar{U}(jw)$ and $Y(jw) = G(jw)U(jw) = (I - H(jw))(I + H(jw))^{-1}U(jw)$. Also note that $\|u\|^2 = \|\bar{u}\|^2 + \|z\|^2 + 2\int_0^\infty z'\bar{u}$, so that

$$\|y\|^2 = \|u\|^2 - 4\int_0^\infty z'\bar{u} \tag{2}$$

Passivity of H is equivalent to

$$\int_0^\infty z'\bar{u} \geq \gamma\|u\|^2$$

which, according to (2) is equivalent to

$$\|y\|^2 < \|u\|^2$$

or $\|G\| < 1$.