Massachusetts Institute of Technology
6.241 Dynamic Systems

Prof. M.A. Dahleh **Midterm Exam**

Instructions

1. During the exam, you can use the course notes, this year's homework and solutions, and any additional handouts given in the class.

2. Please, write down the solution of each problem on a separate page. Be organized and clear. If you write down an argument and you think it is not completely correct, indicate exactly where the problem is. This way I can assess the level of confidence you have in your arguments.

Problem 1 (30 points) This problem consists of *short* computational problems.

1. Let $A = (1 \quad 1 \quad 1)'$ and $b = (1 \quad 2 \quad 1)'$. Find x that minimizes $\|Ax - b\|_2$.

2. Let $A = (1 \quad 1 \quad 1)'$ and $b = (1 \quad 2 \quad 1)'$. Let Δ and e be 3×1 vectors. Find the minimum value of $\|\Delta\|_2^2 + \|e\|_2^2$ such that $(A + \Delta)x = b + e$ for some x.

3. Let $\dot{x} = Ax$ be an unforced system with initial state $x_0 = (1, 1, 1, 1)'$, and A
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
Find $x(t), \forall t \geq 0$.

4. Find a state space realization for the following Multi-input Single-output system:
$$\begin{bmatrix} \dfrac{1}{s-1} & \dfrac{s}{s+2} \end{bmatrix}$$

5. Verify whether the following system is unstable, stable i.s.L., or asymptotically stable:
$$\dot{x}_1 = -2x_1x_2^2, \qquad \dot{x}_2 = -2x_1^2x_2 - 4x_2^3$$

Problem 2 (30 points) Prove or disprove (by a counter example) the following statements.

1. Suppose $\dot{x} = (A+\Delta)x$ and A is known to be a stable symmetric matrix of size $n \times n$. Then

   $$\min_{\Delta}\{\sigma_{max}[\Delta] | \text{system is not asymptotically stable, } \Delta \text{ is a real matrix}\} = \min_{1 \le i \le n} |\lambda_i(A)|$$

   where $\lambda_i(A)$ denotes the ith eigenvalue of A.

2. Let P and Q be positive definite matrices. Then

   $$\max_{x}\{x'Px | x'Qx \le 1\} = \lambda_{max}[Q^{-1}P].$$

   (Hint: Q can be written as $Q = F'F$)

3. Let $x_{k+1} = Ax_k$ and $y_k = Cx_k$ be a discrete time system of order n where C is a matrix of rank $r < n$. If for every x_0 the output satisfies $y_r = 0$, then $dim(\mathcal{N}(A)) \ge r$.

Problem 3 (40 points) Given the discrete-time system

$$x_{k+1} = vw'x_k + bu, \quad y_k = c'x_k$$

where v, w, b, c are all n-dimensional vectors.

a. Answer the following questions for general v, w, b, c:

1. Under what conditions (on v and w) is this system asymptotically stable? What are the eigenvalues of the system.

2. For a zero input and initial condition x_0, compute y_k, $k \geq 0$.

3. Assume that the transition matrix is perturbed from vw' to $v(w+\delta)'$. Find the smallest $\|\delta\|_2$ that results in an unstable system (not asymptotically stable).

b. Assume that v, w, b, c have the following numerical values:

$$v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

We are interested in relocating the eigenvalues of this system. One approach for doing this is to use constant output feedback; i.e., $u = \alpha y$ for some gain α.

1. Is this system asymptotically stable without feedback?

2. What is the resulting transition matrix after feedback?

3. What are the new eigenvalues of the system? Can α be chosen so that the resulting eigenvalues are all inside the unit disc?